Cycles in quasi 4-connected graphs

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Abstract

A graph G is said to be quasi 4-connected if G is 3-connected and for each cutset $K \subseteq V(G)$ with $|K| = 3$, K is the neighbourhood of a vertex of degree three and $G - K$ has precisely two components. It is evident that such graphs need not be 4-connected and yet they exhibit many of the properties of 4-connected graphs. In this paper, we show that, given a set, $N \subseteq E(G)$ of four independent edges, there is a cycle in G containing N. In fact, we show, more generally, that given a "free edge system" F of size at most 4, in a quasi 4-connected graph G, there is a cycle in G containing F. We also consider the existence of cycles through a given set of vertices and avoiding another set of vertices.

1. Introduction.

A graph G is said to be *quasi 4-connected* if G is 3-connected and for each cutset $K \subseteq V(G)$ with $|K| = 3$, K is the neighbourhood of a vertex of degree three and $G - K$ has precisely two components. Thus, cyclically 4-edge-connected cubic graphs are quasi 4-connected. Quasi 4-connected graphs have been studied in [7], [9] and [10]. Through these studies it is apparent that quasi 4-connected graphs exhibit many of the properties of 4-connected graphs (particularly in relation to the presence of certain minors) without necessarily being 4-connected. In this way, the quasi 4-connected property offers a true refinement of the strict vertex connectivity notion. We shall determine further areas in which quasi 4-connected graphs exhibit properties of 4-connected graphs.

The link between the connectivity of a graph and the nature of the cycles it admits has long been studied and has yielded one of the most famous results in graph theory; namely the corollary to the following theorem of Dirac [3].

Theorem 1.1. *(Dirac). If G is an n-connected graph, then given any 2 edges and any $n-2$ vertices there is a cycle in G containing all of these elements.* ∎

Corollary 1.2. *Let G be an n-connected graph and let $N \subseteq V(G)$ have $|N| = n$. The there is a cycle in G containing N.* ∎

It is readily seen that this result is best possible by considering, for example, $K_{n,n+1}$. In the case of 3-connected graphs, this indicates the general form of all stoppers, as was shown by Watkins and Mesner [12] in the following theorem.

Theorem 1.3. *(Watkins and Mesner). A graph G of connectivity n, $n \geq 3$, contains a cycle through any nominated set of $n+1$ vertices unless there is a set of n vertices, the removal of which disconnects G into at least $n+1$ components.* ∎

An edge analogue of Corollary 1.2 has been conjectured by Lovász [8].

Conjecture. *(Lovász). Let N be a set of n independent edges in an n-connected graph G. Then there is a cycle in G containing N, unless N is an odd cutset.*

The conjecture was verified in the case $n = 3$ along with the original mention of the problem. Since that time it has been verified for $n = 4$ by Erdös and Györi [4] and for $n = 5$ by Sanders [11]. Thus we have the following theorem.

Theorem 1.4. *Let N be a set of n independent edges in an n-connected graph G, $n = 3, 4, 5$. Then G has a cycle containing N, unless N is an odd cutset.* ∎

In [5], Häggkvist and Thomassen showed that strengthening the connectivity requirement slightly ensured a cycle through any n edges.

Theorem 1.5. *(Häggkvist & Thomassen). Let N be a set of n independent edges in an $(n+1)$-connected graph G. Then there is a cycle in G containing N.* ∎

It has also been observed that many graphs exhibit properties of more highly connected graphs even though their own connectivity is limited by the presence of trivial small cutsets. Indications of the enhanced "connectivity" present in such graphs may be obtained from properties such as cyclic connectivity and the quasi 4-connected property with which much of this paper is concerned.

In the case of cubic graphs, it is not possible to achieve connectivity higher than 3. However, such graphs may be arbitrarily highly cyclically connected. Making use of the enhanced connectivity afforded, Aldred, Holton and Thomassen [1] showed the following theorem.

Theorem 1.6. *Let G be a cyclically 5-connected cubic graph and let $N \subseteq E(G)$ be an independent set of edges with $|N| \leq 4$. Then there is a cycle in G containing N.* ∎

In addition to this, Aldred and Holton [2] were able to extend the result as follows.

Theorem 1.7. *Let G be a cyclically 6-connected cubic graph and let $N \subseteq E(G)$ be an independent set of edges with $|N| \leq 5$. Then there is a cycle in G containing N.* ∎

Note that in each case the cyclic connectivity assumption precludes the possibility that N incorporates an odd cutset.

These results provide some partial confirmation of a conjecture, analogous to the Lovász Conjecture, put forward by Aldred, Holton and Thomassen (see [1]).

Conjecture (Aldred, Holton & Thomassen). *Let G be a cyclically $n+1$-connected cubic graph and let $N \subseteq E(G)$ be an independent set of edges with $|N| \leq n$. Then there is a cycle in G containing N.*

2. Preliminaries

That n independent edges containing a minimal odd cutset cannot lie on a cycle is immediately obvious. If we consider a more general collection of edges, there are clearly further configurations of edges which cannot lie on a common cycle. The notion of a "free path system" (see [1], [2]) is that of a set of vertices and edges, not necessarily independent, which does not immediately preclude the existence of a cycle containing the set. With this notion in mind we make the following definitions.

Definition: Let $F \subseteq E(G)$ be such that $\langle F \rangle$ is a linear forest (a disjoint union of paths) with at least two components. If v is an endvertex of the path Π in $\langle F \rangle$, then the *free neighbourhood of v with respect to F*, $fN_F(v)$, is defined as follows. $fN_F(v) = \{u \in N_G(v) : u \notin V(\Pi) \text{ and } deg_{\langle F \rangle}(u) \neq 2\}$. In addition, for a set, A, of endvertices of paths in $\langle F \rangle$ we define $fN_F(A) = \cup_{v \in A} fN_F(v)$.

Definition: A *free edge system* of size k in a graph G is a set of edges $F \subseteq E(G)$ with $|F| = k$ such that F contains no odd cutset, there is no contraction of G onto the cubic graph H in Figure 0 below, with $\{e_1, e_2, e_3, e_4\}$ contained in the image of F and either:

(i) $\langle F \rangle$ is a linear forest with at least two components such that if A is a set of endvertices of paths in $\langle F \rangle$, then $|fN_F(A)| \geq |A|$. Furthermore, if $\langle F \rangle$ has at least three components, $|A| = 2$ and A consists of both ends of the same path in $\langle F \rangle$, then $fN_F(A)$ is not just both ends of some component of $\langle F \rangle$;

(ii) $\langle F \rangle \cong P_{k+1}$ (a path on $k+1$ vertices) and neither endvertex has only interior vertices of the path as neighbours; or

(iii) $\langle F \rangle \cong C_k$.

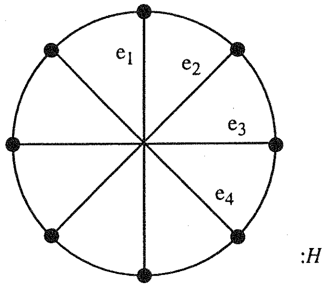

Figure 0

In [1] Theorem 1.4 was generalized for 3-connected graphs to include free path systems of size 3. A free edge system defined above is also a free path system in the sense of [1] which gives us the following result.

Lemma 2.1. *Let G be a 3-connected graph and let F be a free edge system of size at most 3 in G. Then there is a cycle in G containing F.* ∎

Lemma 2.2. *Let G be a 4-connected graph and let F be a free edge system of size at most 4 in G. Then there is a cycle in G containing F.*

Proof. Let G and F be as in the statement of the theorem. We may assume that F is not an independent set in G as, by Theorem 1.4, the required cycle is known to exist in this case. Thus, there is a path segment uvw in $\langle F \rangle$. Form a new graph G' from G by deleting the vertex v and adding the edge uw.The edge set $F' \subseteq E(G')$, $F' = (F - \{uv, vw\}) \cup uw$ forms a free edge system in the 3-connected graph G' and thus, by Lemma 2.1, there is a cycle in G' containing F'. This cycle clearly lifts to a cycle in G containing F. ∎

(Note that we may proceed by induction to show that if the Lovász conjecture is true, then for each free edge system F of size at most n in an n-connected graph G, there is a cycle in G through F.)

In a similar way the following result may be obtained. This result is essentially a corollary of Theorem 1.5.

Lemma 2.3. *Let G be an $n+1$-connected graph and let F be a free edge system in G of size at most n, then there is a cycle in G containing F.* ∎

We shall establish that if we weaken the condition that G is 4-connected in Lemma 2.2, and require only that G is quasi 4-connected, then the conclusion will still follow. To prove this stronger result, we first introduce the following structural definition and lemma.

Definition: Let $G \neq K_4$ be a quasi 4-connected graph and let $v \in V(G)$ be such that $N_G(v) = \{u_1, u_2, u_3\}$ (i.e. $deg_G(v) = 3$).The *quasi reduction of G at v* is the graph $G' = (G - v) + \{u_1u_2, u_2u_3, u_3u_1\}$ (edges added only when not already present in G).

Lemma 2.4. *Let $G \neq K_4$ be a quasi 4-connected graph and let $v \in V(G)$ be such that $N_G(v) = \{u_1, u_2, u_3\}$ (i.e. $deg_G(v) = 3$). Then G', the quasi reduction of G at v, is also quasi 4-connected. Furthermore, unless G is a quasi 4-connected spanning subgraph of G_1 or G_2 (see Figure 1 below), G' has fewer vertices of degree 3 than G.*

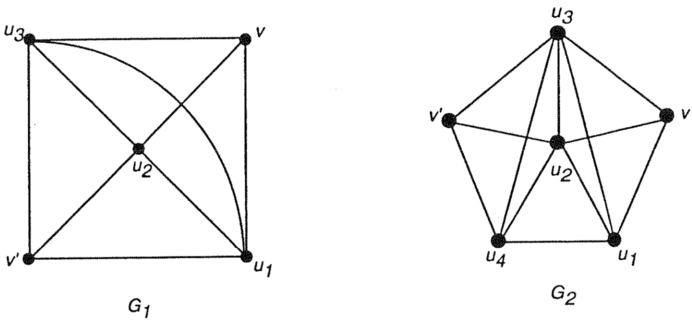

Figure 1.

Proof. Let G, v, G' be as in the statement of the lemma. Suppose, by way of contradiction, that G' is not quasi 4-connected. Then either G' has a vertex of degree 2 or G' has a nontrivial 3-cut. In the former case, we may assume that the vertex is u_1 (as the vertex must be in the neighbourhood of v) and thus $\{u_2, u_3\}$ is a 2-cut in G. This contradicts the assumption that G is quasi 4-connected. So we may assume that G' has a nontrivial 3-cut $K = \{x, y, z\}$. But K must also be a nontrivial 3-cut in G, again contradicting our assumption that G is quasi 4-connected.Thus G' is quasi 4-connected.

If G' were to contain as many vertices of degree 3 as G, then at least one of the vertices in $\{u_1, u_2, u_3\}$, say u_1, must have degree 3 in G'. However, if this were the case, then $N_{G'}(u_1)$ would be a nontrivial 3-cut in G unless G is a spanning subgraph of G_1 or G_2. ∎

3. Cycles in quasi 4-connected graphs.

We are now in a position to analyse how closely the cycles present in quasi 4-connected graphs parallel what we would expect from 4-connected graphs. To begin with we may determine that while a quasi 4-connected graph need only be 3-connected, it must contain a cycle through any four nominated vertices.

Theorem 3.1. *Let G be a quasi 4-connected graph and let $M \subseteq V(G)$ be a set of at most 4 vertices. Then there is a cycle in G containing M.*

Proof. We may assume that $|M| = 4$ as G is 3-connected and Corollary 1.2 ensures the existence of the desired cycle if $|M| \leq 3$. Now, by Theorem 1.3, G contains a cycle through M unless there is a cutset of size 3 in G the deletion of which disconnects G into at least 4 components. This is impossible by the definition of quasi 4-connected. ∎

In the next two results, we see that cycles through edges in quasi 4-connected graphs also parallel those in 4-connected graphs.

First we introduce a lemma which shows that the special quasi 4-connected graphs G_1 and G_2 have the desired cycle properties.

Lemma 3.2. *Let G be a graph on at most 6 vertices and let F be a free edge system of size 4 in G. Then there is a cycle in G containing F.*

Proof. Denoting by P_k a path on k vertices, the only free edge systems of size 4 covering at most six vertices, apart from a 4-cycle, are:

(i) $\langle F \rangle \cong P_3 \cup P_3$;
(ii) $\langle F \rangle \cong P_4 \cup P_2$;
(iii) $\langle F \rangle \cong P_5$.

Note that (i) and (ii) require that $|V(G)| = 6$

In case (i), let $F = \{ab, bc, xy, yz\}$. Then the cycle required exists in G if and only if both ax and cz or both az and cy are present in G. So, without loss of generality, assume that neither ax nor az is present in G. Thus F is not free, as a only has neighbours in its own P_3 or interior to the other.

In case (ii), let $F = \{ab, bc, cd, xy\}$ Then the cycle required exists in G if and only if both ax and dy or both ay and dx are present in G. It now follows that the cycle exists if and only if F is free.

Finally, in case (iii), let $F = \{ab, bc, cd, de\}$ and let x be the possible further vertex in G. The cycle exists if and only if the edge ae is present or, in the case of a further vertex, both ax and xe are present. In the event that neither of these occurs, F cannot be free. ∎

Theorem 3.3. *Let F be a free edge system of size at most 4 in the quasi 4-connected graph G. Then there is a cycle in G containing F.*

Proof. First we note that if $|V(G)| \leq 6$, then by Lemma 3.2, there is a cycle in G through F. Also, if $|F| \leq 3$, then the cycle exists, since G is 3-connected. Consequently, we may assume that $|F| = 4$, $|V(G)| \geq 7$ and proceed by induction on t, the number of vertices of degree 3 in G.

When $t = 0$, G is 4-connected and by Lemma 2.2, there is a cycle in G containing F.

Assume that the result is true for all t such that $0 \leq t \leq n$ and consider the quasi 4-connected graph G with $n+1$ vertices of degree 3. Let $v \in V(G)$ be a vertex of degree 3, with $N_G(v) = \{u_1, u_2, u_3\}$, and let G' be the quasi reduction of G at v. Then, by Lemma 2.4, G' is quasi 4-connected and has fewer vertices of degree three than G. Let $F' \subseteq E(G')$ be defined as follows. If no edge in F is incident with v in G, then $F' = F$. If $u_i v$ and vu_j are edges in F, then $F' = (F - \{u_i v, vu_j\}) \cup \{u_i u_j\}$. In either case it is easy to see that F' is a free edge system in G' with size at most 4. Thus, by induction, there is a cycle in G' through F' and this cycle easily lifts to a cycle in G through F. So we must consider that there is precisely one edge in F incident with v, say $u_1 v$. It is clear that if F consists of four independent edges and $u_j \in N_G(v) - \{u_1\}$ is such that $deg_{\langle F \rangle}(u_j) \leq 1$, then $F' = (F - \{u_1 v\}) \cup \{u_1 u_j\}$ is free in G' and the result follows by induction. Thus there must be a vertex $x \in V(\langle F \rangle$ such that $deg_{\langle F \rangle}(x) = 2$. Let x have neighbours w, y in $\langle F \rangle$. We form G' from G by deleting x and adding the edge wy and $F' = (F - \{wx, xy\}) \cup \{wy\}$. If x is not a neighbour of a vertex of degree 3 in G, other than w or y, then G' is 3-connected and contains a cycle through F', giving the desired cycle in G. As such, we may assume that $x \in N_G(v)$ but $xv \notin F$, say $x = u_2$. It is now straightforward to check that $F' = (F - \{u_1 v\}) \cup \{u_1 u_3\}$ is free in G' and the result follows by induction. ∎

Corollary 3.4. *Let G be a quasi 4-connected graph and $M \subseteq V(G)$, $N \subseteq E(G)$ be such that $|M| + |N| \leq 4$. Then G contains a cycle through M and N if and only if $M \cup N$ can be covered by a free edge system.* ∎

In particular, Corollary 3.3 says that if $|M| = |N| = 2$, then G has a cycle through M and N unless $\langle N \rangle$ is a path on three vertices all of which lie in the neighbourhood of the same vertex of degree 3 in M, giving just the obvious exception to Dirac's Theorem for 4-connected graphs.

Finally, let us consider an interesting variation on the cycles through given vertices problem which was introduced by Hemminger, Plummer and Wilson in [6] and involves specifying two disjoint sets of vertices one of which is to be included in a cycle and the other to be excluded by the cycle. This idea is specified precisely in the next definition

Definition. A graph G is said to be $C(j,k)$ if, for each pair of sets $J, K \subseteq V(G)$ with $J \cap K = \emptyset$, $|J| = j$, $|K| = k$, there is a cycle in G containing J and avoiding K.

From Corollary 1.2, one immediately sees that an n-connected graph G is $C(n-i, i), 0 \leq i \leq n-2$ and clearly, if vertices of degree n are present in G, it cannot be $C(1, n-1)$ (although the obvious exception here is the only one possible). By the same token, a quasi 4-connected graph cannot be $C(2,2)$ if there are vertices of degree 3 present (i.e. if it is not 4-connected).

Theorem 3.5. *Let G be a quasi 4-connected graph and let $J, K \subseteq V(G)$ with $J \cap K = \emptyset$, $|J| = 4-k$, $|K| = k$, $k = 0, 1, 2$. Then there is a cycle in G through J and avoiding K unless:*

(i) $k = 1$ and G has the form shown in Figure 2;
(ii) $k = 2$ and K is contained in the neighbourhood of a vertex $u \in J$ with $deg_G(u) = 3$.

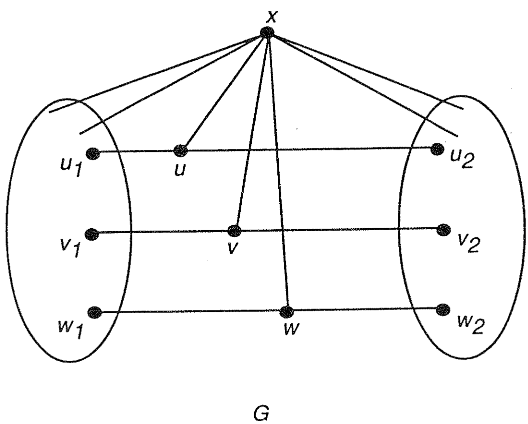

Figure 2

Proof. By Theorem 3.1, the desired cycle exists if $k = 0$, so we may assume that $k = 1$ or 2. If $k = 1$ and thus K consists of a single vertex, x say, then $G - x$ is 3-connected, and hence contains the desired cycle, unless $x \in N_G(z), deg_G(z) = 3$. Thus we assume that $x \in N_G(z), deg_G(z) = 3$ and let $J = \{u, v, w\}$ be a set of three vertices which do not lie on a cycle in $G - x$. For each $z \in N_G(x)$ such that $deg_G(z) = 3$, "suppress" z in $G - x$ (i.e. if $N_G(z) = \{x, y_1, y_2\}$, then delete x, z and add the edge y_1y_2 if not already present) and thereby form a new graph G'. The graph G' so formed is obviously 3-connected and thus contains a cycle through J unless J contains a vertex of degree 3 in the neighbourhood of x. But for each such member of J we require that the edge corresponding to its suppression be included in the cycle. This can readily be done unless all three vertices in J are of degree 3 and lie in the neighbourhood of x. In this case, the structure of G is precisely that depicted in Figure 2.

It remains only to consider the case when $k = 2$. Let $J = \{u, v\}$ and $K = \{w, x\}$ be vertex sets such that there is no cycle in $G - K$ containing J. Thus, by Corollary 1.2, $G - K$ is not 2-connected, from which we may deduce that K is contained in the neighbourhood of a vertex of degree 3. Furthermore, since G is quasi 4-connected, deleting any vertices of degree 1 arising from the deletion of K gives a 2-connected graph. Hence, if neither vertex in J has degree 1 in $G - K$, there is a cycle in $G - K$ through J. This completes the proof. ∎

From Theorem 3.4, we see that the $C(j, k)$ properties of quasi 4-connected graphs are very close to those of 4-connected graphs confirming further the general similarities between cyclic properties of both classes of graphs.

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