More on Embedding Partial Totally Symmetric Quasigroups

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Abstract

In this paper, it is shown that any partial totally symmetric quasigroup of order n can be embedded in a totally symmetric quasigroup of order v, $v \geq 4n+4$, $v \equiv 0 \pmod 4$. This result is combined with an earlier result obtained by Raines and Rodger to show that any partial totally symmetric quasigroup of order n can be embedded in a totally symmetric quasigroup of order v, for all even $v \geq 4n+4$. This bound can be improved to $4n+2$ in most cases.

1 Introduction

A *(partial) quasigroup* is an ordered pair $(Q, \circ)$ where Q is a set and $\circ$ is a binary operation on Q such that for every a, $b \in Q$, there exists (at most one) x, $y \in Q$ satisfying the equations $a \circ x = b$ and $y \circ a = b$. A *totally symmetric quasigroup* is a quasigroup that satisfies the identities $x \circ y = y \circ x$ and $y \circ (x \circ y) = x$ for all x, $y \in Q$. A *partial totally symmetric quasigroup* is a partial quasigroup in which: if $x \circ y$ exists then so does $y \circ x$ and $x \circ y = y \circ x$; and if $x \circ y$ and $y \circ (x \circ y)$ exist then $y \circ (x \circ y) = x$.

(Partial) totally symmetric quasigroups can be represented in graph theoretical terms. Let K_n^+ be the complete graph on n vertices with exactly one loop incident with each vertex (loops are considered to be edges here). Define an *extended triple* to be a loop, a loop with an edge attached (also known as a *lollipop*), or a copy of K_3 (also known as a *triple*). We denote a loop by $\{a, a, a\}$, a lollipop by $\{a, a, b\}$, $a \neq b$, when the loop of the lollipop is incident with vertex a, and a triple by $\{a, b, c\}$, where a, b, and c are distinct. A *(partial) extended triple system* of order n is an ordered pair (V, B), where B is a set of extended triples defined on the vertex set V which partitions (a subset of) the edges of K_n^+. We denote a partial extended triple system and an extended triple system of order n by $PETS(n)$ and $ETS(n)$, respectively. It has been shown (see, for example, [5]) that a (partial) totally symmetric quasigroup of order n is equivalent to a (partial) extended triple system of order n.

D. M. Johnson and N. S. Mendelsohn [6] first investigated extended triple systems and gave necessary conditions for their existence; F. E. Bennett and N. S. Mendelsohn [2] showed the sufficiency of these conditions.

A $PETS(n)(V, B)$ is said to be *embedded* in an $ETS(V', B')$ if $V \subseteq V'$ and $B \subseteq B'$. D. G. Hoffman and C. A. Rodger [5] showed that a complete totally symmetric quasigroup of order n can be embedded in one of order $v > n$ if and only if $v > 2n$, v is even if n is, and $(n, v) \neq (6k + 5, 12k + 12)$. Subsequently, M. E. Raines and C. A. Rodger [9] showed that any partial totally symmetric quasigroup of order n can be embedded in a complete totally symmetric quasigroup of order v, for all $v \geq 4n + 6, v \equiv 2 \pmod 4$ and showed that this bound on v can be lowered to $4n + 2$ in many cases.

The following theorem is the main focus of the paper.

Theorem 1.1 *Any partial totally symmetric quasigroup of order n can be embedded in a totally symmetric quasigroup of order v for all even $v \geq 4n + 4$.*

The technique used to prove Theorem 1.1 follows closely the ideas used in [9], but the details vary considerably. For terms and notation not defined here, we refer the reader to [3].

2 Preliminary Results

We start by stating a famous result due to Turán.

Lemma 2.1 ([12]) *If a simple graph G on n vertices contains no K_3, then $\epsilon(G) \leq \lfloor \frac{n^2}{4} \rfloor$.*

A *near 1-factor* of a graph G is a set of mutually nonadjacent edges in G which saturates all but one vertex of G. We have the following well-known result.

Lemma 2.2 *If n is even (odd), then the edges of K_n can be partitioned into (near) 1-factors.*

Let Γ be any edge-coloring of G. Let $G_\alpha, \alpha \in \Gamma$, denote the set of edges colored α in this edge coloring of G. The edge-coloring is said to be *equalized* if $||G_\alpha| - |G_\beta|| \leq 1$, for all $\alpha, \beta \in \Gamma$.

Lemma 2.3 ([8] [14]) *A graph which has a proper n-edge-coloring has an equalized proper n-edge-coloring.*

A *(partial) symmetric quasi-latin square* of *order* r on the symbols $1, \ldots, n$ is an $r \times r$ array of cells such that

(i) for each i, j, $1 \leq i, j \leq r$, $i \neq j$, if a symbol is in cell (i, j) then it is also in (j, i),

(ii) for each i, j, $1 \leq i, j \leq r$, $i \neq j$, cell (i, j) contains at most one symbol (the diagonal cells can contain any number of symbols), and

(iii) each symbol occurs (at most) exactly once in each row and (at most) exactly once in each column. (This is known as the *latin property*).

Define $N_L(i)$ to be the number of times symbol i occurs in some (partial) symmetric quasi-latin square L.

Theorem 2.4 *Let $n \geq 1$, r be odd, and $x \in \{0, 1\}$. Let L be a partial symmetric quasi-latin square of order $r + x$ on the symbols $1, \ldots, 2n+1$ in which row i contains $r+x-2$ symbols for $1 \leq i \leq r$, and in which row $r+1$ (when $x = 1$) contains $r+x-1$ symbols. Then L can be embedded in the top left corner of a symmetric quasi-latin square L' of order $2n+3$ in which the diagonal cells (i, i), $r+x+1 \leq i \leq 2n+3$, and the near-diagonal cells $(r+2i-1, r+2i)$ and $(r+2i, r+2i-1)$, $1 \leq i \leq n-(r-3)/2$ are empty, without adding any symbols to the cells in L if and only if*

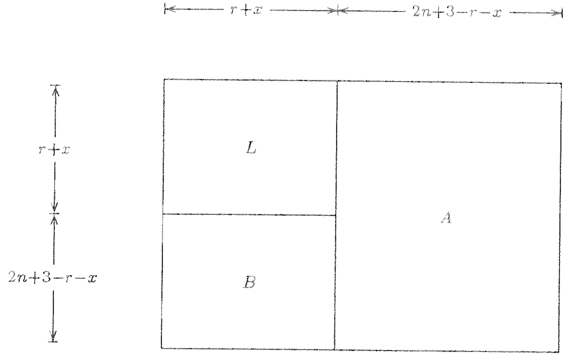
(i) $N_L(i)$ is odd for $1 \leq i \leq 2n+1$, and

(ii) $N_L(i) \geq 2(r+x) - 2n - 3$ for $1 \leq i \leq 2n+1$.

Proof: Necessity. Each symbol must occur $2n+3$ times in L', and no symbol can be placed on the main diagonal of L' outside L. Therefore, we have that symbols are placed in pairs in L' outside L, so each symbol must occur an odd number of times in L; thus (*i*) is necessary. Let A and B be as indicated in Figure 1, and let $N_A(i)$ and $N_B(i)$ be the number of times symbol i occurs in A and B, respectively. Then $N_A(i) = 2n+3-(r+x)$ and $N_B(i) \leq 2n+3-(r+x)$; Therefore, $N_L(i) = N_{L'}(i) - N_A(i) - N_B(i) \geq 2n+3-2(2n+3-(r+x)) = 2(r+x) - 2n - 3$, so (ii) is necessary.

Sufficiency. Suppose $x = 0$, let $r \leq s < n+2$ with s odd, and proceed by induction on s. Assume that s rows and columns have been completed so that each row contains $s-2$ symbols, thus forming L^*. Assume that for $1 \leq i \leq 2n+1$, $N_{L^*}(i)$ is odd, $N_{L^*}(i) \geq 2s - 2n - 3$, and that the appropriate diagonal and near-diagonal cells of L^* are empty. Two steps are necessary.

Step 1: Add row (column) $s+1$ to L^* to form a symmetric quasi-latin square, L_1, as follows. Form a bipartite graph B_1 with bipartition ($X = \{1, \ldots, 2n+1\}$, $Y = \{\rho_1, \ldots, \rho_s\}$) of the vertex set as follows: form the edge $\{i, j\}$ in B_1 if and only if symbol i, for $1 \leq i \leq 2n+1$, does not occur in row j of L^*, $1 \leq j \leq s$. We have that for each vertex $\rho_j \in Y$, $d_{B_1}(\rho_j) = 2n+1-(s-2) = 2n-s+3$, and $d_{B_1}(i) = s - N_{L^*}(i) \leq s-(2s-2n-3) = 2n-s+3$, with equality if $N_{L^*}(i) = 2s-2n-3$. Therefore, B_1 has maximum degree $\Delta(B_1) = 2n-s+3$, so B_1 can be properly $(2n-s+3)$-edge colored by König's theorem [7]. Choose one of these colors, say α. For every edge $\{i, \rho_j\}$ in B_1 with color α, place symbol i in cells $(s+1, j)$ and $(j, s+1)$. We have that row (column) $s+1$ is latin since color α occurs at most once at each



L'

Figure 1:

vertex in X, and each of the cells $(s+1,k)$ and $(k,s+1)$, $1 \leq k \leq s$, contains exactly one symbol since color α occurs exactly once at each vertex in Y. Since L^* is latin, L_1 is latin because of the way in which we defined B_1; a symbol occurs in $(j,s+1)$ or $(s+1,j)$ only if it does not occur previously in row (column) j of L^*, $1 \leq j \leq s$. Since there is no vertex ρ_{s+1} in B_1, cell $(s+1,s+1)$ remains empty, so row i of L_1 contains $s-1$ symbols if $1 \leq i \leq s$, and s symbols if $i = s+1$. In addition, $N_{L_1}(i)$ is odd since by assumption $N_{L^*}(i)$ is odd and since each symbol is added 0 or 2 times in forming L_1 from L^*. Finally, $N_{L_1}(i) \geq 2s-2n-1$, $1 \leq i \leq 2n+1$, since if $N_{L^*}(i) < 2s-2n-1$, then $N_{L^*}(i) = 2s-2n-3$ ($N_{L^*}(i)$ is odd and $N_{L^*}(i) \geq 2s-2n-3$), so $d_{B_1}(i) = \Delta(B_1)$; therefore, i is incident with an edge colored α in B_1 and this means that symbol i occurs in row and column $s+1$.

Step 2: Add row (column) $s+2$ to form a symmetric quasi-latin square L_2 as follows. Form a bipartite graph B_2 with bipartition $(X = \{1,\ldots,2n+1\},\ Y = \{\rho_1,\ldots,\rho_{s+1}\})$ of the vertex set as follows: form the edge $\{i,j\}$ in B_2 if and only if symbol i, for $1 \leq i \leq 2n+1$, does not occur in row j of L_1, $1 \leq j \leq s+1$. We have that for each vertex $\rho_j \in Y$, $1 \leq j \leq s$, $d_{B_2}(\rho_j) = 2n+1-(s-1) = 2n-s+2$. However, row $s+1$ contains s symbols, so $d_{B_2}(\rho_{s+1}) = 2n-s+1$. For each vertex $i \in X$, $d_{B_2}(i) = s+1-N_{L_1}(i) \leq s+1-(2s-2n-1) = 2n-s+2$, with equality if $N_{L_1}(i) = 2s-2n-1$. Therefore, $\Delta(B_2) = 2n-s+2$, so B_2 can be properly $(2n-s+2)$-edge colored. Let α be the color not found at vertex ρ_{s+1}. For every edge $\{i,\rho_j\}$ in B_2 with the color α, place symbol i in cells $(s+2,j)$ and $(j,s+2)$. We have that row (column) $s+2$ is latin since color α occurs at most once at each vertex in X, and each of the cells $(s+2,k)$ and $(k,s+2)$ for $1 \leq k \leq s$ contains exactly one symbol since color α occurs exactly once at each vertex in Y except for ρ_{s+1}. Furthermore, cells $(s+1,s+2)$, $(s+2,s+1)$, and $(s+2,s+2)$ remain empty since B_2 contains no vertex ρ_{s+2} and since vertex ρ_{s+1} is incident with no edge colored α; thus each row of L_2 contains s symbols. Since L_1 is latin, L_2 is latin because of the way in which we defined B_2; a symbol occurs in $(j, s+2)$ or $(s+2, j)$ only if it does not occur previously in row (column) j of L_1, for $1 \leq j \leq s+1$. We have that $N_{L_2}(i)$ is odd since $N_{L_1}(i)$ is odd and since each symbol is added 0 or 2 times in forming L_2 from L_1. In addition, $N_{L_2}(i) \geq 2s-2n+1$, for $1 \leq i \leq 2n+1$, since if $N_{L_i}(i) < 2s-2n+1$, then $N_{L_1}(i) = 2s-2n-1$ ($N_{L_1}(i)$ is odd and $N_{L_1}(i) \geq 2s-2n-1$), so $d_{B_2}(i) = \Delta(B_2)$; hence, symbol i is added to row and column $s+2$. This completes the induction step and the proof if $x = 0$.

If $x = 1$, first apply Step 2 and then apply the proof when $x = 0$. This completes the proof. □

A partial Steiner triple system of order n (PSTS(n)) is an ordered pair (S, T) where T is a set of edge-disjoint copies of K_3, or *triples*, that form a subgraph $G(S)$ of K_n with vertex set S. We define the *leave* of (S, T) to be the complement of $G(S)$ in K_n.

Let $\mu(n)$ denote the maximum possible number of triples in a partial Steiner triple system of order n, PSTS(n).

Lemma 2.5 ([11])

$$\mu(n) = \begin{cases} \lfloor \frac{1}{3} n \lfloor \frac{1}{2}(n-1) \rfloor \rfloor & \textit{for } n \not\equiv 5 \ (\textit{mod } 6) \\ \lfloor \frac{1}{3} n \lfloor \frac{1}{2}(n-1) \rfloor \rfloor - 1 & \textit{for } n \equiv 5 \ (\textit{mod } 6) \end{cases}$$

For a PSTS(n) on the vertex set $\{1, \ldots, n\}$, let $r(i)$ denote the number of triples which contain symbol i. If $|r(i) - r(j)| \leq 1$, for $1 \leq i < j \leq n$, the PSTS(n) is said to be *equitable*. The existence of equitable PSTS(n)s has been settled [1], but here we need the additional property stated in the following lemma.

Lemma 2.6 ([10]) *There exists an equitable partial STS(n) (S, T) with $t(n)$ triples such that the leave contains a 1-factor if n is even and a near 1-factor if n is odd if and only if $t(n) \leq T(n)$, where*

$$T(n) = \begin{cases} \mu(n) & = n(n-2)/6 & \textit{if } n \equiv 0 \ (\textit{mod } 6) \\ \mu(n) - \lfloor n/3 \rfloor & = (n-1)(n-2)/6 & \textit{if } n \equiv 1 \ (\textit{mod } 6) \\ \mu(n) & = n(n-2)/6 & \textit{if } n \equiv 2 \ (\textit{mod } 6) \\ \mu(n) - n/3 & = n(n-3)/6 & \textit{if } n \equiv 3 \ (\textit{mod } 6) \\ \mu(n) - 1 & = (n-4)(n+2)/6 & \textit{if } n \equiv 4 \ (\textit{mod } 6) \\ \mu(n) - (n-5)/3 & = (n-1)(n-2)/6 & \textit{if } n \equiv 5 \ (\textit{mod } 6). \end{cases}$$

A graph G is a *star multigraph* if there is some vertex of G which is incident with every multiple edge of G.

Lemma 2.7 ([4]) *If G is a star multigraph, then $\chi'(G) \leq \Delta(G) + 1$.*

3 Embedding a PETS(n) in an ETS($4n + 4$)

Given any PETS(n) (V, B), define the *deficiency graph*, $G(B)$, to be the graph on the vertex set V whose edge set consists of the edges of K_n^+ not found in any extended triple in B. Let $w(G(B))$ denote the number of vertices of even degree in $G(B)$, and let

$$W(G(B)) = \begin{cases} w(G(B)) & \text{if } \epsilon(G(B)) + w(G(B)) \equiv 0 \pmod 3, \\ w(G(B)) + 2 & \text{if } \epsilon(G(B)) + w(G(B)) \equiv 1 \pmod 3, \\ w(G(B)) + 4 & \text{if } \epsilon(G(B)) + w(G(B)) \equiv 2 \pmod 3. \end{cases}$$

We say that (V, B) is maximal if $G(B)$ contains no extended triples (so $G(B)$ contains no loops).

Lemma 3.1 *Let (V, B) be a maximal $PETS(u), u \geq 2$. Then (V, B) can be embedded in a $PETS\ (2u + 1)(V^*, B^*)$ satisfying:*

(i) $\Delta(G(B^*)) \leq u$

(ii) $W(G(B^*)) \leq u + 1$,

(iii) $\epsilon(G(B^*)) + W(G(B^*)) \leq 3T(u + 2)$, *and*

(iv) $G(B^*)$ *contains at least two vertices of degree at most* $u - 1$.

Proof: Let $V = \{1, \ldots, u\}$ and $V^* = \{1, \ldots, 2u + 1\}$.
Case 1: u is odd (so $w(G(B)) \neq 0$).
Since $w(G(B)) \neq 0$, there is at least one vertex of even degree in $G(B)$. Without loss of generality, assume vertex u has even degree. Furthermore, if $w(G(B)) = u$, then we can assume $d_{G(B)}(u) \leq u - 3$, for if two or more vertices had degree $u - 1$, then (V, B) would no longer be maximal. Define B^* as follows.

(1) $B \subseteq B^*$.

(2a) If $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} \equiv 0 \pmod 3$ or if $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} \equiv 1 \pmod 3$ and $w(G(B)) \leq u - 2$, then B^* contains the lollipops $\{u + i, u + i, u\}$, $1 \leq i \leq u + 1$.

(2b) If $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} \equiv 1 \pmod 3$ and $w(G(B)) = u$, then B^* contains the lollipops $\{u + i, u + i, u\}$, $2 \leq i \leq u - 1$, the lollipop $\{u + 1, u + 1, 2u + 1\}$, and the loops at vertices $2u$ and $2u + 1$.

(2c) If $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} \equiv 2 \pmod 3$, then B^* contains the lollipops $\{u + i, u + i, u\}$, $1 \leq i \leq u$, and the loop at vertex $2u + 1$.

(3) Using Lemma 2.2, partition the edges of K_{u+1} defined on the vertex set $\{u + 1, \ldots, 2u + 1\}$ into the 1-factors $F_1, \ldots, F_u$. Assume without loss of generality that F_u contains the edges $\{u+1, 2u+1\}, \{u+2, 2u\}, \ldots, \{u+\frac{u+3}{2}\}, \{u+\frac{u+5}{2}\}$. For each edge $\{a, b\} \in F_v$, $1 \leq v \leq u - 1$, let B^* contain the triple $\{v, a, b\}$.

Since (V, B) is maximal, $\epsilon(G(B)) \leq \lfloor \frac{u^2}{4} \rfloor$ by Lemma 2.1. From (3) we have that $E(G(B^*))$ contains F_u. Therefore: if (2a) applies then we have $E(G(B^*)) = E(G(B)) + F_u$, so $\epsilon(G(B^*)) = \epsilon(G(B)) + \frac{u+1}{2} \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+1}{2}$; if (2b) applies then we have that $E(G(B^*)) = E(G(B)) + F_u \setminus \{\{u+1, 2u+1\}\} \cup \{\{u, u+1\}, \{u, 2u\}, \{u, 2u+1\}\}$, so $\epsilon(G(B^*)) = \epsilon(G(B)) + (\frac{u+1}{2} - 1) + 3 \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+5}{2}$; if (2c) applies then $E(G(B^*)) = E(G(B)) \cup F_u \cup \{\{u, 2u+1\}\}$, so $\epsilon(G(B^*)) = \epsilon(G(B)) + \frac{u+1}{2} + 1 \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+3}{2}$. In addition, $\Delta(G(B)) \leq u - 1$. However, in cases (2a) and (2c), $d_{G(B^*)}(i) \leq d_{G(B)}(i) + 1 \leq (u-1) + 1 = u$, for $1 \leq i \leq u$, and $d_{G(B^*)}(i) \leq 2$, for $u + 1 \leq i \leq 2u + 1$, so $\Delta(G(B^*)) \leq u$; in case (2b), $d_{G(B^*)}(u) \leq d_{G(B)}(u) + 3 \leq u$ (since $w(G(B)) = u$, we can assume $d_{G(B)}(u) \leq u - 3$), $d_{G(B^*)}(i) = d_{G(B)}(i) \leq u$, for $1 \leq i \leq u - 1$, and $d_{G(B)}(j) \leq 2$, for $u + 1 \leq j \leq 2u + 1$, proving (i).

Also, in the above construction, $w(G(B)) = w(G(B^*))$. In (2a), the vertices $u + 1, \ldots, 2u + 1$ have odd degree in $G(B^*)$ and all vertices $1, \ldots, u$ which have even (odd) degree in $G(B)$ have even (odd) degree in $G(B^*)$; in (2b) the same argument applies for vertices $1, \ldots, u - 1$, but $d_{G(B^*)}(u) = d_{G(B)}(u) + 3$ (so u has odd degree), $d_{G(B^*)}(2u) = 2$, and $d_{G(B^*)}(i) = 1$ for every $i \in \{u + 1, \ldots, 2u - 1, 2u + 1\}$, so $w(G(B^*)) = w(G(B))$; in (2c), a similar argument to the one used in (2a) applies except for the fact that vertex u has odd degree and vertex $2u + 1$ has even degree in $G(B^*)$. In any event, $w(G(B^*)) \leq u$. Clearly, $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} = \epsilon(G(B^*)) + w(G(B^*))$ in (2a); $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} = \epsilon(G(B^*)) + w(G(B^*)) - 2$ in (2b); and, $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} = \epsilon(G(B^*)) + w(G(B^*)) - 1$ in (2c). Therefore, we have in (2a) $W(G(B^*)) = w(G(B^*))$ if $\epsilon(G(B^*)) + w(G(B^*)) = \epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} \equiv 0 \pmod 3$, and if $\epsilon(G(B^*)) + w(G(B^*)) \equiv 1 \pmod 3$ then $W(G(B^*)) = w(G(B^*)) + 2 \leq u$, as $w(G(B^*)) \leq u - 2$; in (2b), $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} = \epsilon(G(B^*)) + w(G(B^*)) - 2 \equiv 1 \pmod 3$, so $\epsilon(G(B^*)) + w(G(B^*)) \equiv 0 \pmod 3$ which means that $w(G(B^*)) = W(G(B^*)) = u$; in (2c), $\epsilon(G(B)) + w(G(B)) + \frac{u+1}{2} = \epsilon(G(B^*)) + w(G(B^*)) - 1 \equiv 2 \pmod 3$, so $\epsilon(G(B^*)) + w(G(B^*)) \equiv 0 \pmod 3$ which means that $w(G(B^*)) = W(G(B^*)) \leq u$, proving (ii).

We now investigate $\epsilon(G(B^*)) + W(G(B^*))$. In (2a), $\epsilon(G(B^*)) \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+1}{2}$ and $W(G(B^*)) \leq u$, so $\epsilon(G(B^*)) + W(G(B^*)) \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+1}{2} + u \leq 3T(u+2)$, when $u \neq 3$; in (2b), $\epsilon(G(B^*)) + W(G(B^*)) \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+5}{2} + u \leq 3T(u+2)$ when $u \geq 7$ (it is also easily shown that $\epsilon(G(B^*)) + W(G(B^*)) \leq 3T(u+2)$ when $u = 5$, for we have that if $w(G(B)) = u$ and if (V, B) is maximal then $\epsilon(G(B)) \leq 5$); in (2c), $\epsilon(G(B^*)) + W(G(B^*)) \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+3}{2} + u \leq 3T(u+2)$ when $u \neq 3$. It is easily verified that if $u = 3$, the constructions provide a $PETS(7)(V^*, B^*)$ such that $\Delta(G(B^*)) \leq 3$, $W(G(B^*)) \leq 4$, and $\epsilon(G(B^*)) + W(G(B^*)) \leq 6 = 3T(5)$, thus proving (iii) for all odd $u \geq 3$. It is also very easily verified from the above constructions that $G(B^*))$ contains at least two vertices of degree at most $u - 1$, so (iv) is satisfied.

Case 2: u is even.

If $w(G(B)) \neq 0$, there are at least two vertices of even degree in $G(B)$. Without loss of generality, we can assume that vertices $u - 1$ and u are two such vertices. Furthermore, if $w(G(B)) = u$, we can assume that $d_{G(B)}(u) \leq u - 2$, for if u is even and all vertices have even degree in $G(B)$, then $\Delta(G(B)) \leq u - 2$. Define B^* as follows.

(1) $B \subseteq B^*$.

(2a) If $\epsilon(G(B))+w(G(B))+1+\frac{u}{2} \equiv 0 \pmod 3$ or if $\epsilon(G(B))+w(G(B))+1+\frac{u}{2} \equiv 1 \pmod 3$ *and* $w \leq u-2$ or if $\epsilon(G(B))+w(G(B))+1+\frac{u}{2} \equiv 2 \pmod 3$ *and* $w \leq u-4$ then B^* contains the lollipops $\{u+i, u+i, i\}$, for $1 \leq i \leq u$ and the loop at vertex $2u+1$.

(2b) If $\epsilon(G(B))+w(G(B))+1+\frac{u}{2} \equiv 1 \pmod 3$ *and* $w(G(B))=u$, then B^* contains the lollipops $\{u+i, u+i, i\}$, for $1 \leq i \leq u-2$, and the loops at vertices $2u-1$, $2u$, and $2u+1$.

(2c) If $\epsilon(G(B))+w(G(B))+1+\frac{u}{2} \equiv 2 \pmod 3$ *and* $w(G(B)) \in \{u-2, u\}$, then B^* contains the lollipops $\{u+i, u+i, i\}$, for $1 \leq i \leq u-1$, and the loops at the vertices $2u$ and $2u+1$.

(3) Using Lemma 2.2, partition the edges of K_{u+1} defined on the vertex set $\{u+1, \ldots, 2u+1\}$ into the near 1-factors $F_1, \ldots, F_{u+1}$, with the property that F_v does not saturate vertex $u+v$. For each edge $\{a,b\} \in F_v$, for $1 \leq v \leq u$, let B^* contain the triple $\{v, a, b\}$ (notice that the edges in F_{u+1} are not yet included in any extended triple in B^*).

Again, since (V, B) is maximal, $\epsilon(G(B)) \leq \lfloor \frac{u^2}{4} \rfloor$ by Lemma 2.1. We have from (3) that $E(G(B^*))$ contains F_{u+1}. Therefore, in (2a), $E(G(B^*)) = E(G(B)) \cup F_{u+1}$, so $\epsilon(G(B^*)) = \epsilon(G(B)) + \frac{u}{2} \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u}{2}$; in (2b), $E(G(B^*)) = E(G(B)) \cup F_{u+1} \cup \{\{2u-1, u-1\}, \{2u, u\}\}$, so $\epsilon(G(B^*)) = \epsilon(G(B)) + \frac{u}{2} + 2 \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+4}{2}$; and in (2c), $E(G(B^*)) = E(G(B)) \cup F_{u+1} \cup \{\{u, 2u\}\}$, so $\epsilon(G(B^*)) = \epsilon(G(B)) + \frac{u}{2} + 1 \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+2}{2}$. In addition, $\Delta(G(B)) \leq u-1$, so in any case $d_{G(B^*)}(u-1) \leq d_{G(B^*)}(u) \leq d_{G(B)}(u) + 1 \leq u$. Clearly, $d_{G(B^*)}(i) \leq u-1$, for $1 \leq i \leq u-2$, and $d_{G(B^*)}(j) \leq 2$, for $u+1 \leq j \leq 2u+1$, (so (iv) is satisfied for $u \geq 4$), so $\Delta(G(B^*)) \leq u$, thus proving (i).

We now investigate $W(G(B^*))$. Clearly, in the above construction, $w(G(B^*)) = w(G(B)) + 1$ (vertex $2u+1$ has degree zero in $G(B^*)$, and this accounts for the extra vertex of even degree in B^*). In (2a), we have that $\epsilon(G(B)) + w(G(B)) + 1 + \frac{u}{2} = \epsilon(G(B^*)) + w(G(B^*))$; clearly in all cases, $W(G(B^*)) \leq u+1$. In (2b), $\epsilon(G(B)) + w(G(B)) + 1 + \frac{u}{2} = \epsilon(G(B^*)) + W(G(B^*)) - 2 \equiv 1 \pmod 3$; therefore, $\epsilon(G(B^*)) + w(G(B^*)) \equiv 0 \pmod 3$, so $w(G(B^*)) = W(G(B^*)) \leq u+1$. In (2c), $\epsilon(G(B)) + w(G(B)) + 1 + \frac{u}{2} = \epsilon(G(B^*)) + w(G(B^*)) - 1 \equiv 2 \pmod 3$, so $\epsilon(G(B^*)) + w(G(B^*)) \equiv 0 \pmod 3$; therefore, $w(G(B^*)) = W(G(B^*)) \leq u+1$, proving (ii).

Now we investigate (iii). In (2a), $\epsilon(G(B^*)) + W(G(B^*)) \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u}{2} + u + 1 \leq 3T(u+2)$, for $u \geq 4$; in (2b), $\epsilon(G(B^*)) + W(G(B^*)) \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+4}{2} + u + 1 = \lfloor \frac{u^2}{4} \rfloor + \frac{3u+6}{2} \leq 3T(u+2)$, for $u \geq 6$; and in (2c), $\epsilon(G(B^*)) + W(G(B^*)) \leq \lfloor \frac{u^2}{4} \rfloor + \frac{u+2}{2} + u + 1 = \lfloor \frac{u^2}{4} \rfloor + \frac{3u+4}{2} \leq 3T(u+2)$, for $u \geq 4$. It now remains to check the values of $\epsilon(G(B^*)) + W(G(B^*))$ when $u = 4$. If $u = 4$, we need only consider the case (2b). If $u = 4$, we fall into case (2b) when $\epsilon(G(B)) = 0$. In this case, we have $\epsilon(G(B)) = 0$ and $w(G(B)) = 4$. However, $\epsilon(G(B^*)) = 4$ and $w(G(B^*)) = 5$, so

$\epsilon(G(B^*)) + w(G(B^*)) = 9 \leq 3T(6)$. In addition, $W(G(B^*)) = 5$, and $\Delta(G(B^*)) = 2$, so (i)-(iii) are satisfied for $u = 4$. Clearly, there are at least two vertices of degree at most 3, so for all even $u \geq 4$, (i)-(iv) are satisfied. □

Proposition 3.2 *Let $u \geq 3$. Any PETS$(2u+1)$ (V^*, B^*) satisfying*

(i) $\Delta(G(B^*)) \leq u$,

(ii) $W(G(B^*)) \leq u + 1$,

(iii) $\epsilon(G(B^*)) + W(G(B^*)) \leq 3T(u+2)$, and

(iv) $G(B^*)$ contains at least two vertices of degree at most $u - 1$

can be embedded in an ETS$(4u+4)$ $(\hat{V}, \hat{B})$.

Proof: We can clearly take (V^*, B^*) to be maximal. We use five types of extended triples to embed $(V^* = \{1, \ldots, 2u+1\}, B^*)$ in $(\hat{V} = \{1, \ldots, 4u+4\}, \hat{B})$:

(a) lollipops $\{a, a, b\}$, $a \geq 2u+2$, $b \leq 2u+1$;

(b) lollipops and loops on vertices in $\{2u+2, \ldots, 4u+4\}$;

(c) triples $\{a, b, c\}$, $a, b \leq 2u+1$, $c \geq 2u+2$;

(d) triples $\{a, b, c\}$, $2u+2 \leq a, b, c \leq 4u+4$; and

(e) triples $\{a, b, c\}$, $a \leq 2u+1$, $b, c \geq 2u+2$.

We let $B^* \subseteq \hat{B}$ and consider each type of extended triple.

Type a: Since $4u+4$ is even, each vertex of $\hat{V}$ must occur in $\hat{B}$ with an odd number of other vertices from $\hat{V}$, so each vertex needs to occur in an odd number of lollipops. We use Type a lollipops to adjust the $w = w(G(B^*))$ vertices of $G(B^*)$ which occur in an even number of lollipops (these are precisely the vertices of even degree in $G(B^*)$). Let $\{x_1, \ldots, x_w\}$ be this set of vertices, and let $\{\{x_i, (2u+1)+i, (2u+1)+i\} | 1 \leq i \leq w\} \subseteq \hat{B}$.

We need to have that the number of edges remaining to be placed in extended triples after the Type b lollipops are defined is divisible by 3; therefore, we man need to add up to four more lollipops as follows. Let $\phi \in \mathbb{Z}_3$ where $\phi \equiv \epsilon(G(B^*)) + w(G(B^*)) \pmod 3$. Let x_{w+i}, for $1 \leq i \leq \phi$ be vertices in $\{1, \ldots, 2u+1\} \setminus \{x_1, \ldots, x_w\}$ (we have $w + \phi \leq 2u+1$ by (ii)). If $\phi \leq 1$, let $\{\{x_{w+i}, 2u+2i+w, 2u+2i+w\}, \{x_{w+i}, 2u+2i+w+1, 2u+2i+w+1\} | 1 \leq i \leq \phi\} \subseteq \hat{B}$. By (iv) we can specify that $d_{G(B^*)}(x_{w+i}) \leq u - 1$. Therefore, $w + 2\phi = W(G(B^*))$ lollipops have been defined.

Type b: Let $\{\{2u+2j+2\phi+w+2, 2u+2j+2\phi+w+2, 2u+2j+2\phi+w+3\} | 0 \leq j \leq u - \phi - \frac{w-1}{2}\} \subseteq \hat{B}$ and let $\hat{B}$ contain the loops on the vertices $\{2u+2j+2\phi+w+3 | 0 \leq j \leq u - \phi - \frac{w-1}{2}\}$. If we consider just the extended triples in $\hat{B}$ defined thus far, we have that every vertex in $\hat{V}$ occurs in $\hat{B}$ with an odd number of other vertices; in addition, each of the $4u+4$ loops is in some extended triple in $\hat{B}$. Finally, for $2u+2 \leq i \leq 4u+4$, vertex i is in a Type b lollipop if and only if it is *not* in a Type a lollipop.

Type c: Form a simple graph H consisting of the edges of $G(B^*)$ together with the edges (not the loops) $\{x_i, 2u+1+\ell\}$, with $i \in \{1, \ldots, w+\phi\}$ and $\ell \in \{1, \ldots, w+2\phi\}$, that are in Type a lollipops. Give H a proper equalized $(u+2)$-edge coloring with the colors $2u+2, \ldots 3u+3$ in which the lollipop edges $\{x_i, 2u+1+\ell\}$ are colored with $2u+1+\ell$, for $1 \leq \ell \leq w+2\phi$, in the following manner. Form a graph H' from H by amalgamating vertices $2u+2, \ldots, 2u+2\phi+w+1$ into a single vertex v. We have that $d_{H'}(i) \leq \Delta(G(B^*))+1 \leq u$, for $1 \leq i \leq 2u+1$, by (i) and (iv), and $d_{H'}(v) \leq w(G(B^*))+2\phi = W(G(B^*)) \leq u+1$ by (ii). Therefore, if $\phi \neq 0$, then H' is a star multigraph; thus, by Lemma 2.3 and Lemma 2.7, H' can be given a proper equalized $(u+2)$-edge-coloring with the colors $2u+2, \ldots, 3u+3$, such that in the corresponding edge-coloring of H, $\{x_i, 2u+1+\ell\}$ is colored $2u+1+\ell$. Clearly, by Vizing's theorem, the same result can be obtained if $\phi = 0$.

For each edge $\{i, j\}$ in $G(B^*)$ colored k, we let $\{i, j, k\} \in \hat{B}$.

Type d: Consider the previously described edge-coloring of H and let δ_x denote the number of edges of H colored x. Let $\{\{2u+2, \ldots, 3u+3\}, T'\}$ be an equitable partial Steiner triple system of order $u+2$ with $(\epsilon(G(B^*))+W(G(B^*)))/3$ triples (by the definition of $W(G(B^*))$, this number is an integer), such that $\delta_x = r(x)$, the number of triples in T' which contain symbol x, $2u+2 \leq x \leq 3u+3$, and such that no triple in T' has an edge in common with a Type b lollipop. Such a PSTS$(u+2)$ exists by Lemma 2.6, since by assumption (iii), $\epsilon(G(B^*))+W(G(B^*))/3 \leq T(u+2)$. Let $T' \subseteq \hat{B}$.

Type e: It now remains to place the remaining edges in triples, using Theorem 2.4. First we form a partial symmetric quasi-latin square L of order $u+2$ on the symbols $1, \ldots, 2u+1$ as follows:

(1) place symbol $j \leq 2u+1$ in cell (i,i) if an edge colored $2u+1+i$ is incident with vertex j in H, and

(2) for $1 \leq i < j \leq 2u+1$, if $\{i+2u+1, j+2u+1\}$ is not an edge of a triple in T' (see Type d) and is not a lollipop edge of Type b then fill cells (i,j) and (j,i) with a symbol in $\{1, \ldots, 2u+1\}$ preserving the latin property, of course; this can be done greedily since at most $u+1$ symbols occur in each row and column.

We now show that every symbol occurs an odd number of times on the main diagonal of L, and, since L is symmetric, altogether an odd number of times in L.

We consider two cases.

Case 1: $d_{G(B^*)}(x) = 2m$.

Vertex x has even degree in $G(B^*)$, so there is a lollipop edge of Type a incident with x colored with some color c in the graph H. Furthermore, since $d_{G(B^*)}(x) = 2m$, x is contained in $2m$ triples of the form $\{x, y, \alpha_k\}$, where α_k is the color of the edge $\{x, y\}$ in H. Consequently, symbol x occurs once in cell (c,c) and in $2m$ distinct cells of the form (α_k, α_k), so symbol x occurs in $2m+1$ main diagonal cells of L. Therefore, we have x occurring an odd number of times in L.

Case 2: $d_{G(B^*)}(x) = 2m+1$.

Vertex x has odd degree in $G(B^*)$, so there are either 0 or 2 lollipop edges of Type a incident with x in the graph H; if there are two such edges, they are given two distinct colors, say c_1 and c_2. Furthermore, x is contained in $2m-1$ triples of the form $\{x, y, \alpha_k\}$, where α_k is the color of the edge $\{x, y\}$ in H. Therefore, symbol x is placed once in cells (c_1, c_1) and (c_2, c_2) and once in $2m-1$ distinct cells of the form (α_k, α_k), so symbol x occurs in $2m+1$ diagonal cells of L. If there are no such lollipop edges, then vertex x is contained in $2m+1$ triples of the form $\{x, y, \alpha_k\}$, where α_k is the color of the edge $\{x, y\}$ in H. Therefore, symbol x is placed in $2m+1$ disti nct cells of the form (α_k, α_k), so symbol x occurs in $2m+1$ diagonal cells of L. Thus, we have x occurring an odd number of times in L.

We also have that for $1 \leq i \leq u+2$, row i of L contains u symbols, except that if $u+2$ is even, then row $u+2$ contains $u+1$ symbols. Suppose first that $u+2$ is odd. Since $r(2u+i+1)$ is the number of triples in T' containing symbol $2u+i+1$, from (1) the number of symbols in cell (i,i) is $2r(2u+i+1)-1$ or $2r(2u+i+1)$ if symbol $2u+i+1$ is in a Type a or Type b lollipop, respectively (we know that exactly one of these possibilities occurs); from (2) the number of $u+1$ off-diagonal cells that remain empty is $2r(2u+i+1)$ or $2r(2u+i+1)+1$ if vertex $2u+i+1$ is in a Type a or Type b lollipop, respectively, so $u+1-2r(2u+i+1)$ or $u-2r(2u+i+1)$ are filled. Nevertheless, for $1 \leq i \leq u+1$, row i of L contains u symbols.

Suppose now that $u+2$ is even; the argument varies slightly. We consider vertex $3u+3$. This vertex must occur in a Type b lollipop, so from (1), the number of symbols in cell $(u+2, u+2)$ of L is $2r(3u+3)$. Row $u+2$, being the last row (column) of L, contains one less empty off-diagonal cell. Therefore, $2r(3u+3)$ of the $u+1$ off-diagonal cells are filled. Thus, row i of L contains u symbols for $1 \leq i \leq u+1$, and row $u+2$ of L contains $u+1$ symbols.

We have that $N_L(i)$ is odd and $N_L(i) \geq 2(u+2) - 2u - 3 = 1$. Furthermore, if $u+2$ is odd, then each row of L contains u symbols, and if $u+2$ is even, then row i of L contains u symbols for $1 \leq i \leq u$, and row $u+2$ contains $u+1$ symbols. Therefore by Theorem 2.4, L can be embedded in the top left corner of a symmetric quasi-latin square L' of order $2u+3$ on the symbols $1, \ldots, 2u+1$ such that cells (i,i), $u+3 \leq i \leq 2u+2$, and cells $(u-x+2i+1, u-x+2i+2)$ and $(u-x+2i+2, u-x+2i+1)$, for $1 \leq i \leq u+1-(u-x+1)/2$ are empty, where $x = 0$ or 1 if $u+2$ is odd or even, respectively. Use L' to form triples with the remaining edges as follows: if symbol i occurs in cells (y,z) and (z,y), $y \neq z$, of L' then let $\{y, z, i\} \in \hat{B}$.

This completes the definition of $\hat{B}$. Now we show that $(\hat{V}, \hat{B})$ is an ETS$(4u+4)$ by proving that every edge occurs in exactly one extended triple in $\hat{B}$.

Consider edges of the form $\{x, y\}$, $x, y \leq 2u+1$. These edges were already in extended triples in B^* or they were colored in $G(B^*)$ and were therefore used in Type c triples. All loops $\{x, x, x\}$ are already in some extended triple in B^* since B^* is assumed to be maximal. Hence, each edge of this form is in exactly one extended triple in $\hat{B}$.

Now consider edges of the form $\{x, y\}$, for $x \leq 2u+1$ and $y \geq 2u+2$. Each symbol $1, \ldots, 2u+1$ occurs exactly once in each row of L', so $\{x, y\}$ occurs in an extended triple of Type a or Type c if x is in a diagonal cell of L', and of Type e otherwise. In any event, $\{x, y\}$ occurs in exactly one extended triple in $\hat{B}$.

Finally, consider edges of the form $\{x, y\}$ where $x, y \geq 2u+2$. Each cell (x, y) in L', $x \neq y$, contains 0 or 1 symbols. Suppose (x, y) contains 0 symbols; then $\{x, y\}$ is in either a Type b or a Type d extended triple. Now suppose (x, y) contains 1 symbol; then $\{x, y\}$ is in a Type e triple. Suppose $x = y$. If (x, x) is filled with an odd number of symbols, then exactly one symbol, say i, is joined to x by the Type a lollipop $\{x, x, i\}$ (there is only one symbol of this type in cell (x, x) because at most one Type a lollipop is incident with vertex x, for $2u+2 \leq x \leq 4u+4$); otherwise, $\{x, x, x+1\}$ or $\{x, x, x\} \in \hat{B}$.

Hence, every edge of the form $\{x, y\}$, for $1 \leq x, y \leq 4u+4$, is contained in exactly one extended triple in $\hat{B}$, and the proof is complete. □

We now prove the following theorem.

Theorem 3.3 *Any PETS(u) (V, B) can be embedded in an ETS(v), for all* $v \geq 4u+4$, *where* $v \equiv 0 \pmod 4$.

Proof: First assume $u \in \{1, 2\}$. Clearly, any $PETS(1)$ can be trivially embedded in an $ETS(v)$ for all $v \geq 8$, $v \equiv 0 \pmod 4$. (This corresponds to the existence of such $ETS(v)$s.) Now if $u = 2$, then $\epsilon(G(B)) = 0$ or 1. If $\epsilon(G(B)) = 0$, then by [5] we can obtain the desired result. If $\epsilon(G(B)) = 1$, then we assume that $B = \{\{1,1,1\}, \{2,2,2\}\}$ and let $B^* = B \cup \{\{1,2,3\}, \{3,3,3\}\}$. This forms an $ETS(3)$ which can be embedded in the desired $ETS(v)$s by [5].

Now suppose $u \geq 3$ and let $v = 4(u+k)+4$, where $k \geq 0$. Embed (V, B) in a maximal PETS$(u+k)$ (V_1, B_1). By Lemma 3.1, (V_1, B_1) can be embedded in a PETS$(2(u+k)+1)$ (V_2, B_2) satisfying (i)–(iv), which by Proposition 3.2 can be embedded in an ETS$(4(u+k)+4)$. □

We also have the following theorem.

Theorem 3.4 ([9]) *Any PETS(u) can be embedded in an ETS(v), for all* $v \geq 4u+6$, $v \equiv 2 \pmod 4$.

This bound on v can be lowered to $4u+2$ in most cases [9]. Combining Theorem 3.3 and Theorem 3.4 gives a much greater result.

Theorem 3.5 *Any PETS(u) can be embedded in an ETS(v) for all even* $v \geq 4u+4$.

Clearly Theorem 1.1 is a corollary of Theorem 3.5.

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