POSITIVE BINARY LABELLING OF GRAPHS

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ABSTRACT:

Let $G = (V,E)$ be a finite undirected graph with vertex set V and edge set E. A mapping $f : E \to \{0,1\}^m$ is called an M-coding of G, if the induced mapping $g : V \to \{0,1\}^m$, defined as $g(v) = \sum_{u \in V, uv \in E} f(uv)$, assigns different vectors to the vertices of G, where all summations are mod 2. An M-coding f is called positive if the zero vector is not assigned to any edge and the induced labelling g does not assign the zero vector to any vertex. Let $m(G)(m^+G))$ be the smallest number m for which an M-coding (positive M-coding) of G is possible. Trivially, $m^+(G) \geq m(G) \geq \lceil \mathrm{Log}_2|V| \rceil$. Recently, Aigner and Triesch proved that $m(G) \leq \lceil \mathrm{Log}_2|V| \rceil + 4$. In a recent paper, we determined $m(G)$. Tuza proved that $m^+(G) \leq m(G) + 2$. In this paper we prove that

$$m^+(G) = \begin{cases} k+1, & \text{if } |V| = 2^k, 2^k-2 \text{ or } 2^k-3 \\ k, & \text{otherwise}, \end{cases}$$

where $k = \lceil \log_2|V| \rceil$.

1. INTRODUCTION

Let $G = (V,E)$ be a finite undirected graph with vertex set V and edge set E. The graph theory literature contains numerous problems concerned with labelling graphs. Such problems typically involve the determination of a function f on $V \cup E$, that satisfies certain conditions. A well-known example is that of graceful labelling [4,6,7]. Most graph labellings are concerned with assigning natural numbers to the vertices and edges of the graph. Recently, labellings with subsets of a finite set have been studied [1,2,3,5,8]. Obviously such labellings can be interpreted as labellings with 0,1-sequences. In this paper we consider such labellings.

More precisely, let $K = GF(2)$ be the 2-element finite field. A mapping $f : E(G) \to K^m$ is called an **M-coding** of G if the induced mapping $g : V(G) \to K^m$, defined as

$$g(v) = \sum_{u \in V, uv \in E} f(uv)$$

assigns different vectors to the vertices of G, where the addition is co-ordinatewise (mod 2) in K^m, m is called the **length** of the coding f. The smallest m for which an M-coding of G exists is denoted by $m(G)$. Trivially $m(G) \geq \lceil \log_2|V| \rceil$. Further, a necessary condition for a graph G to have an M-coding is : every component of G has at least 3 vertices; note that if G has a component consisting of just two vertices u and v, then $g(u) = g(v)$ for any f. We assume throughout this paper that every component of G has at least 3 vertices.

The problem of determining $m(G)$ has been considered by a number of authors. In 1990, Tuza [8] proved that $m(G) \leq 3 \lceil \log_2|V| \rceil$. Later Aigner and Triesch [1] established the better bound :

$$m(G) \leq \lceil \log_2|V| \rceil + 4.$$

Recently, Caccetta and Jia [5] established that :

$$m(G) = \begin{cases} k, & \text{if } |V| \neq 2^k-2 \\ k+1, & \text{otherwise,} \end{cases} \tag{1.1}$$

where $k = \lceil \log_2|V| \rceil$.

A number of labellings with additional properties were proposed by Tuza [8]. One such labelling is the following. An M-coding f is called positive if the zero vector is not assigned to any edge and the induced labelling g does not assign the zero-vector to any vertex. The smallest m for which a positive M-coding exists is denoted by $m^+(G)$.

Tuza [8] proved that :

$$m^+(G) \leq m(G) + 2.$$

Consequently, from (1.1) we obtain that

$$m^+(G) \leq \lceil \log_2|V| \rceil + 3$$

for any graph G. In this paper, we prove that

$$m^+(G) = \begin{cases} k+1, & \text{if } |V| = 2^k, 2^k-2, \text{ or } 2^k-3 \\ k, & \text{otherwise,} \end{cases}$$

where $k = \lceil \log_2|V| \rceil$.

We conclude our introduction by noting that the M-coding problem can be viewed as a set labelling problem by considering the 0,1-vectors as characteristic vectors of subsets of a finite set X. We wish to label the edges of G with subsets of X such that the symmetric differences are distinct at the vertices of G. Here $m(G)$ ($m^+(G)$) becomes the cardinatlity of the smallest set which permits such a labelling (with the empty set excluded).

2. RESULTS

We establish our main results by exhibiting the required labellings inductively. We begin with some notation and terminology.

As stated in the introduction, the M-coding problem can be viewed as a set labelling problem. We use standard set theoretic notation and terminology. Throughout, standard upper case letters will denote sets whilst script upper case letters will denote family of sets. Thus for sets A and B we write :

$$A \backslash B = \{x : x \in A, \text{ and } x \notin B\} \qquad \text{(A not B)}$$

$$A \Delta B = (A \cup B) \backslash (A \cap B) \qquad \text{(symmetric difference)}$$

Let $A_1, A_2, \ldots, A_t$ denote t sets. We write

$$A_1 \Delta A_2 \Delta \ldots \Delta A_t = \sum_{i=1}^{t} \Delta A_i .$$

A family $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ of subsets of 2^X is called a **zero-subset** if

$$\sum_{i=1}^{t} \Delta C_i = \phi.$$

In our work we will often deal with subgraphs of G and utilize the following terminology. For a graph G and set X let

$$g : V(G) \to 2^X$$

be any vertex labelling of G. For a subgraph H of G we define

$$g(H) = \sum_{v \in V(H)} \Delta g(v).$$

Trees will play an important role in our work. Let T be a tree and $v \in V(T)$ and $e \in E(T)$. T-e consists of two components. The component containing v is denoted by $B[T;v,e]$; the other component is denoted by $\bar{B}[T;v,e]$.

The following lemma plays a crucial role in the proof of our main result.

Lemma 2.1 : Let T be a tree with n vertices and X a set. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_n\}$ denote a family of distinct subsets of $2^X \setminus \{\phi\}$. Then for every $v \in V(T)$, there exists a 1-1 mapping

$$g : V(T) \to \mathscr{C}$$

such that

$$g(\bar{B}[T;v,e]) \neq \phi \qquad \text{for every } e \in E(T).$$

Proof : We use induction on n. The result is trivially true for $n = 1$ and 2. So suppose, as our inductive hypothesis that $n \geq 3$ and the assertion is true for all $n' < n$. We will prove that the assertion is true for n.

For $v \in V(T)$, we choose any edge, say $e_0 = vv_1$, incident to v. Let $T_1 = \bar{B}[T;v,e_0]$ and $T_2 = B[T;v,e_0]$ be the two components of $T-e_0$. Let $n_i = |V(T_i)|$, $i = 1,2$. From $\mathscr{C}$ we can choose a subset $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_{n_1}\}$ such that

$$\sum_{i=1}^{n_1} \Delta C'_i \neq \phi .$$

Now by our induction hypothesis, there exists a 1-1 mapping

$$g_1 : V(T_1) \to \mathscr{C}'$$

such that

$$g_1(\bar{B}[T_1;\ v_1,\ e]) \neq \phi$$

for every $e \in E(T_1)$.

Next we consider T_2 and the set $\mathscr{C}'' = \mathscr{C} \setminus \mathscr{C}'$. Our induction hypothesis implies the existence of a 1-1 mapping

$$g_2 : V(T_2) \to \mathscr{C}''$$

such that

$$g_2(\bar{B}[T_2;\ v,\ e]) \neq \phi$$

for every $e \in E(T_2)$.

Define

$$g(u) = \begin{cases} g_1(u), & \text{if } u \in V(T_1) \\ g_2(u), & \text{if } u \in V(T_2). \end{cases}$$

We will establish that g is the required mapping. Let $e \in E(T)$. We distinguish 3 cases according to whether e is in T_1 or in T_2 or is e_0.

If $e \in E(T_1)$, then $\bar{B}[T;v,e] = \bar{B}[T_1;v_1,e]$ and hence $g(\bar{B}[T;v,e]) = g_1(\bar{B}[T_1;v_1,e]) \neq \phi$.

If $e \in E(T_2)$, then $\bar{B}[T;v,e] = \bar{B}[T_2;v,e]$ and hence $g(\bar{B}[T;v,e]) = g_2(\bar{B}[T_2;v,e]) \neq \phi$.

Finally, if $e = e_0$, then $\bar{B}[T;v,e] = T_1$ and hence

$$\begin{aligned} g(\bar{B}[T,v,e]) &= g(T_1) \\ &= g_1(T_1) \\ &\neq \phi \quad \text{(because of the choice of } \mathcal{E}'\text{)}. \end{aligned}$$

This completes the proof of the lemma. □

Our next lemma establishes a positive edge labelling for a tree T given a positive vertex labelling of T. For convenience we introduce the following notation.

Let G be a graph, X a set and f an mapping of E(G) onto 2^X. For $v \in E(G)$ we let

$$S_G^v(f) = \sum_{\substack{uv \in E(G) \\ u \in V(G)}} \Delta f(uv) \tag{2.1}$$

and

$$L_G(f) = \left\{ S_G^v(f) \;:\; v \in V(G) \right\}. \tag{2.2}$$

Note that $S_G^v(f)$ defines a mapping

$$g \,:\, V(G) \to 2^X.$$

This is the mapping induced by f. Further, observe that f is an M-coding of G if and only if the induced mapping g is 1-1.

Lemma 2.2 : Let T be a tree with n vertices and X a set. Lt $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ denote a family of distinct subsets of $2^X \setminus \{\phi\}$ such that

$$\sum_{i=1} \Delta C_i = \phi \,.$$

Then there exists a positive M-coding f of T such that $L_T(f) = \mathcal{C}$.

Proof : Let $v_0 \in V(T)$. By Lemma 2.1, there exists a 1-1 mapping

$$g \,:\, V(T) \to \mathcal{C}$$

such that

$$g(\bar{B}[T; v_0, e]) \neq \phi, \quad \text{for every } e \in E(T).$$

Observe that

$$g(\bar{B}[T; v_0, e]) + g(B[T; v_0, e]) = g(T)$$

$$= \sum_{i=1}^{n} \Delta C_i$$

$$= \phi.$$

Consequently

$$g(\bar{B}[T;v_0,e]) = g(B[T;v_0,e]),$$

for every $e \in E(T)$.

Define the mapping

$$f : E(T) \to 2^X \backslash \{\phi\},$$

as

$$f(e) = g(\bar{B}[T;v_0,e]).$$

Now for each $v \in V(T)$

$$S_T^v(f) = \sum_{\substack{uv \in E(T) \\ u \in V(T)}} \Delta f(uv)$$

$$= \sum_{\substack{uv \in E(T) \\ u \in V(T)}} \Delta g(\bar{B}[T;v_0,e=uv])$$

$$= \sum_{\substack{uv \in E(T) \\ u \in V(T)}} \Delta g(\bar{B}[T;v,uv])$$

$$= \sum_{\substack{v' \neq v \\ v' \in V(T)}} \Delta g(v')$$

$$= \left\{ \sum_{\substack{v' \neq v \\ v' \in V(T)V)}} \Delta g(v') \right\} \Delta \Phi$$

$$= \left\{ \sum_{\substack{v' \neq v \\ v' \in V(T)}} \Delta g(v')\Delta g(v) \right\} \Delta g(v)$$

$$= g(T)\ \Delta g(v)$$

$$= g(v) \qquad \text{(as } g(T) = \phi\text{)}.$$

Consequently

$$L_T(f) = \left\{ S_T^v(f) : v \in V(T) \right\} = \{g(v) : v \in V(T)\} = \mathscr{C},$$

as required. □

Our next lemma was proved in [5] (Corollary of Theorem 1).

Lemma 2.3 : Let X be a set with $|X| = k \geq 2$ and $2^k = 3p + 4q + 5r + 1$. Then $2^X \setminus \{\phi\}$ can be partitioned into p zero-subsets of order 3, q zero-subsets of order 4 and r zero-subsets of order 5. □

We make use of the above lemma to establish the existence of zero-subsets of prescribed order. This results plays an important role in the proof of our main theorem.

Lemma 2.4 : Let X be a set with $|X| = k \geq 2$ and $n_1, n_2, \ldots, n_t$ positive integers satisfying the following conditions :

(i) $\quad n_i \geq 3, \qquad$ for all i,

(ii) $\quad \sum_{i=1}^{t} n_i \leq 2^k - 1$, and

(iii) $\sum_{i=1}^{t} n_i \neq 2^k-2$ or 2^k-3.

Then there exists zero-subsets $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_t$ of $2^X \setminus \{\phi\}$ such that $|\mathscr{C}_i| = n_i$, $1 \le i \le t$, and $\mathscr{C}_i \cap \mathscr{C}_j = \phi$ for $i \neq j$.

Proof : We can write $n_i = 3p_i + 4q_i + 5r_i$, $1 \le i \le t$. Let

$$p = \sum_{i=1}^{t} p_i, \qquad q = \sum_{i=1}^{t} q_i, \qquad r = \sum_{i=1}^{t} r_i,$$

and

$$2^k - \sum_{i=1}^{t} n_i = s.$$

Condition (ii) implies that $s \ge 1$. So

$$2^k = 3p + 4q + 5r + s$$
$$= 3p + 4q + 5r + (s-1) + 1.$$

Now by (iii) $s-1 \ge 3$ or $s-1 = 0$. We can write

$$s-1 = 3p_0 + 4p_0 + 5r_0$$

and thus

$$2^k = 3(p + p_0) + 4(q + q_0) + 5(r + r_0) + 1.$$

Now, by Lemma 2.3, $2^X \setminus \{\phi\}$ can be partitioned into $(p + p_0)$ zero-subsets P_i $(1 \le i \le p + p_0)$ of order 3, $(q + q_0)$ zero-subsets Q_i $(1 \le i \le q + q_0)$ of order 4 and $(r + r_0)$ zero-subsets R_i $(1 \le i \le r + r_0)$ of order 5. We can use these zero-subsets in the obvious way (choosing p_i elements of $\{P_i\}$, q_i elements of $\{Q_i\}$ and r_i elements of $\{R_i\}$) to construct the desired subsets of $\mathscr{C}_i$, $1 \le i \le t$, of $2^X \setminus \{\phi\}$. This completes the proof of the lemma. □

We are now ready to prove our main theorem.

Theorem 1 : Let G be a graph with each component having order at least 3. Then

$$m^+(G) = \begin{cases} k+1, & \text{if } |V| = 2^k,\ 2^k-2 \text{ or } 2^k-3 \\ k, & \text{otherwise,} \end{cases}$$

where $k = \lceil \log |V| \rceil$.

Proof : Let $G_1, G_2, \ldots, G_t$ be the components of G. Choose a spanning tree T_i of G_i, $1 \le i \le t$ and let $F = \bigcup_{i=1}^{t} T_i$; F is a spanning forest of G. Further, let $n = |V(G)|$ and $n_i = |V(G_i)|$, $1 \le i \le t$. We can write

$$2^k = n + d.$$

We consider two cases according to the value of d.

Case 1 : $d \ne 0, 2$ or 3.

We can write

$$\begin{aligned} 2^k &= n + d \\ &= n + n_2 + \ldots + n_t + (d-1) + 1, \end{aligned}$$

with $d-1 \ge 3$ or $d - 1 = 0$. Lemma 2.4 implies the existence of zero-subsets $\mathcal{C}_i$, $1 \le i \le t$, of $2^X \setminus \{\phi\}$ such that

$$|\mathcal{C}_i| = n_i, \qquad \text{for } 1 \le i \le t$$

and

$$\mathcal{C}_i \cap \mathcal{C}_j = \phi \quad \text{for } i \ne j.$$

For each T_i, $1 \le i \le t$, Lemma 2.2 implies the existence of a mapping :

$$f_i : E(T_i) \to 2^X \setminus \{\phi\}$$

such that (see (2.2))

$$L_{T_i}(f_i) = \mathcal{E}_i.$$

Now define

$$f : E(F) \to 2^X \setminus \{\phi\}$$

as

$$f(e) = f_i(e), \qquad \text{for } e \in E(T_i), \quad 1 \le i \le t.$$

Then

$$\begin{aligned} L_F(f) &= \bigcup_{i=1}^{t} L_{T_i}(f) \\ &= \bigcup_{i=1}^{t} L_{T_i}(f_i) \\ &= \bigcup_{i=1}^{t} \mathcal{E}_i. \end{aligned}$$

Consequently, f is a positive M-coding of F. Using this f we now construct the desired mapping for G.

Choose an edge $e_0 \in E(G) \setminus E(F)$. Suppose without any loss of generality that $e_0 \in G_1$. The subgraph $T_1 \cup \{e_0\}$ contains a unique cycle C. Since $C - e_0$ is a path with at most $n_1 - 1$ edges, and $n_1 - 1 \le n - 1 \le 2^k - 2$, we have $\mathcal{A} = \{f(e) : e \in C - e_0\} \ne 2^X \setminus \{\phi\}$.

Now choose $B \in \{2^X \setminus \{\phi\}\} \setminus \mathcal{A}$.

Define the mapping :

$$f_1 : E(F) \cup \{e_0\} \to 2^X$$

as

$$f_1(e) = \begin{cases} B, & \text{if } e = e_0 \\ f(e)\Delta B, & \text{if } e \in C-e_0 \\ f(e), & \text{otherwise.} \end{cases}$$

The choice of B ensures that $f(e) \neq \phi$ for any e. We now prove that

$$S^v_{F+e_0}(f_1) = S^v_F(f)$$

for every $v \in V(G)$. If $v \notin V(C)$, then

$$S^v_{F+e_0}(f_1) = \sum_{\substack{u \in V(G) \\ uv \in E(F+e_0)}} \Delta f_1(uv)$$

$$= \sum_{\substack{u \in V(G) \\ uv \in E(F)}} \Delta f(uv)$$

$$= S^v_F(f),$$

as required.

Suppose now that $v \in V(C)$. Let u_1 and u_2 be the two neighbours of v in C. We have

$$S^{v}_{F+e_0}(f) = S^{v}_{T_1+e_0}(f_1)$$

$$= \sum_{\substack{u\in V(G)\\ uv\in E(T_1+e_0)}} \Delta f_1(uv)$$

$$= \left(\sum_{\substack{u\neq u_1 \text{ or } u_2\\ uv\in E(T_1+e_0)}} \Delta f(uv)\right) \Delta f_1(u_1v)\Delta f_1(u_2v)$$

$$= \left(\sum_{\substack{u\neq u_1 \text{ or } u_2\\ uv\in E(T_1)}} \Delta f(uv)\right) \Delta(f(u_1v)\Delta B)\Delta(f(u_2v)\Delta B)$$

$$= \sum_{\substack{u\in V(T_1)\\ uv\in E(T_1)}} \Delta f(uv)$$

$$= S^{v}_{T_1}(f)$$

$$= S^{v}_{F}(f).$$

Note that in the above, when $u_iv = e_0$ we replace $f(u_iv)$ by ϕ. We have shown that $S^{v}_{F+e_0}(f_1) = S^{v}_{F}(f)$ for every $v \in u(G)$, as required. Consequently

$$L_{F+e_0}(f_1) = L_F(f)$$

and hence f_1 is a positive M-coding of $F + e_0$.

We can repeat the above process, adding the edges of $E(G)\backslash E(F)$ one by one. In each case the cycle considered is with respect to the added edge and the original F. Ultimately, we end up with the required mapping, thus establishing the result for $d \neq 0,2$ or 3.

Case 2 : $d = 0$, 2 or 3.

Here $n = 2^k$, 2^k-2 or 2^k-3. Since $n < 2^{k+1}-3$, the conclusion of Case 1 above clearly implies that $m^+(G) \leq k + 1$. We distinguish these subcases according to the value of n. For $n = 2^k$, every M-coding f of G of length k yields a set $L_G(f)$ which contains ϕ. Consequently $m^+(G) \neq k$ for $n = 2^k$.

For $n = 2^k-2$, $m^+(G) \neq k$ since

$$\begin{aligned} m^+(G) &\geq m(G) \\ &= k + 1. \quad \text{(by (1.1))}. \end{aligned}$$

We next consider the case $n = 2^k-3$. Let X be a set of k elements. Consider the mapping $f : E(G) \to 2^X$ and its induced mapping $g : V(G) \to 2^X$ defined by :

$$g(v) = \sum_{\substack{u \in V \\ uv \in E}} \Delta f(uv).$$

We have

$$\sum_{v \in V} g(v) = \sum_{v \in V} \sum_{\substack{u \in V \\ uv \in E}} \Delta f(uv) = \phi, \tag{2.3}$$

since each edge labelling appears twice in the above sum. Suppose there exists a zero-subset $S_0 = \{A,B,\phi\}$ of order 3. Then as $A \Delta B \Delta \phi = \phi$ we must have $A \Delta B = \phi$ implying that $A = B$. Since this is not possible there cannot exist a zero-subset of order 3 containing ϕ. Consequently we cannot find a zero-subset of $2^X\backslash\{\phi\}$ of order 2^k-3 and thus $m^+(G) \neq k$ (note (2.3)). □

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