What makes a Tree a Straight Skeleton?*

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Abstract

Let G be a cycle-free connected straight-line graph with predefined edge lengths and fixed order of incident edges around each vertex. We address the problem of deciding whether there exists a simple polygon P such that G is the straight skeleton of P. We show that for given G such a polygon P might not exist, and if it exists it might not be unique. For the later case we give an example with exponentially many suitable polygons. For small star graphs and caterpillars we show necessary and sufficient conditions for constructing P.

Considering only the topology of the tree, that is, ignoring the length of the edges, we show that any tree whose inner vertices have degree at least 3 is isomorphic to the straight skeleton of a suitable convex polygon.

1 Introduction

The straight skeleton $\mathcal{S}(P)$ of a simple polygon P is a skeleton structure like the Voronoi diagram, but consists of straight-line segments only. Its definition is based on a so-called *wavefront propagation* process that corresponds to mitered offset curves. Each edge e of P emits a wavefront that moves with unit speed to the interior of P. Initially, the wavefront of P consists of parallel copies of all edges of P. However, during the wavefront propagation, topological changes occur: An *edge event* happens if a wavefront edge shrinks to zero length. A *split event* happens if a reflex wavefront vertex meets a wavefront edge and splits the wavefront into pieces, see Figure 1. The *straight skeleton* $\mathcal{S}(P)$ is defined as the set of loci that are traced out by the wavefront vertices and it partitions P into polygonal faces. Each face $f(e)$ belongs to a unique edge e of P. Each straight-skeleton edge belongs to two faces, say $f(e_1)$ and $f(e_2)$, and lies on the bisector of e_1 and e_2.

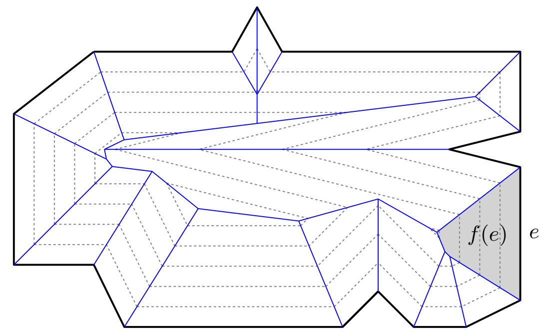

Figure 1: The straight skeleton (thin) of a simple polygon (bold) is defined by the propagating wavefront (dotted).

Straight skeletons have many applications, like automated roof construction, computation of mitered offset curves, topology-preserving collapsing of areas in geographic maps, or solving fold-and-cut problems. See [4] and Chapter 5.2 in [3] for further information and detailed definitions.

Although straight skeletons were introduced to computational geometry in 1995 by Aichholzer et al. [1], their roots actually go back to the 19th century. In textbooks about the construction of roofs (see e.g. [6], pages 86–122) using the angle bisectors (of the polygon defined by the ground walls) was suggested to design roofs where rainwater can run off in a controlled way. This construction is called *Dachausmittlung* and became rather popular. See [5] for related and partially more involved methods to obtain roofs from the ground plan of a house. In this book detailed explanations of the constructions and drawings of the resulting roofs can be found.

Maybe not surprisingly, none of this early work mentions the ambiguity of the non-algorithmic definition of the construction. It can be shown that using solely bisector graphs does not necessarily lead to a unique roof

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construction, and actually does not even guarantee a plane partition of the interior of the defining boundary. See [1] for a detailed explanation and examples.

An interesting inverse problem was motivated to us by Lior Pachter and investigations started in [2]: Which graphs are the straight skeleton of some polygon? In other words, how can straight skeletons be characterized among all graphs?

2 Finding straight skeletons of given topology

First of all, the straight skeleton $\mathcal{S}(P)$ of a simple polygon is known to be connected and cycle-free. Hence, we can rephrase our question as follows: which trees T are realized as straight skeletons of simple polygons? At first we concentrate on the topological structure of trees only and ignore their geometry.

Theorem 1 *For any tree T, whose inner vertices have at least degree 3, there exists a feasible (convex) polygon P such that $\mathcal{S}(P)$ possesses the same topology as T.*

Note that within convex polygons the straight skeleton and the Voronoi diagram are identical. Hence, by the above theorem, any tree is also isomorphic to the Voronoi diagram of a suitable convex polygon.

Proof. We first choose any inner vertex v of T to be the root of T. Then we construct a regular polygon P_1 with $d(v)$ sides, where $d(v)$ denotes the degree of v. The straight skeleton $\mathcal{S}(P_1)$ is a star graph comprising one inner node and $d(v)$ incident edges, which correspond to v and its incident edges in T. It remains to attach the corresponding subtrees from T to each leaf vertex of $\mathcal{S}(P_1)$, if there are any.

In the remainder of the proof we describe an inductive step by which we locally transform a polygon P_i to P_{i+1} in order to attach to a leaf vertex u of $\mathcal{S}(P_i)$ a missing number $k = d(u) - 1$ of incident edges, where $d(u)$ denotes the degree of the vertex of T that corresponds to u. Applying this technique recursively — e.g., in a breadth-first search fashion starting from v — gives us finally a polygon P_m, where m denotes the number of inner nodes of T, whose straight skeleton $\mathcal{S}(P_m)$ is topologically equivalent to T by construction. Furthermore, in our induction step we guarantee that all polygons $P_1, \ldots, P_m$ remain convex.

For the induction step, we consider a leaf vertex u in $\mathcal{S}(P_i)$ and we denote by e the incident straight-skeleton edge of u. As u is a leaf vertex of $\mathcal{S}(P_i)$ it is also a polygon vertex of P_i. Hence, e lies on the straight-skeleton faces $f(s_1)$ and $f(s_2)$ of the two incident polygon edges s_1 and s_2 of u. Since P_i is convex by induction, all faces of $\mathcal{S}(P_i)$ are convex, too. Hence, the projection lines of the mid point u' of e onto s_1 and s_2 are completely contained in $f(s_1)$ and $f(s_2)$, respectively. Let us consider the circular arc C that is centered at u' and tangential to s_1 and s_2 such that C forms a round convex cap of the corner u of P_i, see Figure 2. In the induction step, we locally bevel the corner u of P_i by any convex polygonal chain with $k \geq 2$ vertices that is tangential to C. By that the edge e is truncated to u' and we obtain k additional straight-skeleton edges $e_1, \ldots, e_k$ that are incident to u', as desired. The resulting polygon P_{i+1} is again convex and a locally beveled version of P_i. Note that the remaining straight skeleton of P_i remains unchanged for P_{i+1}. □

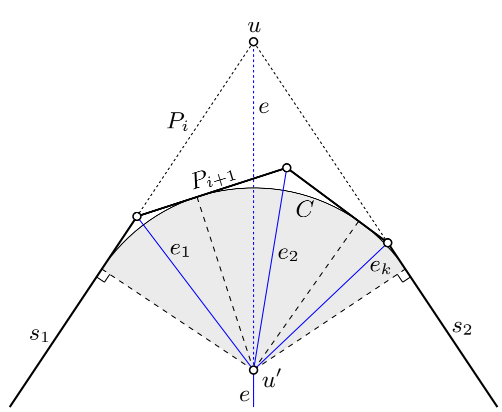

Figure 2: The induction step in order to add k edges to a terminal vertex of $\mathcal{S}(P_i)$, with $k = 3$, by beveling the corner u of P_i.

3 Abstract geometric trees

The original problem motivated by Lior Pachter and for which investigations started in [2] does not only ask for a specific topology of $\mathcal{S}(G)$, but also asks for certain geometric requirements that are to be fulfilled by $\mathcal{S}(P)$. In particular, we want to find a polygon P for which (i) $\mathcal{S}(P)$ has a specific topology, (ii) the edges of $\mathcal{S}(P)$ have a specific length and (iii) the cyclic order of incident edges at vertices of $\mathcal{S}(P)$ is given.

To give a more formal problem definition we denote with *abstract geometric graphs* the set of combinatorial graphs, where the length of each edge and the cyclic order of incident edges around every vertex is predefined (and cannot be altered). Let $\mathcal{G}$ be the set of cycle-free connected abstract geometric graphs. Denote with $E(G)$ an embedding of $G \in \mathcal{G}$ in the plane, that is, the vertices of G are points in $\mathbb{R}^2$ and the edges of G are straight-line segments of the predefined length, connecting the corresponding points and respecting the predefined cyclic order of incident edges around each vertex. Further, denote with $P_{E(G)}$ the polygon resulting from connecting the leaves of G (with straight-line segments) in cyclic order for the embedding $E(G)$. We call a simple polygon $P_{E(G)}$ *suitable* if its straight skeleton $\mathcal{S}(P_{E(G)}) = E(G)$, for the embedding $E(G)$. If there

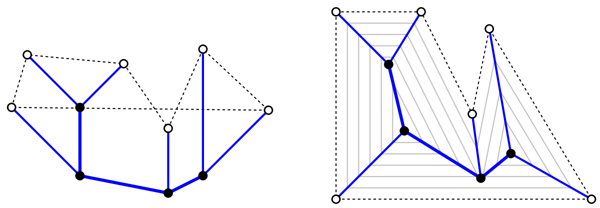

Figure 3: Example of a feasible cycle-free connected abstract geometric graph G (leaves of G are shown as white dots). Left: Arbitrary embedding $E(G)$ and (non-simple) polygon $P_{E(G)}$ (dotted). Right: Suitable polygon $P_{E'(G)}$ for a different embedding $E'(G)$, which is equal to $\mathcal{S}(P_{E'(G)})$. A set of wavefronts of $P_{E'(G)}$ at different points in time are depicted in gray.

exists a suitable polygon for a graph $G \in \mathcal{G}$, we call G *feasible*, see Figure 3.

The obvious questions which arise from these definitions are: Which graphs of $\mathcal{G}$ are feasible? Are the suitable polygons for feasible graphs unique modulo rigid motions? How can one construct a suitable polygon for a feasible graph?

3.1 Star graphs

All polygon edges whose straight-skeleton faces contain a common vertex u (of the straight skeleton) have equal orthogonal distance t to u, because their wavefront edges reach u at the same time t. That is, the supporting lines of those polygon edges are tangential to the circle with center u and radius t. Thus, in this section we consider a subset of $\mathcal{G}$, the so called star graphs. A *star graph* $S_n \in \mathcal{G}$, for $n \geq 3$ has $(n+1)$ vertices, one vertex u with degree n and n leaves $v_1, \ldots, v_n$ ordered counter-clockwise around u. The length of each edge uv_i, with $1 \leq i \leq n$, is denoted by l_i. W.l.o.g. let $l_1 = \max_i l_i$. Observe that the polygon $P_{E(S_n)}$ is star shaped and $v_i v_{i+1}$ (with $v_{n+k} := v_{1+(k-1) \bmod n}$) are its edges.

Observation 1 *If $S_n \in \mathcal{G}$ is a feasible star graph and $P_{E(S_n)}$ is a suitable polygon of S_n, then (1) all straight-skeleton faces are triangles, (2) two consecutive vertices v_i, v_{i+1} can not both be reflex, (3) $l_i < l_{i\pm1}$ for each reflex vertex v_i of $P_{E(S_n)}$, and (4) all edges of $P_{E(S_n)}$ have equal orthogonal distance t to u, with $t \in (0, \min_i l_i]$.*

As a given $S_n \in \mathcal{G}$ is possibly not feasible and a suitable polygon may not be known or might not exist, we define a polyline $L_{S_n}(t, A)$: The vertices $v_1, \ldots, v_{n+1}$ of $L_{S_n}(t, A)$ are the leaves, $v_1, \ldots, v_n$, of S_n, in the same order as for S_n, and one additional vertex v_{n+1} succeeding v_n. The vertices $v_1, \ldots, v_n, v_{n+1}$ have the corresponding distances (predefined in S_n) $l_1, \ldots, l_n, l_1$ to u. A is an assignment for each vertex whether it should

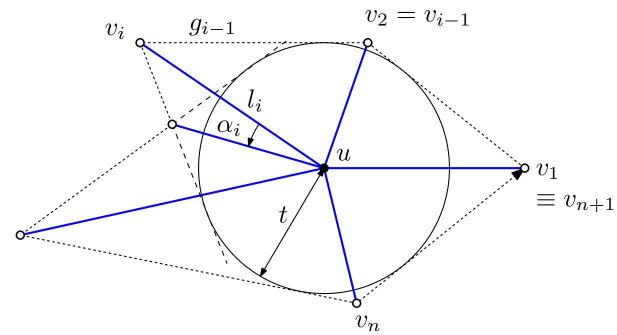

Figure 4: Construction of $L_{S_n}(t, A)$ (and $E(S_n)$) for a given S_n and a fixed distance t and assignment A.

be convex or reflex, as seen from u. As $l_1 = \max_i l_i$, v_1 and v_{n+1} are always convex (fact (3) in Observation 1). For the remaining vertices any convex/reflex assignment, which respects the facts (2) and (3) in Observation 1, can be considered. The edges of $L_{S_n}(t, A)$ have equal orthogonal distance t to u. Of course, not all possible combinations of t and an arbitrary embedding $E(S_n)$ allow such a polyline, but it is possible to construct $L_{S_n}(t, A)$ and $E(S_n)$ simultaneously for a fixed $t \in (0, \min_i l_i]$.

For a fixed assignment A and a fixed $t \in (0, \min_i l_i]$ we construct $L_{S_n}(t, A)$ (and $E(S_n)$) in the following way. Consider the circle C with center u and radius t. Start with v_1 at polar coordinate $(l_1, 0)$, with u as origin. For each v_i, $i = 2 \ldots (n+1)$, consider a tangent g_{i-1} to C (such that the vertices will be placed counter-clockwise around the circle) through v_{i-1}. If v_{i-1} is convex, then there exist two points with distance l_i (l_1 for v_{n+1}) on g_{i-1}. If v_i is assigned to be reflex, then v_i is placed on the point closer to v_{i-1}, and if v_i is assigned to be convex, then v_i is placed on the other point. If v_{i-1} is reflex, then there exists only one applicable point for placing v_i on g_{i-1}. See Figure 4.

The $L_{S_n}(t, A)$ constructed this way is unique (for fixed t and A), and may be not simple (e.g. when circling C many times), simple but not closed ($v_{n+1} \not\equiv v_1$), or simple and closed ($v_{n+1} \equiv v_1$). In the latter case, the construction reveals a witness pair (t, A) for the existence of some $E(S_n)$, a suitable polygon $P_{E(S_n)}$, and thus the feasibility of S_n.

It is easy to see that for each suitable polygon $P_{E(S_n)}$, there exists a polyline $L_{S_n}(t, A)$ (just duplicate the vertex v_1). Hence, deciding feasibility of S_n is equivalent to finding an assignment A and a $t \in (0, \min_i l_i]$ such that $L_{S_n}(t, A)$ is closed and simple. For a polyline $L_{S_n}(t, A)$ and a corresponding embedding $E(S_n)$, we denote with α_i, $i = 1 \ldots n$, the counter-clockwise angle at u, spanned by uv_i and uv_{i+1}. Note that for a suitable polygon $P_{E(S_n)}$ α_i can be defined the same way, with $v_{n+1} \equiv v_1$. It is easy to see that the sum of all α_i is 2π if and only if $L_{S_n}(t, A)$ is closed and simple.

Lemma 2 *Let $S_n \in \mathcal{G}$, distance $t \in (0, \min_i l_i]$ and assignment A be fixed, and let $L_{S_n}(t, A)$ be the resulting polyline. Then $\alpha_A(t) := \sum_{i=1}^{n} \alpha_i$ can be expressed as*

$$\alpha_A(t) = 2 \sum_{\substack{i=1 \\ v_i \text{ convex}}}^{n} \arccos\frac{t}{l_i} - 2 \sum_{\substack{i=1 \\ v_i \text{ reflex}}}^{n} \arccos\frac{t}{l_i}\ . \quad (1)$$

Proof. Recall that v_1 and v_{n+1} are convex by the assumption $l_1 = l_{n+1} = \max_i l_i$. It is easy to see that $\alpha_i = \arccos\frac{t}{l_i} + \arccos\frac{t}{l_{i+1}}$ if v_i is convex and v_{i+1} is convex, $\alpha_i = \arccos\frac{t}{l_i} - \arccos\frac{t}{l_{i+1}}$ if v_i is convex and v_{i+1} is reflex, and $\alpha_i = -\arccos\frac{t}{l_i} + \arccos\frac{t}{l_{i+1}}$ if v_i is reflex and v_{i+1} is convex. If v_i is reflex then $\alpha_{i-1} + \alpha_i$ sums up to $\left(\arccos\frac{t}{l_{i-1}} + \arccos\frac{t}{l_i}\right) + \left(\arccos\frac{t}{l_i} + \arccos\frac{t}{l_{i+1}}\right) - 4\arccos\frac{t}{l_i}$, because $v_{i\pm1}$ are both convex (fact (2) in Observation 1). Thus, summing over all α_i results in the claimed formula. □

We define a suitable polygon to be *unique* if it is the only suitable polygon modulo rigid motions. For the following result we use the first derivative of α_A:

$$\alpha'_A(t) = 2 \sum_{\substack{i=1 \\ v_i \text{ reflex}}}^{n} \frac{1}{\sqrt{l_i^2 - t^2}} - 2 \sum_{\substack{i=1 \\ v_i \text{ convex}}}^{n} \frac{1}{\sqrt{l_i^2 - t^2}}. \quad (2)$$

Lemma 3 *A suitable convex polygon for a star graph S_n exists if and only if $\sum_i \arccos\frac{\min_i l_i}{l_i} \le \pi$. If a suitable convex polygon exists then it is unique.*

Proof. As all vertices are assumed to be convex, we obtain $\alpha_A(0) = n\pi > 2\pi$. Furthermore, we observe that $\alpha_A(t)$ is monotonically decreasing since $\alpha'_A(t) < 0$ for all $t \in (0, \min_i l_i]$. Hence, there is a $t \in (0, \min_i l_i]$ with $\alpha(t) = 2\pi$ if and only if $\alpha_A(\min_i l_i) \le 2\pi$ which is $\sum_i \arccos\frac{\min_i l_i}{l_i} \le \pi$. If this is the case the solution is unique as $\alpha(t)$ is monotonic. □

For $n = 3$, $\alpha_A(0) = 3\pi$ and $\alpha_A(\min_i l_i) < 2\pi$, and thus we immediately get the following corollary.

Corollary 4 *For every S_3 there exists a unique suitable convex polygon.*

Considering star graphs with $n = 5$, we show in the following lemma that they are not always feasible, and that suitable polygons (if they exist) are not always unique.

Lemma 5 *There exist infeasible star graphs, $S_n \in \mathcal{G}$. Further, there exist feasible star graphs for which multiple suitable polygons exist.*

Proof. To prove the first claim consider a star graph with $n = 5$, $l_1 = l_2 = l_3 = l_4 = 1$, and $l_5 = 0.25$. There exist only two possible assignments: either all vertices convex or all but v_5 convex. It is easy to check that for both assignments $\sum_i \alpha_i > 2\pi$, for every $t \in (0, \min_i l_i]$. To prove the second claim consider a star graph with $n = 5$, $l_1 = l_3 = 1$, $l_2 = 0.6$, $l_4 = 0.79$, and $l_5 = 0.75$. Assign all vertices convex, except for v_2. Then $\sum_i \alpha_i$ evaluates to 2π for $t \approx 0.537$ and $t \approx 0.598$. Hence, there exist (at least) two different suitable polygons for this star graph. □

Note that for the latter example two suitable polygons exist that even share the same reflexivity assignment. In the following we discuss sufficient and necessary conditions for the feasibility of a star graph S_4. By Lemma 3 we know in which cases suitable convex polygons exist. The remaining cases are solved by the following lemma.

Lemma 6 *Consider an S_4 for which no suitable convex polygon exists. A suitable non-convex polygon exists if and only if $\frac{1}{\min_i l_i} < \sum_{j=1, l_j \ne \min_i l_i} \frac{1}{l_j}$.*

Proof. First of all, if a polyline has two or more reflex vertices assigned then $\alpha_A(t) < 2\pi$, as each positive summand in Equation (1) is bound by $\pi/2$. Hence, we only need to consider polylines with exactly one reflex vertex, which implies $\alpha_A(0) = 2\pi$.

For simplicity, we may reorder v_i and l_i such that $l_4 = \min_i l_i$. We show that for suitable non-convex polygons v_4 needs to be reflex. Assume to the contrary that some v_k, with $1 \le k \le 3$, is reflex. In this case we obtain that $\alpha'_A(t) < 0$ as $1/\sqrt{l_4^2 - t^2}$ dominates $1/\sqrt{l_k^2 - t^2}$ for all $t \in [0, l_4)$. But since $\alpha_A(0) = 2\pi$ we see that $\alpha_A(t) < 2\pi$ for all $t \in (0, \min_i l_i]$.

Observe that the assumption in the lemma, that no suitable convex polygon exists, is equivalent to $\alpha_A(l_4) > 2\pi$. Recall that $\alpha_A(0) = 2\pi$. Hence, if $\alpha'_A(0) < 0$ then there exists a $t \in (0, l_4)$ such that $\alpha_A(t) = 2\pi$, as α_A is continuously differentiable.

Finally, we show that if $\alpha'_A(0) \ge 0$ then $\alpha'_A(t) > 0$ for all $t \in (0, l_4)$. Hence, there is no $t \in (0, l_4]$ such that $\alpha_A(t) = 2\pi$. From Equation (2) we get that $\alpha'_A(t) > 0$ is equivalent to

$$\frac{1}{\sqrt{l_4^2 - t^2}} > \sum_{i=1}^{3} \frac{1}{\sqrt{l_i^2 - t^2}} \quad \Leftrightarrow \quad 1 > \sum_{i=1}^{3} \sqrt{1 - \frac{l_i^2 - l_4^2}{l_i^2 - t^2}}$$

The right side of this equivalence is true since

$$1 \ge \sum_{i=1}^{3} \sqrt{1 - \frac{l_i^2 - l_4^2}{l_i^2}} > \sum_{i=1}^{3} \sqrt{1 - \frac{l_i^2 - l_4^2}{l_i^2 - t^2}}, \quad (3)$$

where the first inequality is given by $\alpha'_A(0) \ge 0$ and the second inequality holds for all $t \in (0, l_4)$.

To conclude, we have shown that if no suitable convex polygon exists for some S_4, then a suitable non-convex polygon exists for this S_4 if and only if $\alpha'(0) < 0$, which is equivalent to $\frac{1}{\min_i l_i} < \sum_{j=1, l_j \ne \min_i l_i} \frac{1}{l_j}$, as claimed in the lemma. □

3.2 Caterpillar graphs

The techniques developed in the previous section can be generalized to so-called *caterpillar graphs*. A caterpillar graph $G \in \mathcal{G}$ is a graph that becomes a path if all its leaves (and their incident edges) are removed. We call this path the *backbone* of G. Figure 3 shows a caterpillar graph whose backbone comprises three backbone edges.

In general, a caterpillar graph has m backbone vertices, consecutively denoted by $v_0^1, \dots, v_0^m$. We denote the adjacent vertices of a backbone vertex v_0^i, with k_i incident edges, by $v_1^i, \dots, v_{k_i}^i$, such that $v_{k_i}^i = v_0^{i+1}$ for $1 \le i < m$. Furthermore, we denote by l_j^i the length of the edge $v_0^i v_j^i$, see Figure 5. Let us consider a polygon P whose straight skeleton $\mathcal{S}(P)$ forms a caterpillar graph.

Observation 2 *All edges of P whose straight-skeleton faces contain the same backbone vertex v_0^i have identical orthogonal distance to v_0^i.*

We denote this orthogonal distance by r_i. Hence, the supporting lines of the corresponding polygon edges are tangents to the circle of radius r_i centered at v_0^i, see Figure 5.

Lemma 7 *The radii $r_2, \dots, r_m$ of a suitable polygon $P_{E(G)}$ for some given caterpillar graph G are determined by r_1 and the predefined edge lengths of G according to the following recursions, for $1 \le i < m$:*

$$r_{i+1} = r_i + l_{k_i}^i \sin \beta_i$$
$$\beta_i = \beta_{i-1} + (1 - {}^{k_i}\!/_2)\pi + \sum_{\substack{j=1 \\ v_j^i \neq v_0^{i-1}}}^{k_i-1} \begin{cases} \arcsin \frac{r_i}{l_j^i} & v_j^i \text{ is convex} \\ \pi - \arcsin \frac{r_i}{l_j^i} & v_j^i \text{ is reflex} \end{cases}$$

For $i = 1$ we define that $\beta_0 = 0$ and $v_j^1 \neq v_0^0$ being true for all $1 \le j < k_1$.

Proof. Denote with e one of the two edges of $P_{E(G)}$ whose faces of $\mathcal{S}(P_{E(G)})$ contain the edge $v_0^i v_0^{i+1}$. The supporting line of e is tangential to the circles at v_0^i and v_0^{i+1}. Considering the shaded right-angled triangle in Figure 5, we obtain $r_{i+1} - r_i = l_{k_i}^i \cdot \sin \beta_i$.

Consider the polygon P_i' (bold in Figure 5) which comprises the edges of $P_{E(G)}$ whose faces of $\mathcal{S}(P_{E(G)})$ contain v_0^i, trimmed by two additional edges orthogonal to $v_0^{i-1} v_0^i$ and $v_0^i v_0^{i+1}$, respectively. P_i' comprises k_i+2 vertices (k_1+1 for P_1') and hence, the sum of inner angles equals $k_i \pi$ ((k_1−1)π for P_1'). On the other hand, we can express this sum as follows (also for P_1'), which implies the second recursion:

$$k_i \pi = 2\pi + 2\beta_{i-1} - 2\beta_i + 2 \sum_{\substack{j=1 \\ v_j^i \neq v_0^{i-1}}}^{k_i-1} \begin{cases} \arcsin \frac{r_i}{l_j^i} & v_j^i \text{ is convex} \\ \pi - \arcsin \frac{r_i}{l_j^i} & v_j^i \text{ is reflex} \end{cases}$$

□

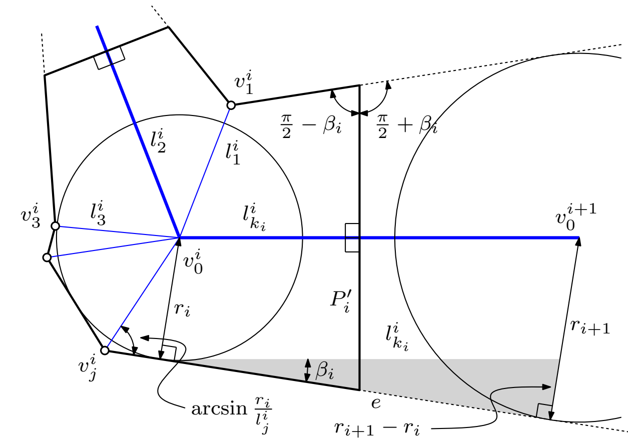

Figure 5: A section of a polygon P for which $\mathcal{S}(P)$ is a caterpillar graph.

Corollary 8 *The sum of the inner angles of $P_{E(G)}$ with convexity assignment A is a function*

$$\alpha_A(r_1) = 2 \sum_{j=1}^{n} \begin{cases} \arcsin \frac{r_{v_j}}{l_j} & v_j \text{ is convex} \\ \pi - \arcsin \frac{r_{v_j}}{l_j} & v_j \text{ is reflex} \end{cases}, \qquad (4)$$

where r_{v_j} denotes the radius of the circle at the backbone vertex that is adjacent to v_j and l_j denotes the length of the incident edge of G.

The previous corollary provides us with a tool in order to find suitable polygons $P_{E(G)}$ for caterpillar graphs G. We know that for any suitable polygon $P_{E(G)}$ the identity $\alpha_A(r_1) = (n-2)\pi$ must hold. Hence, we can determine all suitable polygons $P_{E(G)}$ as follows: for all 2^n possible assignments A we determine all r_1 such that $\alpha_A(r_1) = (n-2)\pi$.

For any such pair (A, r_1) we construct a polyline $v_1, \dots, v_n, v_{n+1}$ by a similar method as outlined for star graphs: shooting rays tangential to circles centered at the backbone vertices v_0^i. In order to switch over from v_0^i to v_0^{i+1}, we consider the previously constructed ray, which needs to be tangential to the two circles centered at both, v_0^i and v_0^{i+1}, respectively. As the length of the edge $v_0^i v_0^{i+1}$ is given, the center v_0^{i+1} of the next circle is uniquely determined, cf. Figure 5. If there is any non-backbone edge with length $l_j^i < r_i$ then there is no suitable polygon for that particular pair (A, r_1). For each candidate polyline we check whether it is closed, simple and forms a suitable polygon. Note that all suitable polygons can be constructed by the above method.

Lemma 9 *There is at most a finite number of suitable polygons $P_{E(G)}$ for a caterpillar graph G.*

Proof. As α_A is analytic, there are no accumulation points in the set $\{r_1 : \alpha_A(r_1) = (n-2)\pi\}$. Otherwise, α_A would be identical to $(n-2)\pi$. In other words, there is only a finite number of possible pairs (A, r_1) that correspond to a suitable polygon. □

Lemma 10 *There exists a caterpillar graph with $3m$ vertices having 2^{m-2} suitable polygons.*

Proof. We consider a caterpillar graph with m backbone vertices, for which the two outer backbone vertices are of degree three and the $m-2$ inner backbone vertices are of degree four. We set the length of the non-backbone edges to $\sqrt{2}/4$ and the length of the backbone edge $e_k = v_0^k v_0^{k+1}$ to $3/4 \cdot 2^{k-1}$. We embed the backbone as a rectilinear path and at each v_0^k, with $1 < k < m$, we either make a left or right turn from e_{k-1} to e_k. This gives us 2^{m-2} possible embeddings, see Figure 6. We now consider the polygon P shown in Figure 6, which forms a rectilinear hose (a mitered offset curve) around the embedding of G with thickness $\frac{1}{2}$. It remains to show that P is not self-overlapping.

For each edge e_k we consider the axis-parallel square $A_k \supseteq e_k$ with side length 2^k that has v_0^{k+1} as a midpoint of one of its sides. By our choice of edge lengths we observe that $A_{k-1} \subseteq A_k$ and A_{k-1} and A_k share a common vertex. We say that a polygon edge belongs to e_i if e_i is contained in its straight-skeleton face. Let s denote a polygon edge that belongs to e_{k+1}. By construction, s cannot overlap with polygon edges belonging to e_k. Using an induction-type argument, all polygon edges belonging to e_i, with $i < k$, are contained in the one axis-parallel half of A_k that does not contain v_0^{k+1}. Hence, s does not intersect polygon edges that belong to $e_1, \ldots, e_k$. It follows by induction that P is not self-overlapping. □

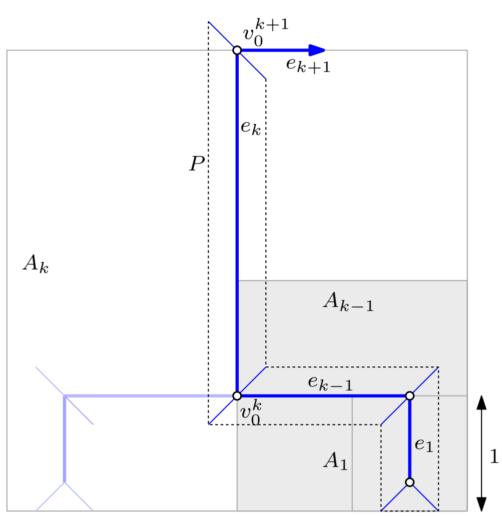

Figure 6: For the above caterpillar graph an exponential number of polygons exist by making left or right turns at backbone vertices.

4 Conclusion

In this work, we considered the inverse problem of computing the straight skeletons of simple polygons. First, we proved that each tree, whose inner vertices have degree at least 3, can be realized as the straight skeleton of a convex polygon. The constructive proof also provides the outline for an algorithm to construct a suitable polygon.

Next, we considered a more restrictive version of the original question by predefining the lengths of the straight-skeleton edges and their circular orders at straight-skeleton vertices. For star graphs we showed that there exists a unique suitable convex polygon for every S_3. Further, we derived that for general star graphs feasibility is neither guaranteed nor unique, and we gave sufficient and necessary conditions for the existence of a suitable polygon for all S_4. Furthermore, we gave a simple necessary and sufficient condition that tells us when a suitable convex polygon exists for a star graph.

Concerning the more general caterpillar graphs we provided a basic method for constructing all suitable polygons and we proved that the number of suitable polygons is finite. Furthermore, we showed that an n-vertex caterpillar graph may possess $2^{n/3-2}$ suitable polygons. Finding a tight upper bound on the number of suitable polygons is an open question. Finally, the major open questions concern arbitrary trees: How to decide feasibility? Are there at most finitely many feasible polygons or is there even an entire continuum of feasible polygons? After all, a partial result is given in [2]: there are at most $2n-5$ suitable convex polygons for an arbitrary abstract geometric tree with n leaves.

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