The Fine Structure of Threefold Directed Triple Systems.(*)

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Abstract. The fine structure of a threefold directed triple system is the vector (c_1,c_2,c_3), where c_i is the number of directed triples appearing precisely i times in the system. We determine necessary and sufficient conditions for a vector to be the fine structure of a threefold directed triple system.

1. Introduction. A λ-fold triple system of order v, denoted $TS(v,\lambda)$, is a pair $(V,\mathcal{A})$ where V is a v-set and $\mathcal{A}$ is a collection of 3-subsets (called blocks or triples) of V, such that each 2-subset of V is contained in exactly λ triples.

A λ-fold directed triple system of order v, denoted $DTS(v,\lambda)$, is a pair $(V,\mathcal{B})$ where V is again a v-set while $\mathcal{B}$ is a collection of ordered 3-subsets (called directed or transitive triples) of V, such that each ordered pair of distinct elements of V is contained in exactly λ directed triples. We note that each directed triple contains 3 ordered 2-subsets and the ordered 2-subsets contained in the directed triple (a,b,c) are (a,b), (a,c) and (b,c). A directed triple system is also called a transitive triple system

(*) Research supported by MURST and GNSAGA-CNR.

in the literature. It is well-known that there exists a DTS(v,λ) if and only if either $\lambda\equiv 1,2 \pmod 3$ and $v\equiv 0,1 \pmod 3$, or $\lambda\equiv 0 \pmod 3$ and $v\neq 2$ [1].

These definitions permit $\mathcal{A}$ and $\mathcal{B}$ to contain repeated triples and repeated directed triples respectively. For each $i=1,2,\ldots,\lambda$, let $\mathcal{B}_i$ denote the set of (directed) triples appearing i-times in ($\mathcal{B}$) $\mathcal{A}$. Let $|\mathcal{B}_i|=c_i$. Then ic_i is the number of i-times repeated (directed) triples contained in the multiset $i\mathcal{B}_i$ (the union of i copies of $\mathcal{B}_i$). It is easy to verify that the number of (directed) triples contained in a (DTS(v,λ)) TS(v,λ) is ($c_1+2c_2+\ldots+\lambda c_\lambda=\lambda v(v-1)/3$) $c_1+2c_2+\ldots+\lambda c_\lambda=\lambda v(v-1)/6$.

Designs with repeated blocks have interesting applications in statistics [6], so the structure of repeated (directed) triples in (directed) triple systems has been widely studied. Lindner and Rosa [8] implicitly determined the possible number of repeated triples in a TS$(v,2)$ for $v\equiv 1,3 \pmod 6$. Rosa and Hoffman [11] later extended this determination to the case $v\equiv 0,4 \pmod 6$. Lindner and Wallis [9], and independently Fu [7] determined the possible number of repeated directed triples in a DTS$(v,2)$.

For $\lambda>2$, the following questions concerning repeated (directed) blocks arise:

1) Determine the possible support sizes, or number of distinct (directed) blocks, in a (DTS(v,λ)) TS(v,λ). Remark that this question is to determine the possible values for the sum $c_1+c_2+\ldots+c_\lambda$. This problem is completely solved: see [1] and its references for TS(v,λ) and [10] for DTS(v,λ).

2) Given a (DTS(v,λ)) TS(v,λ), the *fine structure* of the

system is the vector $(c_1,\ldots,c_\lambda)$, where c_i is the number of (directed) triples repeated exactly i times. It is a very interesting problem to produce necessary and sufficient conditions for a vector to be the fine structure of a (DTS(v,λ)) TS(v,λ). This problem is solved for TS$(v,3)$, [2] and [3].

Our aim in this paper is to determine the fine structures for DTS$(v,3)$.

Put

$$s(v) = \begin{cases} \dfrac{v(v-1)}{3} & \text{if } v\equiv 0,\ 1 \pmod 3 \\[2ex] \dfrac{v(v-1)-2}{3} & \text{if } v\equiv 2 \pmod 3. \end{cases}$$

Since any two of $\{c_1,c_2,c_3\}$ determine the third, we follow [2] by adopting the following notation for the fine structure: (t,s) is said to be the fine structure of a DTS$(v,3)$ if $c_2=t$ and $c_3=s(v)-s$. We first need to know the pairs (t,s) which can possibly arise as fine structures. We use the notation *DFine*(v) for the set of fine structures which actually arise in DTS$(v,3)$ systems. For every $v\geq 3$, we define

$$DAdm(v)=\begin{cases} \{(t,s)\ :\ 0\leq t\leq s\leq s(v),\ s\neq 1,\ (t,s)\notin\{(1,2),(1,3)\}\} & \text{if } v\equiv 0,1 \pmod 3 \\[2ex] \{(t,s)\ :\ 0\leq t\leq s\leq s(v),\ s\neq 0,1,2,3,4,5 & \text{if } v\equiv 2 \pmod 3 \end{cases}$$

Main theorem. *DFine*(v)=*DAdm*(v) for every $v\geq 8$. *DFine*$(3)=\{(0,0),$ $(0,2),\ (2,2)\}$, *DFine*(4)=*DAdm*$(4)\setminus\{(0,2),(0,3)\}$, *DFine*$(5)$=*DAdm*$(5)$, *DFine*$(6)$=*DAdm*$(6)\setminus\{(0,2),\ (0,3),\ (0,4),\ (1,4)\}$, *DFine*$(7)$=*DAdm*$(7)\setminus$

$\{(0,3),\ (1,4),\ (0,5)\}$.

In section 2, we prove that $DFine(v) \subseteq DAdm(v)$ and we establish some necessary conditions for $v=4,6,7$. In section 3, we introduce recursive standard constructions. In section 4, we describe the determination of fine structures for small values of v, then we combine all the results to prove the Main Theorem.

2. Necessary conditions.

Let $(V,\mathcal{B})$ be a DTS(v,3). Sometimes we will call blocks the directed triples of $\mathcal{B}$. For each $x \in V$, let $\mathcal{B}(x)$ denote the multiset of blocks in $\mathcal{B}_1 \cup 2\mathcal{B}_2$ containing x and let $d(x)=|\mathcal{B}(x)|$. Let $(V^*,\mathcal{B}^*)$ denote the partial threefold directed triple system such that $V^*=\{x \in V \mid d(x)>0\}$ and $\mathcal{B}^*=\mathcal{B}_1 \cup 2\mathcal{B}_2$. For each $\alpha \in V^*$ define the multiset $V(\alpha)$ in the following way: if $x \in V^*$, $x \neq \alpha$, and x appears in exactly $3n$ directed triples of $\mathcal{B}(\alpha)$, then put n copies of x in $V(\alpha)$. Let ρ_n be the number of elements $x \in V^*$ with $d(x)=3n$.

Lemma 2.1. If $x \neq y$, $d(x)=d(y)=3$, then $|\mathcal{B}(x) \cap \mathcal{B}(y)|=0$.

Proof. See Lemma 2.2 of [10]. ∎

It is easy to prove the following Lemma.

Lemma 2.2. If $d(x)=3$ then there are $y,z \in V^*$ such that $\mathcal{B}_2$ contains either (x,y,z) or (y,z,x) and the directed pair (z,y) appears in three directed triples of $\mathcal{B}^*$. Moreover $c_2 \geq \rho_1$.

Lemma 2.3. It is $(0,2)\in DFine(3)$, $(0,2)\notin DFine(v)$ for $v=4,6$ and $(1,2)\notin DFine(v)$ for every $v\equiv 0,1 \pmod 3$.

Proof. Suppose that $(t,2)\in DFine(v)$. Then $c_1+2c_2=6$ and $\rho_1+2\rho_2=6$. It follows that $(\rho_1,\rho_2)\in\{(6,0), (4,1), (2,2), (0,3)\}$. It is easy to eliminate the cases $(\rho_1,\rho_2)\in\{(6,0), (4,1)\}$. From $(\rho_1,\rho_2) = (2,2)$ it follows $c_2=2$, and from $(\rho_1,\rho_2) = (0,3)$ it follows either $c_2=0$ or $c_2=2$. Then $(1,2)\notin DFine(v)$. Let $V^*=\{0,1,2\}$ and $\mathcal{B}^*=\{120, 102, 210, 021, 201, 012\}$. Clearly $(V^*,\mathcal{B}^*)$ is a DTS(3,3) such that $(0,2)\in DFine(3)$. Moreover it is easy to check that there is not a partial threefold directed triple system verifying $(\rho_1,\rho_2)=(0,3)$ and not isomorphic to $(V^*,\mathcal{B}^*)$. Since for $v=4,6$ there is not a DTS(v,3) that embeds $(V^*,\mathcal{B}^*)$, it follows that $(0,2)\notin DFine(v)$ for $v=4,6$. ∎

Lemma 2.4. It is $(0,3)\notin DFine(v)$ for $v=4,6,7$ and $(1,3)\notin DFine(v)$ for every $v\equiv 0,1 \pmod 3$.

Proof. Suppose $(t,3)\in DFine(v)$, $t\le 1$. From Lemma 2.2 it follows $\rho_1\le 1$. Since $\rho_3\le 1$ the equation $\rho_1+2\rho_2+3\rho_3=9$ has the solutions: $(\rho_1,\rho_2,\rho_3)=(1,4,0)$, $(0,3,1)$. Lemma 2.2 eliminates the case $(1,4,0)$.

Let $(\rho_1,\rho_2,\rho_3)=(0,3,1)$, $V^*=\{1,2,3,4\}$, $d(1)=d(2)=d(3)=6$ and $d(4)=9$. At first suppose $t=1$. Clearly if a directed triple of $\mathcal{B}^*$ meets the ordered pair ij, $i,j\in\{1, 2, 3\}$, then ji does not appear in any directed triple of $\mathcal{B}^*$. Suppose 12 appears in the two times repeated block γ, then one of the following cases arises: (i) $\gamma=124$, (ii) $\gamma=142$, (iii) $\gamma=412$ (we denote the directed triple (a,b,c) by abc). Case (i): We can suppose either $412\in\mathcal{B}^*$ or $142\in\mathcal{B}^*$.

Then 324, 432, $342 \in \mathcal{B}^*$. The remaining blocks of $\mathcal{B}^*$ contain 34 one time repeated, 43 two times repeated and 13 three times repeated. It is easy to see that this is impossible. Similarly we can eliminate the cases (i) and (ii).

At last suppose $t=0$. Let $\mathcal{B}^*=\{124, 142, 412, 341, 431, 314, 234, 423, 243\}$. It is easy to verify that there is not a partial threefold directed triple system such that $(\rho_1,\rho_2)=(0,3)$ and not isomorphic to $(V^*,\mathcal{B}^*)$. Since for $v=4,6,7$ there is not a DTS$(v,3)$ that embeds $(V^*,\mathcal{B}^*)$, it follows that $(0,3)\notin DFine(v)$ for $v=4,6,7$. ∎

Lemma 2.5. It is $(0,4),(1,4)\notin DFine(6)$, $(1,4)\notin DFine(7)$.

Proof. Suppose $(t,4)\in DFine(v)$, $v=6,7$ and $t\le 1$. By $\rho_4\le 1$ and Lemmas 2.1 and 2.2, the equation $\rho_1+2\rho_2+3\rho_3+4\rho_4=12$ has the following solutions:

	ρ_1	ρ_2	ρ_3	ρ_4
(1)	1	1	3	0
(2)	1	4	1	0
(3)	0	0	4	0
(4)	0	3	2	0
(5)	0	6	0	0
(6)	0	4	0	1
(7)	1	2	1	1
(8)	0	1	2	1

At first suppose $v=6$. Let $V=\{0,1,\ldots,5\}$. Since $|\mathcal{B}_3|=6$, it is easy to see that $|V^*|\ge 5$. Moreover $|V^*|=5$ implies that $d(x)\ge 6$ for every $x\in V^*$. Therefore the cases (1), (3), (7) and (8) are impossible.

Case (2): Let $d(0)=3$ and $d(1)=9$. If $1\notin V(0)$ then by Lemma 2.2, $t\geq 2$. Suppose that 12 meets a block of $\mathcal{B}(0)$. By Lemma 2.2, the directed triples containing 21 are in $\mathcal{B}^*$. Then it is impossible to construct $\mathcal{B}_3$ containing exactly four triples meeting 0, two triples meeting 1 and three triples meeting 2.

Case (4): Let $d(1)=d(2)=d(3)=6$ and $d(4)=d(5)=9$. Then $\mathcal{B}^*$ contains triples meeting the ordered pairs 45 and 54. This implies that $(V,\mathcal{B}_3)$ has triples meeting 04, 40, 05 and 50. Therefore $\mathcal{B}_3$ contains a triple meeting 1, 2 and 3. Suppose it is 123. Then $\mathcal{B}^*$ contains triples meeting two of the following three ordered pairs 21, 31 and 32. This is impossible.

Case (5): The following possibilities arise: j) $V(1)=\{2,2,3,3\}$. Then $|\mathcal{B}_3|>6$; jj) $V(1)=\{2,2,3,4\}$. Then $t\geq 2$; jjj) $V(1)=\{2,3,4,5\}$. Let b_1, b_2 and b_3 denote the directed triples of $\mathcal{B}(1)$ meeting 2. If b_1, b_2, $b_3\in\mathcal{B}_1$ then $V(2)=\{1,3,4,5\}$. Therefore $d(0)=3$, impossible. If $b_1\in\mathcal{B}_1$ and $b_2=b_3\in\mathcal{B}_2$ then $t\geq 2$.

Case (6): Let $V\setminus V^*=\{1\}$ and let 0 be the element of V^* such that $d(0)=12$. Clearly both the directed pairs 01 and 10 are in blocks of $\mathcal{B}_3$. This is impossible.

Let $v=7$, $V=\{0,1,\ldots,6\}$ and suppose $(1,4)\in DFine(7)$. To eliminate cases (1), (3), (5), (7) and (8) proceed as for $v=6$.

Case (2): Let $d(0)=3$ and $d(1)=9$. If $1\notin V(0)$ then by Lemma 2.2, $t\geq 2$. Suppose that 12 meets a block of $\mathcal{B}(0)$. By Lemma 2.2, the directed triples containing 21 are in $\mathcal{B}^*$. Suppose that $3\in V(2)$. Then there are $b_1,b_2\in\mathcal{B}_3$ such that $1,3\in b_1$ and $2,3\in b_2$. It follows $t\geq 2$.

Case (4): Let $d(1)=d(2)=d(3)=6$ and $d(4)=d(5)=9$. If each block meeting both 4 and 5 contains the same element $x \in \{1,2,3\}$, then $t \geq 2$. To complete the proof we can proceed as in the analogous case for v=6.

Case (6): Let $V^*=\{0,1,\ldots,4\}$ and $d(0)=12$, Obviously $(V^*,\mathcal{B}^*)$ can not be the union of two sub-DTS(3,3). Then for every ordered pair ij, $i,j \in V^* \setminus \{0\}$, appearing in some directed triple of $\mathcal{B}^*$ the ordered pair ji is not contained in any block of $\mathcal{B}^*$. Suppose $\mathcal{B}_2=\{120\}$. Then either $102 \in \mathcal{B}_1$ or $012 \in \mathcal{B}_1$. In the first case there are three directed triples of $\mathcal{B}^*$ containing 01. Then $|\mathcal{B}_2| \geq 2$. It is easy to see that also the case $012 \in \mathcal{B}_1$ implies $|\mathcal{B}_2| \geq 2$. Similarly we can eliminate the cases $\mathcal{B}_2=\{102\}$ and $\mathcal{B}_2=\{012\}$. ∎

Lemma 2.6. $(0,5) \notin DFine(7)$.

Proof. Let $(V,\mathcal{B})$ be a DTS(7,3) having the fine structure (0,5). Since, by Lemma 2.2 it is $\rho_1=0$, the equation $\rho_1+2\rho_2+3\rho_3+4\rho_4+5\rho_5=15$ has the following solutions:

	ρ_2	ρ_3	ρ_4	ρ_5
(1)	5	0	0	1
(2)	1	3	1	0
(3)	0	5	0	0
(4)	4	1	1	0
(5)	6	1	0	0
(6)	3	3	0	0

It is trivial to eliminate cases (1), (2) and (3).

To eliminate the remaining cases we prove at first the following *Proposition (P)*: There is not a partial $(V^*,\mathcal{B}^*)$ satisfying one of (4), (5) or (6) and such that: i) there is a directed triple $a_1 \in \mathcal{B}^*$ meeting 12, $d(1)=d(2)=6$; ii) the directed triples of $\mathcal{B}(1)$ meet five distinct elements.

In fact the ordered pair 12 meets other two elements of $\mathcal{B}^*$, say a_2 and a_3. Suppose $\alpha \in a_2$ and $\beta \in a_3$. Let $3 \in a_1$, by i) it is $\alpha \neq 3$ and $\beta \neq 3$. If $\alpha=\beta$ then both the ordered pairs $\alpha 2$ and 2α appear in directed triples of $\mathcal{B}^*$. By i) this is impossible. If $\alpha \neq \beta$ then $3,\alpha,\beta \in \mathcal{B}(1) \cup \mathcal{B}(2)$. Then the directed triples of $\mathcal{B}^* \setminus (\mathcal{B}(1) \cup \mathcal{B}(2))$ meet an element $x \notin \{1,2,3,\alpha,\beta\}$. This implies $d(\alpha) \in \{8,11\}$.

Case (4): Let $V^*=\{1,2,...,6\}$, and let $d(i)=6$ for $i=1,...,4$, $d(5)=9$ and $d(6)=12$. Note that for every $\alpha,\beta \in \{1,...,4\}$ both $\alpha\beta$ and $\beta\alpha$ can not appear in blocks of $\mathcal{B}^*$, otherwise $|\mathcal{B}_3|>9$. It is possible to suppose that 12 meets a block of $\mathcal{B}(5)$. The blocks of $\mathcal{B}(1)$ ($\mathcal{B}(2)$) meet the elements $2,5,6,\gamma \in V^*$, ($1,5,6,\delta \in V^*$). Proposition (P) implies that $\gamma,\delta \in \{5,6\}$. Then 1 and 2 do not appear in any triple of $\mathcal{B}(3) \cup \mathcal{B}(4)$. It follows that both the directed pairs 34 and 43 meet blocks of $\mathcal{B}^*$.

Case (5): Let $V^*=\{1,2,...,7\}$ and let $d(i)=6$ for $i=1,...,6$, $d(7)=9$. From Proposition (P) it follows that if $d(\alpha)=6$ then $V(\alpha)$ contains at least one repeated element. The following cases arise: (a): $V(1)=\{2,2,7,7\}$. Then $\mathcal{B}_3>9$. (b): $V(1)=\{2,3,7,7\}$. Then $V(7)=\{1,1,2,2,3,3\}$ and $(V^*,\mathcal{B}^*)$ has a sub-DTS(3,3) on the elements 4,5,6. It is easy to see that it is impossible to embed $(V^*,\mathcal{B}^*)$ in $(V,\mathcal{B})$. (c): $V(1)=\{2,2,3,3\}$. Proceeding as in (b) we can eliminate this case. (d): $V(1)=\{2,2,3,7\}$. It is easy to see that there are blocks of $\mathcal{B}(1)$ meeting both 23 and 32, a contradiction.

Case (6): Let $V^*=\{1,2,...,6\}$ and let $d(1)=d(2)=d(3)=6$, $d(4)=d(5)=d(6)=9$. Clearly $|V(1) \cap \{4,5,6\}| \geq 2$ and there is $b \in \mathcal{B}^*$ such that $|b\cap\{1,2,3\}|\geq 2$. Let $1,2\in b$. By proposition (P), it is possible to suppose $V(1)=\{2,4,4,5\}$. Then $V(2)=\{1,4,4,5\}$. Since $d(6)=9$ this is impossible. ∎

Theorem 2.1. $DFine(v)\subseteq DAdm(v)$.

Proof. Let $v\equiv 2 \pmod 3$. It follows from Lemma 2.7 of [10] that $c_3\leq s(v)-6$. Since $c_3=s(v)-s$ we obtain $s\geq 6$. From $c_1\geq c_2$ and $c_1+2c_2+3c_3=v(v-1)$, it follows $c_2\leq \left\lfloor\frac{v(v-1)}{3}\right\rfloor-c_3=s$.

Let $v\equiv 0,1 \pmod 3$. It is easy to see that $s\neq 1$ and $c_3\leq s(v)$. Lemmas 2.3 and 2.4 complete the proof. ∎

3. Recursive Constructions.

In this section we will denote by $(V,\mathcal{B})$ a DTS(v,3) on the v-set $V=\{\infty_1,\infty_2,\ldots,\infty_v\}$.

Put $U=V\cup Z_\tau$, $\tau\in\{v+1,v+4\}$. For every $i\in\{1,2,\ldots,v\}$ and $j\in Z_\tau\setminus\{0\}$ let

$F_j=\{(a,b)\mid a,b\in Z_\tau,\ b-a\equiv j \pmod\tau\}$,

$\infty_i F_j=\{(\infty_i,a,b)\mid (a,b)\in F_j\}$,

$F_j\infty_i=\{(a,b,\infty_i)\mid (a,b)\in F_j\}$,

$\langle\infty_i F_j\rangle=\{(a,\infty_i,b)\mid (a,b)\in F_j\}$.

Lemma 3.1 Let (t,s) be the fine structure of $(V,\mathcal{B})$. Then for each $\alpha\in\{0,1,\ldots,v\}$, $(t,s+\alpha(v+1))$ is the fine structure of a DTS(2v+1,3) $(W,\mathcal{E})$.

Proof. Let $W=V\cup Z_{v+1}$ and let $\mathcal{D}$ be the multiset of directed triples constructed by putting on it either $\infty_i F_i\cup F_i\infty_i\cup\langle\infty_i F_i\rangle$ or $3\langle\infty_i F_i\rangle$ for each $i=1,2,\ldots,v$. Let $\mathcal{E}$ be the multiset of directed triples of $\mathcal{B}$ and $\mathcal{D}$. ∎

Lemma 3.2 Let (t,s) be the fine structure of $(V,\mathcal{B})$. Then for each $\beta=1,2,\ldots,v$, and for each $\alpha=\beta,\beta+1,\ldots,v$, $(t+\beta(v+1),s+\alpha(v+1))$ is the fine structure of a DTS$(2v+1,3)$ $(W,\mathcal{E})$.

Proof. Let $W=V\cup Z_{v+1}$. At first we suppose $\beta\geq 2$. Let $\mathcal{D}$ be the multiset of directed triples constructed by putting on it $2<\infty_\beta F_\beta>\cup<\infty_1 F_\beta>$, $2<\infty_i F_i>\cup<\infty_{i+1}F_i>$ for each $i=1,2,\ldots,\beta-1$, and, if $\beta<v$, either $\infty_i F_i \cup F_i\infty_i \cup <\infty_i F_i>$ or $3<\infty_i F_i>$ for each $i=\beta+1,\beta+2,\ldots,v$. Let $\mathcal{E}$ be the multiset of directed triples of $\mathcal{B}$ and $\mathcal{D}$.

Now let $\beta=1$. If v is odd then $F_{\frac{v+1}{2}}=F'\cup F''$, $F'=\{(i,\frac{v+1}{2}+i) \mid i=0,1,\ldots,\frac{v-1}{2}\}$ and $F''=\{(\frac{v+1}{2}+i,i) \mid i=0,1,\ldots,\frac{v-1}{2}\}$. Let $\mathcal{C}$ be the multiset of directed triples $2\infty_{\frac{v+1}{2}}F'\cup 2F''\infty_{\frac{v+1}{2}}\cup F'\infty_{\frac{v+1}{2}}\cup\infty_{\frac{v+1}{2}}F''$. If v is even then $F_{\frac{v}{2}}=F'\cup F''\cup\{(\frac{v+2}{2},0)\}$, $F'=\{(i+1,i+\frac{v+2}{2}) \mid i=0,1,\ldots,\frac{v-2}{2}\}$ and $F''=\{(i+1,i+\frac{v+2}{2}) \mid i=\frac{v+2}{2},\frac{v+2}{2}+1,\ldots,v\}$. Let $\mathcal{C}$ be the multiset of directed triples $2\infty_{\frac{v}{2}}F'\cup 2F''\infty_{\frac{v}{2}}\cup F'\infty_{\frac{v}{2}}\cup\infty_{\frac{v}{2}}(F''\setminus\{(0,\frac{v}{2})\})\cup\{(\frac{v+2}{2},\infty_{\frac{v}{2}},0),\ (\frac{v+2}{2},\infty_{\frac{v}{2}},0),\ (0,\infty_{\frac{v}{2}},\frac{v}{2}),\ (\infty_{\frac{v}{2}},\frac{v+2}{2},0)\}$.

Using the differences F_i, with $i\neq\frac{v+1}{2}$ if v is odd and $i\neq\frac{v}{2}$ if v is even, we can proceed as in the case $\beta\geq 2$ to construct a multiset $\mathcal{D}$ of directed triples. Let $\mathcal{E}$ be the multiset of directed triples of $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. ∎

Lemma 3.3. Let (t,s) be the fine structure of $(V,\mathcal{B})$. Then $(t,s+(v+1)(v+4))$ and $(t+\varepsilon(v+4),s+(\alpha+\varepsilon)(v+4))$, for each $\alpha=0,1,\ldots,v$

and $\varepsilon=0,1$, are the fine structures of a DTS(2v+4,3) $(W,\mathcal{E})$.

Proof. Let $W=V\cup Z_{v+4}$. Let $\mathcal{C}_1=\{(j,j+1,j+3) \mid j\in Z_{v+4}\}$, $\mathcal{C}_2=\{(j,j+2,j+3) \mid j\in Z_{v+4}\}$ and $\mathcal{C}_3=\{(j,j+3,j+4) \mid j\in Z_{v+4}\}$.

Let $\mathcal{D}_1$ be the multiset of directed triples constructed by putting on it either $3\mathcal{C}_1$ or $2\mathcal{C}_1\cup\mathcal{C}_2$ and, for each $i=4,5,\ldots,v+3$, either $3<\infty_{i-3}F_i>$ or $\infty_{i-3}F_i\cup F_i\infty_{i-3}\cup<\infty_{i-3}F_i>$. Let $\mathcal{E}$ be the multiset of directed triples of $\mathcal{B}$ and $\mathcal{D}_1$. It is easy to see that $(W,\mathcal{E})$ has the fine structure $(t+\varepsilon(v+4),s+(\alpha+\varepsilon)(v+4))$ for each $\alpha=0,1,\ldots,v+1$ and $\varepsilon=0,1$.

Let $\mathcal{D}_2$ be the multiset of directed triples constructed by putting on it $\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$, $\infty_1F_4\cup F_4\infty_1\cup<\infty_1F_2>$ and, for each $i=5,6,\ldots,v+3$, $\infty_{i-3}F_i\cup F_i\infty_{i-3}\cup<\infty_{i-3}F_i>$. Let $\mathcal{E}$ be the multiset of directed triples of $\mathcal{B}$ and $\mathcal{D}_2$. It is easy to see that $(W,\mathcal{E})$ has the fine structure $(t,s+(v+1)(v+4))$. ∎

Lemma 3.4. Let (t,s) be the fine structure of $(V,\mathcal{B})$. Then, for each $\beta=2,3,\ldots,v$, $\alpha=\beta,\beta+1,\ldots,v$ and $\varepsilon=0,1$, $(t+\beta(v+4),s+(v+1)(v+4))$ and $(t+(\beta+\varepsilon)(v+4),s+(\alpha+\varepsilon)(v+4))$, are the fine structures of a DTS(2v+4,3) $(W,\mathcal{E})$.

Proof. Let $W=V\cup Z_{v+4}$ Define $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ as in the proof of Lemma 3.3. Let $\mathcal{D}_1$ be the multiset of directed triples constructed by putting on it either $3\mathcal{C}_1$ or $2\mathcal{C}_1\cup\mathcal{C}_2$, $2<\infty_\beta F_{\beta+3}>\cup<\infty_\beta F_4>$, $2<\infty_j F_{j+3}>\cup<\infty_j F_{j+4}>$ for each $j=1,2,\ldots,\beta-1$, and, if $\beta<v$, either $3<\infty_i F_{i+3}>$ or $\infty_i F_{i+3}\cup F_{i+3}\infty_i\cup<\infty_i F_{i+3}>$ for each $i=\beta+1,\beta+2,\ldots,v$. Let $\mathcal{E}$ be the multiset of directed triples of $\mathcal{B}$ and $\mathcal{D}_1$. It is easy to see that $(W,\mathcal{E})$ has the fine structure $(t+(\beta+\varepsilon)(v+4),s+(\alpha+\varepsilon)(v+4))$. Let $\mathcal{D}_2$ be the multiset of directed triples constructed by putting on it $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, $2<\infty_\beta F_{\beta+3}> \cup <\infty_\beta F_2>$, $2<\infty_j F_{j+3}> \cup <\infty_j F_{j+4}>$ for each $j=1,2,\ldots,\beta-1$, and , if $\beta<v$, $\infty_i F_{i+3} \cup F_{i+3}\infty_i \cup <\infty_i F_{i+3}>$ for each $i=\beta+1,\beta+2,\ldots,v$. Let $\mathcal{E}$ be the multiset of directed triples of $\mathcal{B}$ and $\mathcal{D}_2$. It is easy to see that $(W,\mathcal{E})$ has the fine structure $(t+\beta(v+4),s+(v+1)(v+4))$. ∎

Theorem 3.1. Let $v\geq 7$. If $DFine(v)=DAdm(v)$ then $DFine(2v+1)=DAdm(2v+1)$.

Proof. By Lemma 3.1, if $\alpha=0$, then $DFine(v)\subseteq DFine(2v+1)$. Let $\rho=0,1,\ldots,v$ and $h=1,2,\ldots,v$, then from Lemma 3.1 it follows that $(t,s(v)-v+h(v+1)+\rho)\in DFine(2v+1)$ for each $t=0,1,\ldots,s(v)-v+\rho$. Let $\rho=0,1,\ldots,v$, $h=1,2,\ldots,v$ and $\beta=1,2,\ldots,h$, then from Lemma 3.2 it follows that $(t,s(v)-v+\rho+h(v+1))\in DFine(2v+1)$ for each $t=\beta(v+1),\beta(v+1)+1,\ldots,s(v)-v+\rho+\beta(v+1))$. Since $s(v)-v\geq v$ for $v\geq 7$, the proof is completed. ∎

Theorem 3.2. Let $v\geq 9$. If $DFine(v)=DAdm(v)$ then $DFine(2v+4)=DAdm(2v+4)$.

Proof. By Lemma 3.3, if $\alpha=\varepsilon=0$, then $DFine(v)\subseteq DFine(2v+1)$. Let $\rho=0,1,\ldots,v+3$, $h=0,1,\ldots,v$ and $\beta=0,1,\ldots,h+1$. By Lemmas 3.3 and 3.4, it is $(t+\beta(v+4),s(v)+1+h(v+4)+\rho)\in DFine(2v+4)$ for each $t=0,1,\ldots,s(v)-v-3+\rho$. Since $s(v)-v-3+\beta(v+4)\geq(\beta+1)(v+4)-1$ for $v\geq 9$, the proof is completed. ∎

For every v_1, $v_2\geq 3$ put $DFine(v_1) + DFine(v_2)= \{(t_1 + t_2,s_1 + s_2) \mid (t_1,s_1)\in DFine(v_1)$ and $(t_2,s_2)\in DFine(v_2)\}$ and $kDFine(v_1) = DFine(v_1) +\ldots+ DFine(v_1)$ k times.

Theorem 3.3. If there is a pairwise balanced design (PBD) of order v having k_i blocks of size v_i for every $i=1,2,\ldots,h$, then $k_1 DFine(v_1) + k_2 DFine(v_2) +\ldots+ k_h DFine(v_h) \subseteq DFine(v)$.

Proof. Let $\mathcal{B}$ the blocks of the PBD. For each block in $\mathcal{B}$ of size v_i, form a DTS(v_i). ∎

4. Solutions for small orders.

Lindner and Wallis [9], and independently Fu [7], proved the following Lemma.

Lemma 4.1. For $v\equiv 0,1 \pmod 3$, there exist two DTS(v,1) on the same element set intersecting in k triples if and only if $k \in \{0,1,\ldots,s(v)-2,s(v)\}$.

For the proof of the following Lemma it is possible to see the survey [4].

Lemma 4.2. For all $v\equiv 0,1 \pmod 3$ there exists a large set of disjoint DTS(v,1).

As a trivial consequence of Lemmas 4.1 and 4.2 we obtain the following result.

Theorem 4.1. Let $v\equiv 0,1 \pmod 3$. For each s, $0\le s\le s(v)$, $s\ne 1$ it is $(s,s)\in DFine(v)$.

From Lemma 2.3 and Theorems 2.1 and 4.1 it follows that:

Theorem 4.2. $DFine(3)=\{(0,0),\ (0,2),\ (2,2)\}$.

Theorem 4.3 $DFine(4)=DAdm(4)\setminus\{(0,2),\ (0,3)\}$.

Proof. Let $V=\{1,2,3,4\}$. If $\mathcal{A}_1=\{124,\ 231,\ 342,\ 413,\ 142,\ 213,\ 324,\ 431,\ 421,\ 132,\ 243,\ 314\}$ then $(V,\mathcal{A}_1)$ produces $(0,4)$. If $\mathcal{B}_1=\{123,\ 241,\ 314,\ 432,\ 412,\ 134,\ 243,\ 142,\ 413,\ 234\}$ and $\mathcal{B}_2=\{321\}$, then $(V,\mathcal{B}_1\cup 2\mathcal{B}_2)$ produces $(1,4)$. If $\mathcal{C}_1=\{431,\ 342,\ 341,\ 432,\ 321,\ 142,\ 413,\ 234\}$ and $\mathcal{C}_2=\{123,\ 214\}$ then $(V,\mathcal{C}_1\cup 2\mathcal{C}_2)$ produces $(2,4)$. If $\mathcal{D}_1=\{342,\ 341,\ 432,\ 142,\ 213,\ 324\}$ and $\mathcal{D}_2=\{123,\ 214,\ 431\}$ then $(V,\mathcal{D}_1\cup 2\mathcal{D}_2)$ produces $(3,4)$. If $\mathcal{E}_1=\{241,\ 431,\ 341,\ 314,\ 342\}$, $\mathcal{E}_2=\{214,\ 432\}$ and $\mathcal{E}_3=\{123\}$ then $(V,\mathcal{E}_1\cup 2\mathcal{E}_2\cup 3\mathcal{E}_3)$ produces $(2,3)$. Lemmas 2.3, 2.4 and Theorems 2.1, 4.1 complete the proof. ■

Theorem 4.4. $DFine(6)=DAdm(6)\setminus\{(0,2),(0,3),(0,4),(1,4)\}$.

Proof. Let $V=\{1,2,\ldots,6\}$. If $\mathcal{A}_1=\{534,\ 354,\ 345,\ 154,\ 235\}$, $\mathcal{A}_2=\{145,\ 253\}$ and $\mathcal{A}_3=\{651,\ 562,\ 241,\ 132,\ 316,\ 426,\ 643\}$ then $(V,\mathcal{A}_1\cup 2\mathcal{A}_2\cup 3\mathcal{A}_3)$ produces $(2,3)$.

If $\mathcal{M}_1=\{451,\ 415,\ 541,\ 542,\ 452,\ 524,\ 423,\ 513\}$, $\mathcal{M}_2=\{243,\ 153\}$ and $\mathcal{M}_3=\{614,\ 346,\ 162,\ 635,\ 256,\ 321\}$ then $(V,\mathcal{M}_1\cup 2\mathcal{M}_2\cup 3\mathcal{M}_3)$ produces $(2,4)$. Replacing in this DTS(6,3) some opportune blocks we can produce new fine structures. For example if $\mathcal{M}_1^{(1)}=(\mathcal{M}_1\setminus\{542,\ 524,\ 423\})\cup\{243\}$ and $\mathcal{M}_2^{(1)}=(\mathcal{M}_2\setminus\{243\})\cup\{423,\ 524\}$ then $(V,\mathcal{M}_1^{(1)}\cup\mathcal{M}_2^{(1)}\cup 3\mathcal{M}_3)$ produces $(3,4)$.

If $\mathcal{B}_1=\{126,\ 162,\ 612,\ 236,\ 263,\ 623,\ 346,\ 364,\ 634,\ 456,\ 465,$

645, 516, 561, 651} and $\mathcal{B}_3$={214, 153, 431, 325, 542} then (V,$\mathcal{B}_1\cup 3\mathcal{B}_3$) produces (0,5). Replacing opportune blocks, we obtain (t,5) for t=2,4.

If $\mathcal{C}_1$={452, 425, 542, 543, 534, 453, 143, 134, 314, 152, 125, 215, 132}, $\mathcal{C}_2$={321} and $\mathcal{C}_3$={264, 365, 461, 516, 623} then (V,$\mathcal{C}_1\cup 2\mathcal{C}_2\cup 3\mathcal{C}_3$) produces (1,5). Replacing opportune blocks, we obtain (3,5).

If $\mathcal{L}_1$={135, 163, 153, 154, 164, 146, 263, 236, 235, 254, 245, 246, 356, 536, 563, 465, 645, 654} and $\mathcal{L}_3$={521, 612, 341, 432} then (V,$\mathcal{L}_1\cup 2\mathcal{L}_2$) produces (0,6). Replacing opportune blocks, we obtain (t,6) for t=1,2,...,5.

If $\mathcal{D}_1$={124, 241, 214, 431, 314, 341, 126, 162, 621, 236, 263, 623, 436, 364, 643, 456, 465, 645, 516, 561, 651} and $\mathcal{D}_3$={153, 325, 542} then (V,$\mathcal{D}_1\cup 3\mathcal{D}_3$) produces (0,7). Replacing opportune blocks, we obtain (t,7) for t=1,2,...,6.

If $\mathcal{E}_1$={214, 421, 124, 341, 431, 314, 542, 524, 452, 126, 162, 621, 236, 263, 623, 436, 364, 643, 456, 465, 654, 516, 561, 651} and $\mathcal{E}_3$={153, 325} then (V,$\mathcal{E}_1\cup 3\mathcal{E}_3$) produces (0,8). Replacing opportune blocks, we obtain (t,8) for t=1,2,...,6.

If $\mathcal{F}_1$={421, 612, 236, 314, 436, 364, 634, 524, 645, 516}, $\mathcal{F}_2$={214, 162, 263, 431, 542, 465, 561} and $\mathcal{F}_3$={153, 325} then (V,$\mathcal{F}_1\cup 2\mathcal{F}_2\cup 3\mathcal{F}_3$) produces (7,8).

If $\mathcal{G}_1$={214, 421, 124, 341, 431, 314, 542, 524, 425, 126, 162, 621, 326, 263, 623, 436, 364, 643, 456, 465, 654, 516, 561, 651, 325, 352, 235} and $\mathcal{G}_3$={153} then (V,$\mathcal{G}_1\cup 3\mathcal{G}_3$) produces (0,9). Replacing opportune blocks, we obtain (t,9) for t=1,2,...,8 and (t,10) for t=0,1,...,9.

Lemmas 2.3, 2.4, 2.5 and Theorems 2.1, 4.1 complete the proof. ∎

Theorem 4.5. $DFine(7)=DAdm(7)\setminus\{(0,3),(0,5),(1,4)\}$.

Proof. Starting from the Steiner triple system of order 7, Theorem 3.3 implies that if $(t,s)\in DAdm(7)$ and t and s are even then $(t,s)\in DFine(7)$.

Let $V=\{0,1,\ldots,6\}$. If $\mathcal{A}_1=\{106, 250, 056, 506, 560\}$, $\mathcal{A}_2=\{160, 205\}$ and $\mathcal{A}_3=\{012, 653, 213, 304, 352, 361, 403, 154, 246, 642, 451\}$ then $(V,\mathcal{A}_1\cup 2\mathcal{A}_2\cup 3\mathcal{A}_3)$ produces (2,3).

If $\mathcal{B}_1=\{012, 431, 532, 123, 312, 321\}$, $\mathcal{B}_2=\{021, 413, 523\}$ and $\mathcal{B}_3=\{615, 630, 516, 036, 140, 354, 624, 045, 426, 250\}$ then $(V,\mathcal{B}_1\cup 2\mathcal{B}_2\cup 3\mathcal{B}_3)$ produces (3,4).

If $\mathcal{C}_1=\{012, 102, 120, 230, 203, 023, 340, 304, 034, 410, 401, 014, 541\}$, $\mathcal{C}_2=\{514\}$ and $\mathcal{C}_3=\{605, 506, 532, 216, 624, 315, 425, 613, 436\}$ then $(V,\mathcal{C}_1\cup 2\mathcal{C}_2\cup 3\mathcal{C}_3)$ produces (1,5). Replacing opportune blocks we obtain the fine structures: (t,5) for t=2,3,4; (t,6) for t=1,3,5; (t,7) for t=1,2,...,6; (t,8) for t=1,3,5,7; (t,9) for t=2,3,...,8.

If $\mathcal{G}_1=\{105, 026, 356, 123, 602, 635, 150, 312, 056, 065, 560, 650, 513, 153, 351, 531, 263, 623, 362, 326, 102\}$, $\mathcal{G}_3=\{201, 034, 430, 524, 425, 614, 416\}$ then $(V,\mathcal{G}_1\cup 3\mathcal{G}_3)$ produces (0,7). Replacing opportune blocks we obtain the fine structures (0,9), (1,9) and (t,10) for t=1,3,5,7,9.

If $\mathcal{H}_1=\{021, 102, 120, 230, 203, 023, 430, 304, 034, 463, 436, 346, 410, 401, 014, 154, 541, 514, 351, 315, 135, 631, 613, 136, 261, 216, 126, 605, 056, 650, 506, 065, 560\}$ and $\mathcal{H}_3=\{245, 642,$

532}, then $(V,\mathcal{H}_1\cup 3\mathcal{H}_3)$ produces (0,11). Replacing opportune blocks we obtain the fine structures: (t,11) for t=1,2,...,10; (t,12) for t=1,2,...,11; (t,13) for t=0,1,...,12; (2t+1,14) for t=0,1,...,6.

Lemmas 2.3, 2.4, 2.5, 2.6 and Theorems 2.1, 4.1 complete the proof. ∎

Lemma 4.3 It is $(0,3)\in DFine(v)$ for every $v\equiv 0$, 1 (mod 3), $v\geq 9$.

Proof. By Lemmas 3.1 and 3.3 it is sufficient to prove that $(0,3)\in DFine(v)$ for v=9, 10, 12, 13, 15, 16, 18. Let V={0,1,...,8}.

If $\mathcal{B}_1$={124, 142, 412, 341, 431, 314, 234, 423, 243} and $\mathcal{B}_3$={547, 648, 740, 845, 046, 521, 163, 320, 617, 718, 810, 015, 625, 726, 827, 028, 538, 735, 836, 037, 560} then $(V,\mathcal{B}_1\cup 3\mathcal{B}_3)$ produces $(0,3)\in DFine(9)$.

Let $S=\{0,1,\ldots,\vartheta\}$, ϑ=6,7,9,10. For each $\tau=\vartheta+1$ and $j\in Z_\tau\setminus\{0\}$ put $F_j=\{(a,b)\mid a,b\in Z_\tau,\ b-a\equiv j \pmod{\tau}\}$. If ϑ=6 then put $P_j=F_j$ for j=1, 4, 5; P_2={13, 21, 32, 40, 54, 65, 06}, P_3={14, 25, 36, 43, 50, 61, 02} and P_6={10, 24, 35, 46, 51, 62, 03}. If ϑ=7 then put $P_j=F_j$ for j=1, 4, 5; P_2={13, 21, 32, 46, 50, 64, 75, 07}, P_3={14, 20, 36, 47, 53, 65, 71, 02}, P_6={17, 25, 31, 43, 54, 60, 72, 06}, P_7={10, 24, 35, 42, 57, 61, 76, 03}. If ϑ=9 then put $P_j=F_j$ for j=1, 3, 4, 5, 6, 7; P_2={13, 21, 32, 46, 57, 64, 75, 80, 98, 09}, P_8={19, 20, 35, 43, 54, 68, 76, 87, 91, 02}, P_9={10, 24, 31, 42, 53, 65, 79, 86, 97, 08}. If ϑ=10 then put $P_j=F_j$ for j=1, 3, 4, 5, 6, 7, 8, and if we put a=10, P_2={13, 21, 32, 46, 57, 64, 75, 8a, 90, a8, 09}, P_9={1a, 20, 31, 43, 54, 65, 76, 87, 98, a9, 02}, P_{10}={10, 24, 35, 42, 53, 68, 79, 86, 97, a1, 0a}. Suppose ϑ=6. Let

$(W,\mathcal{B})$, $W=\{\infty_1,\infty_2,\infty_3\}$, be a DTS(3,3) for which it is $(0,0)\in DFine(3)$. Let $\mathcal{A}_1=\langle\infty_1 P_2\rangle\cup\langle\infty_2 P_3\rangle\cup\langle\infty_3 P_6\rangle$, $\mathcal{A}_2=\{1\infty_1 3,\ 2\infty_1 1,\ 3\infty_1 2\}$, $\mathcal{A}_3=\{1\infty_1 3,\ 13\infty_1,\ \infty_1 13,\ 2\infty_1 1,\ 21\infty_1,\ \infty_1 21,\ 3\infty_1 2,\ 32\infty_1,\ \infty_1 32\}$, $\mathcal{A}_4=\{(1+i,2+i,6+i) \mid i=0,1,\dots,6\}$. If $\mathcal{A}=3\mathcal{A}_1\cup\mathcal{A}_3\cup 3\mathcal{A}_4\setminus 3\mathcal{A}_2$ then $(V\cup W,\mathcal{B}\cup\mathcal{A})$ produces $(0,3)\in DFine(10)$. For $\vartheta=7,9,10$ let $(W,\mathcal{B})$ be a DTS$(\vartheta-3,3)$ for which it is $(0,0)\in DFine(\vartheta-3)$. Proceeding as above we obtain the proof for v=12,16,18. By Lemma 3.1 (with the differences P_j instead of F_j) construct a DTS(v,3), v=6,7, such that $(0,0)\in DFine(v)$. Proceeding as above we obtain $(0,3)\in DFine(v)$ for v=13, 15. ∎

Theorem 4.6. *DFine*(9)=*DAdm*(9).

Proof. Starting from the Steiner triple system of order 9, Theorem 3.3 implies that if $(t,s)\in DAdm(9)$ and t and s are even then $(t,s)\in DFine(9)$.

Let V={0,1,...,8}. If $\mathcal{A}_1$={076, 687, 876, 768, 826, 286, 268, 718, 781, 871, 182, 812, 128}, $\mathcal{A}_2$={067} and $\mathcal{A}_3$={083, 370, 341, 456, 480, 604, 432, 724, 385, 105, 573, 147, 584, 502, 136, 275, 201, 623, 651}, then $(V,\mathcal{A}_1\cup 2\mathcal{A}_2\cup 3\mathcal{A}_3)$ produces (1,5).

Lemma 4.3 and Theorem 4.1 produce (0,3), $(s,s)\in DFine(9)$ for each s=0,2,3,...,24. By Lemmas 3.1 and 3.2 (with opportune changes in some cases) it is possible to complete the proof. For the sake of brevity we show the changes only in three cases. Let F_{11}={13, 32, 21}, F_{12}={40, 04}, F_{21}={12, 24, 41}, F_{22}={30, 03}, F_{31}={14, 31, 43}, F_{32}={02, 20}, F_{41}={23, 34, 42}, F_{42}={10, 01} and $F_i=F_{i1}\cup F_{i2}$ for $i=1,2,3,4$. At last let $W=\{\infty_i \mid i=1,2,3,4\}$. Case I: Let (1,4) be the fine structure of the DTS(4,3). If $\mathcal{A}_1=\infty_1 F_{11}\cup\langle\infty_1 F_{11}\rangle\cup F_{11}\infty_1$ and $\mathcal{A}_3=\bigcup_{i=2}^{4}\langle\infty_i F_i\rangle\cup\langle\infty_1 F_{12}\rangle$, then

$(W\cup Z_5, \mathcal{B}\cup\mathcal{A}_1\cup 3\mathcal{A}_3)$ produces $(1,7)$. Case II: Let $(2,3)$ be the fine structure of the DTS$(4,3)$ $(W,\mathcal{B})$. If $\mathcal{A}_1=\bigcup_{i=2}^{4}(<\infty_i F_i>\cup\infty_i F_i\cup F_i\infty_i)\cup\infty_1 F_{11}\cup<\infty_1 F_{11}>\cup F_{11}\infty_1\cup\{\infty_1 04, 40\infty_1\}$ and $\mathcal{A}_2=\{\infty_1 40,\ 04\infty_1\}$, then $(W\cup Z_5, \mathcal{B}\cup\mathcal{A}_1\cup 2\mathcal{A}_2)$ produces $(4,23)$. Case III: Let $(W,\mathcal{B})$ be the DTS$(4,3)$ constructed in Theorem 4.3 producing $(2,3)\in DFine(4)$ where we put ∞_i instead of i (remember that the 2-times repeated blocks are $\infty_2\infty_1\infty_4$ and $\infty_4\infty_3\infty_2$). Let $\mathcal{A}=\bigcup_{i=1}^{4}(\infty_i F_i\cup<\infty_i F_i>\cup F_i\infty_i)$ and let $\mathcal{C}=\mathcal{A}\cup\mathcal{B}\cup\{123,\ \infty_2\infty_1 1,\ \infty_2\infty_4 2,\ \infty_1\infty_4 3,\ 310,\ \infty_4\infty_3 1,\ \infty_4\infty_2 0,\ \infty_3\infty_2 3\}\setminus\{\infty_2\infty_1\infty_4,\ \infty_4\infty_3\infty_2,\ \infty_2 12,\ \infty_4 23,\ \infty_1 13,\ \infty_3 31,\ \infty_2 30,\ \infty_4 10\}$. Then $(W\cup Z_5, \mathcal{C})$ produces $(0,23)$. ∎

Theorem 4.7. *DFine*$(10)=$*DAdm*(10).

Proof. Since there is a PBD on 10 elements with 9 blocks of size 3 and 3 blocks of size 4, by Theorem 3.3 and Lemma 4.3 it remains to prove that $(0,s)$, $(1,s)\in DFine(10)$ for each odd s, $3<s<30$. Let $F_j=\{(a,b)\mid a,b\in Z_7,\ b-a\equiv j \pmod 7\}$, $j\in Z_7\setminus\{0\}$, and let $V=\{\infty_1,\ \infty_2,\ \infty_3\}$. Let $G_{11}=\{12,\ 23,\ 31\}$, $G_{12}=\{46,\ 50,\ 64,\ 05\}$, $G_{21}=\{24,\ 42\}$, $G_{22}=\{13,\ 35,\ 56,\ 60,\ 01\}$, $G_{51}=\{16,\ 20,\ 61,\ 02\}$, $G_{52}=\{34,\ 45,\ 53\}$, $G_1=G_{11}\cup G_{12}$, $G_2=G_{21}\cup G_{22}$ and $G_5=G_{51}\cup G_{52}$. Clearly $F_1\cup F_2\cup F_5=G_1\cup G_2\cup G_5$. With F_3, F_4 and F_6 form the block set $\mathcal{A}=\{(1+i, i, 4+i)\mid i=0,1,\ldots,6 \pmod 7\}$. Let $(V,\mathcal{B})$ be a DTS$(3,3)$ such that either $(0,0)$ or $(0,2)\in DFine(3)$. If $\mathcal{C}=<\infty_1 G_1>\cup<\infty_3 G_{52}>$ and $\mathcal{D}=\infty_2 G_2\cup<\infty_2 G_2>\cup G_2\infty_2\cup\infty_3 G_{51}\cup<\infty_3 G_{51}>\cup G_{51}\infty_3$, then $(V\cup Z_7, \mathcal{B}\cup\mathcal{D}\cup 3\mathcal{C}\cup 3\mathcal{A})$ produces either $(0,11)$ or $(0,13)\in DFine(10)$. Similarly we can prove that $(0,s)\in DFine(10)$ for each $s=5,7,9$.

Let $(V,\mathcal{B})$ be a DTS$(3,3)$ such that $\{\infty_1\infty_2\infty_3,\ \infty_2\infty_3\infty_1\}\subseteq\mathcal{B}$ and $(0,2)\in DFine(3)$. If $\mathcal{A}_1=<\infty_1 G_{12}>\cup<\infty_2 G_{22}>\cup<\infty_3 G_{51}>$, $\mathcal{A}_2=\{12\infty_1,\ 3\infty_1 1,$

$42\infty_2$, $4\infty_3 5$, $53\infty_3\}$, $\mathcal{E}=\{\infty_1\infty_2 2$, $\infty_1\infty_3 3$, $\infty_2\infty_3 4$, $\infty_2\infty_3 4$, $\infty_3\infty_1 3$, $\infty_2\infty_1 2$, $\infty_1 23$, $\infty_2 24$, $\infty_3 34\}$ and $\mathcal{F}=\{234\}$, then $(V\cup Z_7, \mathcal{B}_1\cup\mathcal{E}\cup 2\mathcal{F}\cup 3\mathcal{A}\cup 3\mathcal{A}_1\cup 3\mathcal{A}_2)$ produces $(1,5)$.

If $\mathcal{A}_3=\mathcal{A}_1\setminus\{4\infty_1 6,\ 6\infty_1 4\}$ and $\mathcal{E}_1=\mathcal{E}\cup\{\infty_1 46,\ 4\infty_1 6,\ 46\infty_1,\ 64\infty_1,\ 6\infty_1 4,\ \infty_1 64\}$, then $(V\cup Z_7, \mathcal{B}_1\cup\mathcal{E}_1\cup 2\mathcal{F}\cup 3\mathcal{A}\cup 3\mathcal{A}_3\cup 3\mathcal{A}_2)$ produces $(1,7)$. Analogously it is possible to prove that $(1,s)\in DFine(10)$ for $s=9,11,13$.

Let $(V,\mathcal{B})$ be a DTS(3,3) such that $(0,2)\in DFine(3)$. If $\Gamma=\{104,$ $041,$ $140,$ $304,$ $340,$ $430,$ $415,$ $154,$ $541,$ $521,$ $251,$ $215,$ $362,$ $326,$ $632,$ $603,$ $063,$ $036,$ $526,$ $652,$ $265\}$, $\mathcal{C}_1=\langle\infty_1 G_1\rangle$ and $\mathcal{D}_1=\infty_2 G_2 \cup \langle\infty_2 G_2\rangle \cup G_2\infty_2 \cup \infty_3 G_5 \cup \langle\infty_3 G_5\rangle \cup G_5\infty_3 \cup \{\infty_2 65,\ \infty_2 06,\ \infty_2 10,\ \infty_3 43,\ 43\infty_3,\ \infty_3 54\}\setminus\{\infty_2 56,\ \infty_2 60,\ \infty_2 01,\ \infty_3 34,\ 34\infty_3,\ \infty_3 45\}$, then $(V\cup Z_7, \mathcal{B}\cup\Gamma\cup\mathcal{D}_1\cup 3\mathcal{C}_1)$ produces $(0,23)$. Similarly we can prove that $(0,s)\in DFine(10)$ for $s=15,17,19,21$.

Let $(V,\mathcal{B})$ be a DTS(3,3) such that $\mathcal{B}_3=\{\infty_1\infty_2\infty_3,\ \infty_3\infty_2\infty_1\}$. If $\Gamma_1=\Gamma \cup \{504,\ 310,\ 240,\ 134\}\setminus\{140,\ 304\}$, $\mathcal{D}_2=\infty_1 G_1 \cup \langle\infty_1 G_1\rangle \cup G_1\infty_1 \cup \infty_2 G_2 \cup \langle\infty_2 G_2\rangle \cup G_2\infty_2 \cup \infty_3 G_3 \cup \langle\infty_3 G_3\rangle \cup G_3\infty_3 \cup \{\infty_1\infty_2\infty_3,\ \infty_1\infty_2 3,\ \infty_1\infty_2 1,\ \infty_1\infty_3 5,\ \infty_1\infty_3 0,\ \infty_2\infty_3 2,\ \infty_2\infty_3 4,\ \infty_2 65,\ \infty_2 10,\ 43\infty_3,\ \infty_3 43,\ \infty_2 06\}\setminus\{\infty_2 01,\ \infty_3 34,\ 34\infty_3,\ \infty_3 45,\ \infty_2 60,\ \infty_2 56,\ \infty_1 31,\ \infty_2 13,\ \infty_1 50,\ \infty_3 20,\ \infty_2 24\}$ and $\mathcal{C}_2=\{\infty_3\infty_2\infty_1\}$, then $(V\cup Z_7, \Gamma_1\cup\mathcal{D}_2\cup 3\mathcal{C}_2)$ produces $(0,29)$.

If $\mathcal{D}_3=\mathcal{D}_2\setminus\{\infty_1 46,\ 4\infty_1 6,\ 46\infty_1,\ \infty_1 64,\ 6\infty_1 4,\ 64\infty_1\}$ and $\mathcal{C}_3=\mathcal{C}_2 \cup \{4\infty_1 6,\ 6\infty_1 4\}$, then $(V\cup Z_7, \Gamma_1\cup\mathcal{D}_3\cup 3\mathcal{C}_3)$ produces $(0,27)$. Analogously we obtain $(0,25)\in DFine(10)$. If in the above cases we replace the blocks 362 and 526 with 326 and 562, we obtain $(1,s)\in DFine(10)$ for each $s=15,\ 17,\ 19,\ 21,\ 23,\ 25,\ 27,\ 29$. ∎

Theorem 4.8. *DFine*(12)=*DAdm*(12).

Proof. Since there is a PBD on 12 elements with 16 blocks of size 3 and 3 blocks of size 4, by Theorem 3.3 and Lemma 4.3 it remains to prove that $(0,s)$, $(1,s)\in DFine(12)$ for each odd s, $3<s<44$. Let $F_j=\{(a,b) \mid a,b\in Z_8,\ b-a\equiv j \pmod 8\}$, $j\in Z_8\setminus\{0\}$, and let $V=\{\infty_1, \infty_2, \infty_3, \infty_4\}$. Let $G_{11}=\{13, 21, 32\}$, $G_{12}=\{46, 54, 65\}$, $G_{13}=\{70, 07\}$, $G_2=\{24, 35, 57, 12, 43, 60, 76, 01\}$, $G_4=\{71, 02, 10, 23, 34, 45, 56, 67\}$, $G_{71}=\{15, 51\}$, $G_{72}=\{26, 62\}$, $G_{73}=\{37, 73\}$, $G_{74}=\{40, 04\}$, $G_1=G_{11}\cup G_{12}\cup G_{13}$, $G_7=G_{71}\cup G_{72}\cup G_{73}\cup G_{74}$. Clearly $F_1\cup F_2\cup F_4\cup F_7=G_1\cup G_2\cup G_4\cup G_7$. With F_3, F_5 and F_6 form the block set $\mathcal{A}=\{(1+i,7+i,4+i) \mid i=0,1,\ldots,7\}$. Let $(V,\mathcal{B})$ be a DTS(4,3) such that either $(0,0)$ or $(0,4)$ or $(1,4)\in DFine(4)$. Let $\alpha(i)=i$ for $i=1,2$, $\alpha(4)=3$ and $\alpha(7)=4$. For each G_i (or G_{ij}) put in $\mathcal{C}$ either the directed triples $\langle\infty_{\alpha(i)}G_i\rangle$ 3-times repeated or $\infty_{\alpha(i)}G_i \cup \langle\infty_{\alpha(i)}G_i\rangle \cup G_i\infty_{\alpha(i)}$. Then $(V\cup Z_8,\mathcal{B}\cup\mathcal{C}\cup 3\mathcal{A})$ produces $(0,s_1)$, $(1,s_2)\in DFine(12)$ for each odd s_1, $5\le s_1\le 33$, and for each odd s_2, $7\le s_2\le 33$. Suppose that the DTS(12,3) which produces $(0,3)$ above constructed contains the directed triples $\infty_1G_{11}\cup\langle\infty_1G_{11}\rangle\cup G_{11}\infty_1$. If we change the directed triples $\{2\infty_3 3, 2\infty_3 3, 3\infty_3 4, 3\infty_3 4, \infty_1 32\}$ with $\{32\infty_3, 23\infty_3, \infty_3 34, \infty_3 34, \infty_1 23\}$, then $(1,5)\in DFine(12)$.

If $\mathcal{A}_1=\{174, 741, 417, 205, 025, 520, 316, 163, 631, 427, 247, 742, 530, 350, 035, 614, 641, 164, 752, 725, 527, 063, 036, 306\}$ and $\mathcal{C}_1=\mathcal{C}\cup\{\infty_2 42, \infty_2 53, 53\infty_2, \infty_2 75, 17\infty_3, \infty_3 20, \infty_1 31\}\setminus\{\infty_1 13, \infty_2 24, \infty_2 35, 35\infty_2, \infty_2 57, 71\infty_3, \infty_3 02\}$, then $(V\cup Z_8,\mathcal{B}\cup\mathcal{C}_1\cup\mathcal{A}_1)$ produces $(0,s)$ for each $s=35, 37, 39, 41$.

Suppose the set of 3-times repeated blocks is $\langle\infty_1G_{13}\rangle$. Let $\mathcal{B}_1=\{\infty_1\infty_2\infty_4, \infty_4\infty_3\infty_1, \infty_3\infty_4\infty_2, \infty_4\infty_1\infty_3, \infty_1\infty_4\infty_2, \infty_3\infty_2\infty_4, \infty_2\infty_1\infty_3, \infty_4\infty_2\infty_1, \infty_2\infty_4\infty_3, \infty_3\infty_1\infty_4\}$ and $\mathcal{C}_2=\mathcal{C}_1\cup\{\infty_2\infty_1 0, \infty_1 0\infty_2, \infty_2\infty_3 1, \infty_3 1\infty_2, \infty_3 7\infty_1, 7\infty_1\infty_3, \infty_3 01, 710, 701\}\setminus\{7\infty_1 0, 7\infty_1 0, \infty_2 01, 01\infty_2, \infty_3 10, 7\infty_3 1,$

$\infty_3 71\}$, then $(V \cup Z_8, \mathcal{B}_1 \cup \mathcal{C}_2 \cup \mathcal{A}_1)$ produces (0,43). If in the above five DTS(12,3) we change the blocks 025 and 520 with 205 and 502, it follows $(1,s) \in DFine(12)$ for each s=35, 37, 39, 41, 43. ∎

Theorem 4.9. *DFine*(v)=*DAdm*(v) for v=13,15,16,18.

Proof. Since there are PBD on v elements with blocks of size 3 and blocks of size 4 [5], by Theorem 3.3 and Lemma 4.3 it remains to prove that (0,s), $(1,s) \in DFine(v)$ for each odd s, $3<s<s(v)$. For v=13,15 we can proceed as in Theorem 4.6 using the costruction w—>2w+1, w=6,7. The ingredients are a DTS(w,3) having an opportune fine structure and the following edge decompositions of the complete directed graph on w+1 elements: For w+1=7, let P_1={12, 23, 31, 46, 64, 50, 05}, P_2={13, 24, 35, 42, 56, 60, 01}, P_3={16, 61, 20, 34, 45, 53, 02}, P_4={14, 25, 36, 40, 51, 62, 03}, P_5={15, 26, 30, 41, 52, 63, 04}, P_6={10, 21, 32, 43, 54, 65, 06}. For w+1=8, let P_1={12, 23, 31, 45, 56, 67, 70, 04}, P_2={13, 24, 35, 42, 57, 60, 71, 06}, P_3={14, 25, 36, 47, 50, 61, 72, 03}, P_4={15, 26, 37, 40, 51, 64, 73, 02}, P_5={16, 27, 30, 41, 52, 63, 74, 05}, P_6={17, 20, 34, 46, 53, 62, 75, 01}, P_7={10, 21, 32, 43, 54, 65, 76, 07}. For v=16 the proof follows from the existence of a PBD with three blocks of size 6 and the remaining blocks of size 3. For v =18 the proof follows from the existence of a PBD with 3 blocks of size 4, one block of size 6 and the remaining blocks of size 3 [11]. ∎

Theorem 4.10. *DFine*(5)=*DAdm*(5).

Proof. Let v={0,1,...,4}. If $\mathcal{B}_1$={012, 123, 234, 340, 401, 024, 130, 241, 302, 413, 031, 142, 203, 314, 420, 043 104, 210, 321, 432}, then $(V,\mathcal{B}_1)$ produces $(0,6)$. If $\mathcal{B}_1^{(1)}=\mathcal{B}_1\cup\{032, 431, 014\}\setminus\{130, 302, 413, 031, 104\}$ and $\mathcal{B}_2^{(1)}=\{130\}$, then $(V,\mathcal{B}_1^{(1)}\cup 2\mathcal{B}_2^{(1)})$ produces $(1,6)$. If $\mathcal{C}_1=\{123, 304, 410, 024, 341, 314, 420, 043, 401, 321\}$ and $\mathcal{C}_2=\{432, 012, 203, 130, 214\}$, then $(V,\mathcal{C}_1\cup 2\mathcal{C}_2)$ produces $(5,6)$. If $\mathcal{C}_1^{(1)}=\mathcal{C}_1\cup\{431, 432, 342\}\setminus\{341\}$ and $\mathcal{C}_2^{(2)}=\mathcal{C}_2\setminus\{432\}$, then $(V,\mathcal{C}_1^{(1)}\cup 2\mathcal{C}_2^{(2)})$ produces $(4,6)$. If $\mathcal{C}_1^{(2)}=\mathcal{C}_1^{(1)}\{021, 012, 312\}\setminus\{321\}$ and $\mathcal{C}_2^{(2)}=\mathcal{C}_2^{(1)}\setminus\{012\}$, then $(V,\mathcal{C}_1^{(2)}\cup 2\mathcal{C}_2^{(2)})$produces $(3,6)$. If $\mathcal{C}_1^{(3)}=\mathcal{C}_1^{(2)}\cup\{203, 230, 034\}\setminus\{304\}$ and $\mathcal{C}_2^{(3)}=\mathcal{C}_2^{(2)}\setminus\{203\}$, then $(V,\mathcal{C}_1^{(3)}\cup 2\mathcal{C}_2^{(3)})$ produces $(2,6)$. If $\mathcal{D}_1=\{402, 204, 142, 342, 243, 241, 301, 103\}$ and $D_2=\{321, 401, 034, 143, 102, 230\}$ then $(V,\mathcal{D}_1\cup 2\mathcal{D}_2)$ produces $(6,6)$. ∎

Theorem 4.11. *DFine*(8)=*DAdm*(8).

Proof. Let $V=\{0,1,...,7\}$. If $\mathcal{B}_1=\{126, 157, 257, 346, 375, 475, 567, 765\}$, $\mathcal{B}_2=\{125, 167, 276, 347, 356, 465\}$ and $\mathcal{B}_3=\{143, 421, 532, 624, 731, 203, 510, 630, 740, 054, 061, 072\}$, then $(V,\mathcal{B}_1\cup 2\mathcal{B}_2\cup 3\mathcal{B}_3)$ produces $(6,6)$. To obtain $(t,6)$ for $t=2,3,4,5$ change opportunely in some directed triples of $\mathcal{B}_1\cup 2\mathcal{B}_2$ the directed pair 67 with 76 and (or) 56 with 65. For example changing the blocks 276 and 567 with 156 and 265 we obtain $(5,6)$.

If $\mathcal{C}_1=\{125, 126, 127, 156, 157, 167, 256, 257, 267, 345, 346, 347, 365, 367, 375, 465, 476, 475, 576, 765\}$ then $(V,\mathcal{C}_1\cup 3\mathcal{B}_3)$ produces $(0,6)$. If $\mathcal{C}_1^{(1)}=\mathcal{C}_1\cup\{265, 275\}\setminus\{256, 257, 576, 765\}$ and $\mathcal{C}_2^{(1)}=\{576\}$, then $(V,\mathcal{C}_1^{(1)}\cup 2\mathcal{C}_2^{(1)}\cup 3\mathcal{B}_3)$ produces $(1,6)$.

If $\mathcal{D}_1=\{152, 126, 127, 165, 167, 175, 256, 257, 267, 345, 346, 347, 536, 537, 376, 465, 475, 476, 567, 765, 325, 352, 532\}$ and

$\mathcal{D}_3$={143, 421, 624, 731, 203, 510, 630, 740, 054, 061, 072}, then $(V,\mathcal{D}_1\cup 3\mathcal{D}_3)$ produces (0,7). If $\mathcal{D}_1^{(1)}=\mathcal{D}_1\cup\{125\}\setminus\{152, 325\}$ and $\mathcal{D}_2^{(1)}=\{352\}$, then $(V,\mathcal{D}_1^{(1)}\cup 2\mathcal{D}_2^{(1)}\cup 3\mathcal{D}_3)$ produces (1,7).

If $\mathcal{E}_1$={126, 157, 257, 346, 375, 745, 567, 765, 470}, $\mathcal{E}_2$={125, 167, 276, 347, 356, 465, 740} and $\mathcal{E}_3$={143, 421, 532, 624, 731, 203, 510, 630, 054, 061, 072}, then $(V,\mathcal{E}_1\cup 2\mathcal{E}_2\cup 3\mathcal{E}_3)$ produces (7,7). If $\mathcal{E}_1^{(1)}=\mathcal{E}_1\cup\{347, 374, 475\}\setminus\{745\}$ and $\mathcal{E}_2^{(1)}=\mathcal{E}_2\setminus\{347\}$, then $(V,\mathcal{E}_1^{(1)}\cup 2\mathcal{E}_2^{(1)}\cup 3\mathcal{E}_3)$ produces (6,7). If $\mathcal{E}_1^{(2)}=\mathcal{E}_1\cup\{167, 176, 276, 267\}$ and $\mathcal{E}_2^{(2)}=\mathcal{E}_2\setminus\{167, 276\}$, then $(V,\mathcal{E}_1^{(2)}\cup 2\mathcal{E}_2^{(2)}\cup 3\mathcal{E}_3)$ produces (5,7). If $\mathcal{E}_1^{(3)}=\mathcal{E}_1^{(1)}\cup\{167, 176, 276, 267\}$ and $\mathcal{E}_2^{(3)}=\mathcal{E}_2^{(1)}\setminus\{167, 276\}$, then $(V,\mathcal{E}_1^{(3)}\cup 2\mathcal{E}_2^{(3)}\cup 3\mathcal{E}_3)$ produces (4,7). If $\mathcal{E}_1^{(4)}=\mathcal{E}_1^{(2)}\cup\{356, 465, 365, 456\}$ and $\mathcal{E}_2^{(4)}=\mathcal{E}_2^{(2)}\setminus\{356, 465\}$, then $(V,\mathcal{E}_1^{(4)}\cup 2\mathcal{E}_2^{(4)}\cup 3\mathcal{E}_3)$ produces (3,7). If $\mathcal{E}_1^{(5)}=\mathcal{E}_1^{(3)}\cup\{356, 465, 365, 456\}$ and $\mathcal{E}_2^{(5)}=\mathcal{E}_2^{(3)}\setminus\{356, 465\}$, then $(V,\mathcal{E}_1^{(5)}\cup 2\mathcal{E}_2^{(5)}\cup 3\mathcal{E}_3)$ produces (2,7).

Let $\mathcal{G}_1$={125, 126, 127, 165, 176, 175, 256, 257, 267, 435, 436, 347, 365, 367, 735, 465, 476, 457, 567, 576, 314, 134, 143, 713, 371, 731} and $\mathcal{G}_3$={421, 532, 624, 061, 203, 510, 630, 740, 054, 072}, then $(V,\mathcal{G}_1\cup 3\mathcal{G}_3)$ produces (0,8). If $\mathcal{G}_1^{(1)}=\mathcal{G}_1\cup\{376\}\setminus\{367, 576, 567\}$ and $\mathcal{G}_2=\{567\}$, then $(V,\mathcal{G}_1^{(1)}\cup 2\mathcal{G}_2\cup 3\mathcal{G}_3)$ produces (1,8). If $\mathcal{G}_1^{(2)}=\mathcal{G}_1\cup\{345, 375\}\setminus\{435, 735, 134, 371, 143, 731\}$ and $\mathcal{G}_2^{(2)}=\{143, 731\}$, then $(V,\mathcal{G}_1^{(2)}\cup 2\mathcal{G}_2^{(2)}\cup 3\mathcal{G}_3)$ produces (2,8).

Let $\mathcal{H}_1$={126, 157, 257, 436, 735, 475, 567, 765, 134, 371} and $\mathcal{H}_2$={125, 167, 276, 347, 356, 465, 143, 731}, then $(V,\mathcal{H}_1\cup 2\mathcal{H}_2\cup 3\mathcal{G}_3)$ produces (8,8). If $\mathcal{H}_1^{(1)}=\mathcal{H}_1\cup\{346, 347, 437\}\setminus\{436\}$ and $\mathcal{H}_2^{(1)}=\mathcal{H}_2\setminus\{347\}$, then $(V,\mathcal{H}_1^{(1)}\cup 2\mathcal{H}_2^{(1)}\cup 3\mathcal{G}_3)$ produces (7,8). Similarly we can prove that $(t,8)\in DFine(8)$ for each t=3,4,5,6.

Let $\mathcal{L}_1$={152, 126, 127, 156, 157, 167, 256, 257, 267, 453, 436, 347, 365, 367, 753, 465, 476, 475, 576, 765, 314, 134, 143, 713, 371, 731, 352, 325, 532} and $\mathcal{L}_3=\mathcal{G}_3\setminus\{532\}$, then $(V,\mathcal{L}_1\cup 3\mathcal{L}_3)$ produces (0,9). If $\mathcal{L}_1^{(1)}=\mathcal{L}_1\cup\{125\}\setminus\{152, 325, 352\}$ and $\mathcal{L}_2^{(1)}=\{352\}$, then $(V,\mathcal{L}_1^{(1)}\cup 2\mathcal{L}_2^{(1)}\cup 3\mathcal{L}_3)$ produces (1,9). If $\mathcal{L}_1^{(2)}=\mathcal{L}_1^{(1)}\cup\{346\}\setminus\{436, 134, 143\}$ and $\mathcal{L}_2^{(2)}=\mathcal{L}_2^{(1)}\cup\{143\}$, then $(V,\mathcal{L}_1^{(2)}\cup 2\mathcal{L}_2^{(2)}\cup 3\mathcal{L}_3)$ produces (2,9). If $\mathcal{L}_1^{(3)}=\mathcal{L}_1^{(2)}\cup\{175, 165\}\setminus\{765, 156, 157, 576\}$ and $\mathcal{L}_2^{(3)}=\mathcal{L}_2^{(2)}\cup\{576\}$, then $(V,\mathcal{L}_1^{(3)}\cup 2\mathcal{L}_2^{(3)}\cup 3\mathcal{L}_3)$ produces (3,9). If $\mathcal{L}_1^{(4)}=\mathcal{H}_1\cup\{753, 352\}\setminus\{735\}$ and $\mathcal{L}_2^{(4)}=\mathcal{H}_2\cup\{532\}$, then $(V,\mathcal{L}_1^{(4)}\cup 2\mathcal{L}_2^{(4)}\cup 3\mathcal{L}_3)$ produces (9,9). If $\mathcal{L}_1^{(5)}=\mathcal{L}_1^{(4)}\cup\{346, 347, 437\}\setminus\{436\}$ and $\mathcal{L}_2^{(5)}=\mathcal{L}_2^{(4)}\setminus\{347\}$, then $(V,\mathcal{L}_1^{(5)}\cup 2\mathcal{L}_2^{(5)}\cup 3\mathcal{L}_3)$ produces (8,9). Similarly we obtain $(t,9)\in DFine(8)$ for each t=4,5,6,7.

In the following we give a complete solution for the cases (0,s), (s,s), s=10,11,...,18. Proceeding as above it is possible to obtain the remaining cases.

Let $\mathcal{M}_1=\mathcal{L}_1\cup\{412, 421, 413, 142, 216, 217\}\setminus\{126, 127, 143\}$ and $\mathcal{M}_3=\mathcal{L}_3\setminus\{421\}$, then $(V,\mathcal{M}_1\cup 3\mathcal{M}_3)$ produces (0,10). Let $\mathcal{M}_1^{(1)}=\mathcal{L}_1^{(4)}\cup\{216, 412\}\setminus\{126\}$ and $\mathcal{M}_2^{(1)}=\mathcal{L}_2^{(4)}\cup\{421\}$, then $(V,\mathcal{M}_1^{(1)}\cup 2\mathcal{M}_2^{(1)}\cup 3\mathcal{M}_3)$ produces (10,10).

Let $\mathcal{N}_1=\mathcal{M}_1\cup\{624, 642, 462, 241, 124, 645\}\setminus\{421, 142, 465\}$ and $\mathcal{N}_3=\mathcal{M}_3\setminus\{624\}$, then $(V,\mathcal{N}_1\cup 3\mathcal{N}_3)$ produces (0,11). Let $\mathcal{N}_1^{(1)}=\mathcal{M}_1^{(1)}\cup\{642, 241\}\setminus\{421\}$ and $\mathcal{N}_2^{(1)}=\mathcal{M}_2^{(1)}\cup\{624\}$, then $(V,\mathcal{N}_1^{(1)}\cup 2\mathcal{N}_2^{(1)}\cup 3\mathcal{N}_3)$ produces (11,11).

Let $\mathcal{P}_1=\mathcal{L}_1\cup\{725, 726, 637, 457, 567, 023, 203, 230, 027, 207, 072, 360, 630, 603\}\setminus\{257, 267, 367, 475, 576\}$ and $\mathcal{P}_3=\mathcal{L}_3\setminus\{203, 630, 072\}$, then $(V,\mathcal{P}_1\cup 3\mathcal{P}_3)$ produces (0,12). Let $\mathcal{P}_1^{(1)}=\mathcal{L}_1^{(4)}\cup\{725, 457, 603, 027, 230\}\setminus\{257, 475\}$ and $\mathcal{P}_2^{(1)}=\mathcal{L}_2^{(4)}\cup\{630, 072, 203\}$, then $(V,\mathcal{P}_1^{(1)}\cup 2\mathcal{P}_2^{(1)}\cup 3\mathcal{P}_3)$ produces (12,12).

Let $\mathcal{Q}_1=\mathcal{P}_1\cup\{061, 601, 306, 617, 016\}\backslash\{360, 167\}$ and $\mathcal{Q}_3=\mathcal{P}_3\backslash\{061\}$, then $(V,\mathcal{Q}_1\cup 3\mathcal{Q}_3)$ produces $(0,13)$. Let $\mathcal{Q}_1^{(1)}=\mathcal{P}_1^{(1)}\cup\{063, 601\}\backslash\{603\}$ and $\mathcal{Q}_2^{(1)}=\mathcal{P}_2^{(1)}\cup\{061\}$, then $(V,\mathcal{Q}_1^{(1)}\cup 2\mathcal{Q}_2^{(1)}\cup 3\mathcal{Q}_3)$ produces $(13,13)$.

Let $\mathcal{R}_1=\mathcal{Q}_1\cup\{517, 106, 150, 510, 501\}\backslash\{157, 016\}$ and $\mathcal{R}_3=\mathcal{Q}_3\backslash\{510\}$, then $(V,\mathcal{R}_1\cup 3\mathcal{R}_3)$ produces $(0,14)$. Let $\mathcal{R}_1^{(1)}=\mathcal{Q}_1^{(1)}\cup\{517, 150\}\backslash\{157\}$ and $\mathcal{R}_2^{(1)}=\mathcal{Q}_2^{(1)}\cup\{510\}$, then $(V,\mathcal{R}_1^{(1)}\cup 2\mathcal{R}_2^{(1)}\cup 3\mathcal{R}_3)$ produces $(14,14)$.

Let $\mathcal{S}_1=\mathcal{R}_1\cup\{216, 217, 341, 421, 412, 142\}\backslash\{126, 127, 314\}$ and $\mathcal{S}_3=\mathcal{R}_3\backslash\{421\}$, then $(V,\mathcal{S}_1\cup 3\mathcal{S}_3)$ produces $(0,15)$. Let $\mathcal{S}_1^{(1)}=\mathcal{R}_1^{(1)}\cup\{412, 216\}\backslash\{126\}$ and $\mathcal{S}_2^{(1)}=\mathcal{R}_2^{(1)}\cup\{421\}$, then $(V,\mathcal{S}_1^{(1)}\cup 2\mathcal{S}_2^{(1)}\cup 3\mathcal{S}_3)$ produces $(15,15)$.

Let $\mathcal{T}_1=\mathcal{S}_1\cup\{374, 746, 470, 740, 407, 702\}\backslash\{347, 476, 072\}$ and $\mathcal{T}_3=\mathcal{S}_3\backslash\{740\}$, then $(V,\mathcal{T}_1\cup 3\mathcal{T}_3)$ produces $(0,16)$. Let $\mathcal{T}_1^{(1)}=\mathcal{S}_1^{(1)}\cup\{746, 470\}\backslash\{476\}$ and $\mathcal{T}_2^{(1)}=\mathcal{S}_2^{(1)}\cup\{740\}$, then $(V,\mathcal{T}_1^{(1)}\cup 2\mathcal{T}_2^{(1)}\cup 3\mathcal{T}_3)$ produces $(16,16)$.

Let $\mathcal{U}_1=\mathcal{T}_1\cup\{547, 051, 054, 504, 045\}\backslash\{457, 501\}$ and $\mathcal{U}_3=\mathcal{T}_3\backslash\{054\}$, then $(V,\mathcal{U}_1\cup 3\mathcal{U}_3)$ produces $(0,17)$. Let $\mathcal{U}_1^{(1)}=\mathcal{T}_1^{(1)}\cup\{105, 504\}\backslash\{150\}$ and $\mathcal{U}_2^{(1)}=\mathcal{T}_2^{(1)}\cup\{054\}$, then $(V,\mathcal{U}_1^{(1)}\cup 2\mathcal{U}_2^{(1)}\cup 3\mathcal{U}_3)$ produces $(17,17)$.

Let $\mathcal{V}_1=\mathcal{U}_1\cup\{162, 241, 642, 264, 624\}\backslash\{126, 421\}$, then $(V,\mathcal{V}_1)$ produces $(0,18)$. Let $\mathcal{V}_1^{(1)}=\mathcal{U}_1^{(1)}\cup\{162, 264\}\backslash\{126\}$ and $\mathcal{V}_2^{(1)}=\mathcal{U}_2^{(1)}\cup\{624\}$, then $(V,\mathcal{V}_1^{(1)}\cup 2\mathcal{V}_2^{(1)})$ produces $(18,18)$. ∎

Theorem 4.12. *DFine*(11)=*DAdm*(11).

Proof. Lemma 3.1 produces $(t,s)\in DFine(11)$ for $s=6k$, $k=0,1,\ldots,6$ and $0\le t\le s$.

Let $V=\{0,1,\ldots,9,a\}$. Let $\mathcal{D}_1$ be the block set defined in the above Theorem 4.10. If $\mathcal{G}_1=\{062, 649, 631, 96a, a68, 860, 083, 841,$

438, 218, 209, 910, 139, 942, 3a0, *a*23, 24*a*, 1*a*4} and $\mathcal{G}_2$={051, 85*a*, *a*59, 958, 879, 973, *a*71, 074, 782, 70*a*, 540}, then $(V,\mathcal{D}_1\cup 3\mathcal{G}_1\cup 3\mathcal{G}_2)$ produces (0,7).

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the block sets defined in the above Theorem 4.10. If $\mathcal{G}_1^{(1)}=\mathcal{G}_1\cup\{a32\}\setminus\{a23\}$ and $\mathcal{G}_2^{(1)}$={071, 723, 87*a*, 978, *a*79, 859, 953, *a*51, 054, 582}, then $(V,\mathcal{E}_1\cup 2\mathcal{E}_2\cup 3\mathcal{G}_1^{(1)}\cup 3\mathcal{G}_2^{(1)})$ produces (7,7). Proceeding as in the Theorem 4.10 we prove that $(t,7)\in DFine(11)$ for each t=1,2,...,6.

To settle the cases (t,s), for each s=8,9,...,34 and $0\le t\le s$, it is possible to embed a DTS(5,3) on $W=\{\infty_1, \infty_2,\dots,\infty_5\}$ having an opportune fine structure in a DTS(11,3) as in Theorem 4.6. (Note that F_2={13, 35, 51}∪{24, 40, 02} and F_3={14, 41}∪{25, 52}∪{30, 03}).

To settle the cases (t,35), it is possible to suppose that the DTS(11,3) $(V,\mathcal{B})$ producing (t,34) above constructed contains the three-times repeated blocks $\infty_3 14$ and $41\infty_4$. Moreover it is possible to suppose that $\infty_3\infty_4\infty_1$, $\infty_3\infty_1\infty_4\in\mathcal{B}$. Then it is possible to replace in $\mathcal{B}$ the blocks $\infty_3\infty_1\infty_4$, $\infty_3 14$, $\infty_1 45$, $\infty_4 15$ with $\infty_3\infty_1 4$, $\infty_3\infty_4 1$, $\infty_1\infty_4 5$, 145 and or the blocks $\infty_3\infty_4\infty_1$, $\infty_3 14$, $\infty_4 42$, $\infty_1 12$ with the following ones $\infty_3\infty_4 4$, $\infty_3\infty_1 1$, $\infty_4\infty_1 2$, 142. ∎

Theorem 4.13. *DFine*(14)=*DAdm*(14).

Proof. Let V={0,1,...,9,*a*,*b*,*c*,*d*}. If $\mathcal{A}_1$={*d*8*b*, *dac*, 8*ac*, 90*b*, 9*ca*, *c*0*a*, 0*c*8, *abc*, *cba*}, $\mathcal{A}_2$={*d*8*a*, *dbc*, 8*cb*, 90*c*, 9*ab*, 0*ba*, *c*08} and $\mathcal{A}_3$={*a*8*d*, *a*30, 7*a*9, 1*a*4, 2*a*5, 3*a*6, 5*a*1, 6*a*2, 4*a*7, *bd*0, *b*98, 1*b*6, 2*b*7, 3*b*1, 4*b*2, 5*b*3, 6*b*4, 7*b*5, *c*9*d*, 1*c*7, 2*c*1, 3*c*2, 4*c*3, 5*c*4, 6*c*5, 7*c*6, *d*69, 05*d*, 1*d*2, 2*d*3, 3*d*4, 4*d*5, 7*d*1, 6*d*7, 819, 480, 185, 286, 387, 582, 683, 784, 092, 395, 294, 596, 491, 973, 103, 046, 720, 507, 601}, then $(V,\mathcal{A}_1\cup 2\mathcal{A}_2\cup 3\mathcal{A}_3)$ produces (7,7). If

$\mathcal{A}_1^{(1)}=\mathcal{A}_1\cup\{9ab,\ 9ba,\ cab\}\setminus\{cba\}$ and $\mathcal{A}_2^{(1)}=\mathcal{A}_2\setminus\{9ab\}$, then $(V,\mathcal{A}_1^{(1)}\cup 2\mathcal{A}_2^{(1)}\cup 3\mathcal{A}_3)$ produces $(6,7)$. Similarly we can prove that $(t,7)\in DFine(14)$ for each $t=2,3,4,5$.

Let $\mathcal{B}_1=\{da8,\ d8b,\ 8dc,\ dab,\ dac,\ dbc,\ 8ca,\ 8ba,\ 8cb,\ 90a,\ 90b,\ 90c,\ 9ab,\ 9ac,\ 9bc,\ 0ca,\ 0ba,\ 0cb,\ abc,\ cba,\ ad8,\ a8d,\ 8ad\}$ and $\mathcal{B}_3=\mathcal{A}_3\cup\{c08\}\setminus\{a8d\}$, then $(V,\mathcal{B}_1\cup 3\mathcal{B}_3)$ produces $(0,7)$. If $\mathcal{B}_1^{(1)}=\mathcal{B}_1\cup\{d8a\}\setminus\{da8,\ a8d,\ 8ad\}$ and $\mathcal{B}_2=\{a8d\}$, then $(V,\mathcal{B}_1^{(1)}\cup 2\mathcal{B}_2\cup 3\mathcal{B}_3)$ produces $(1,7)$.

To complete the proof we can proceed similarly to the above Theorem by embedding a DTS(5,3) on $W=\{\infty_1,\ \infty_2,\dots,\infty_5\}$ having an opportune fine structure in a DTS(14,3) (see also Lemmas 3.3 and 3.4). The ingredients are: $P_1=\{10,\ 25,\ 34,\ 43,\ 58,\ 67,\ 76,\ 82,\ 01\}$, $P_2=\{18,\ 20,\ 32,\ 47,\ 54,\ 65,\ 71,\ 86,\ 03\}$, $P_3=\{12,\ 23,\ 31,\ 45,\ 56,\ 64,\ 78,\ 80,\ 07\}$, $P_4=\{14,\ 21,\ 36,\ 42,\ 53,\ 60,\ 75,\ 87,\ 08\}$, $P_5=\{16,\ 27,\ 38,\ 40,\ 51,\ 62,\ 73,\ 84,\ 05\}$ and one of the following block sets: either $3\{(j,j+2,j+6)\mid j\in Z_9\}$, or $2\{(j,j+2,j+6)\mid j\in Z_9\}\cup\{(j,j+4,j+6)\mid j\in Z_9\}$, or $\{(j,j+2,j+6)\mid j\in Z_9\}\cup\{(j,j+6,j+2)\mid j\in Z_9\}\cup\{(j,j+4,j+6)\mid j\in Z_9\}$. ■

Proof of the Main theorem. The results of the section 4 and Theorems 2.1, 3.1 and 3.2 prove the Main theorem.

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(Received 3/7/96)