Note on parameters of balanced ternary designs

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Abstract

Conditions under which constant block size implies constant replication number and vice versa are established for balanced ternary designs.

This note is motivated by a note of almost the identical title of W. D. Wallis [2]. In a balanced binary design with constant block size, the replication number must be constant, but the converse is not true. Wallis [2] proved the following partial converse:

Theorem 1. If the positive integers v, b, r, and λ are such that vr = bk and $\lambda(v-1) = r(k-1)$ for some integer k, and if there is an (r, λ)-design on v treatments with b blocks, then the design has constant block size k.

We assume that the reader is familiar with the definition of ternary design and related notation including non-equireplicate balanced ternary designs. See for example Billington [1]. A balanced ternary design with constant block size does not imply constant replication. For example, consider the balanced ternary design 112, 134, 134, 223, 224 with $\Lambda = 2$ and block size $K = 3$ but whose replication number is not constant. On the other hand, if we have a design with constant replication including ρ_1, the number of times an element occurs singly, and ρ_2, the number of times an element occurs doubly, this does not imply constant block size. For example the balanced ternary design 1125, 1344, 13, 224, 2335, 455 is a balanced ternary design with $\Lambda = 2$ and the replication number $R = 4$, $\rho_1 = 2$, $\rho_2 = 1$ but the block size is not constant.

We prove the following result which is interesting in the sense that similar conditions like the binary case are required in the ternary case for constant ρ_1 and ρ_2 to imply constant block size, but one more condition is required to prove the converse.

Theorem 2. Suppose the positive integers V, B, $R=\rho_1+2\rho_2$, K, Λ are such that

$$VR = BK \qquad (1)$$

$$\Lambda(V-1) = \rho_1(K-1) + 2\rho_2(K-2) \qquad (2)$$

and suppose there is a balanced ternary design on V elements, B blocks with balance Λ. If the balanced ternary design is equireplicate with constant ρ_1 and ρ_2 then it has constant block size K. Conversely, suppose in the balanced ternary design element i occurs r_{i1} times singly and r_{i2} times doubly and

$$\sum_{i=1}^{V}(r_{i1}+2r_{i2})^2 = VR^2;$$

if the design has constant block size then it is equireplicate with $r_{i1} = \rho_1$ and $r_{i2} = \rho_2$ for each element i.

Proof: Suppose that a balanced ternary design with replication number R and balance Λ exists, with B_i blocks of size K_i where i is from a finite index set. We have

$$\Sigma B_i = B \qquad (3)$$

$$\Sigma B_i K_i = VR \qquad (4)$$

$$\Sigma B_i K_i(K_i-1) = \Lambda V(V-1) + 2V\rho_2 \qquad (5)$$

From (1) and (4), the mean of the block sizes $(\Sigma B_i K_i)/(\Sigma B_i)$ is K.
From (4) and (5),

$$\Sigma B_i K_i^2 = VR + \Lambda V(V-1) + 2V\rho_2 \qquad (6)$$

Now

$$\begin{aligned}\Sigma B_i(K_i-K)^2 &= \Sigma B_i K_i^2 + \Sigma B_i K^2 - 2\Sigma B_i K_i K \\ &= \Sigma B_i K_i^2 - \Sigma B_i K^2 = \Sigma B_i K_i^2 - BK^2 \\ &= VR + \Lambda V(V-1) + 2V\rho_2 - VRK \qquad \text{(from (6) and (1))} \\ &= V[\rho_1(K-1) + 2\rho_2(K-2)] - VR(K-1) + 2V\rho_2 \\ &= VR(K-1) - 2V\rho_2 - VR(K-1) + 2V\rho_2 \\ &= 0.\end{aligned}$$

Therefore $K_i = K$ for all i.

To prove the converse, we need the added condition

$$\sum_{i=1}^{V}(r_{i1}+2r_{i2})^2 = VR^2.$$

Observe

$$\begin{aligned}\Sigma((r_{i1}+2r_{i2}) - R)^2 &= \Sigma(r_{i1}+2r_{i2})^2 - 2R\Sigma(r_{i1}+2r_{i2}) + VR^2\\ &= VR^2 - VR^2 = 0.\end{aligned}$$

This implies that $(r_{i1} + 2r_{i2}) = R$, for all i. Now it is well known that a balanced ternary design with constant K and R (i.e. equireplicate BTD) is regular, therefore $r_{i1} = \rho_1$ and $r_{i2} = \rho_2$ for some constant ρ_1 and ρ_2 as required.

References

[1] Elizabeth J. Billington, Balanced n-ary designs: A combinatorial survey and some new results, Ars Combinatoria, **17A** (1984), 37-72.

[2] W.D. Wallis, Note on the parameters of balanced designs, Australasian Journal of Combinatorics, **8** (1993), 75-76.

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