ON DEFECTIVE COLOURINGS OF TRIANGLE-FREE GRAPHS

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Abstract: A graph is **(m,k)-colourable** if its vertices can be coloured with m colours such that the maximum degree of the subgraph induced on vertices receiving the same colour is at most k. The **k-defective chromatic number** $\chi_k(G)$ of a graph G is the least positive integer m for which G is (m,k)-colourable. In this paper we obtain bounds for $\chi_1(G) + \chi_1(\overline{G})$ and $\chi_1(G) \cdot \chi_1(\overline{G})$ when G ranges over the class of all triangle-free graphs of order p.

1. Introduction

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part we follow the notation of Chartrand and Lesniak [5]. For a graph G, we denote the vertex set and the edge set of G by V(G) and E(G) respectively. The complement of a graph G is denoted by $\overline{G}$ and the size of G is denoted by $\varepsilon(G)$. For a positive integer n, P_n is a path of order n and C_n is a cycle of order n. For a subset U of V(G), the subgraph of G induced on U is denoted by G[U] and the subgraph induced on V(G) - U is denoted by G - U.

Let G be a graph and X a subset of V(G). For a vertex u of G, let N(u) denote the set of all neighbours of u in G and let $N_X(u) = N(u) \cap X$. Let N[u] denote the **closed neighbourhood** of u, that is, $N(u) \cup \{u\}$. A graph G is said to have the **property tP_3** if the maximum number of vertex disjoint paths of order 3 in G is t. G is said to have the **property D(1,s)** if G has a C_4 and s vertex disjoint paths of order 3 each, such that the vertex set of the C_4 is disjoint from the vertices of the s paths of

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order 3.

Let F be a graph. A graph G is said to be **F-free**, if it does not contain F as an induced subgraph. A graph is said to be **triangle-free** if it is K_3-free. The generalized Ramsey number R(K(1,m),K(1,n)) is the least positive integer p such that for every graph G of order p either G contains K(1,m) as a subgraph or $\overline{G}$ contains K(1,n) as a subgraph. We extend this definition to the class of triangle-free graphs. For positive integers m and n, we define R′(K(1,m),K(1,n)) as the least positive integer p such that if G is a triangle-free graph of order p either G contains K(1,m) as a subgraph or $\overline{G}$ contains K(1,n) as a subgraph. It is easy to see that

$$R'(K(1,m),K(1,n)) \leq R(K(1,m),K(1,n)) = R(K(1,n),K(1,m)).$$

A subset U of V(G) is said to be **k-independent** if the maximum degree of G[U] is at most k and U is said to be **maximal k-independent** if U is k-independent and $U \cup \{x\}$ is not a k-independent set for any $x \in V(G) - U$. The size of a largest k-independent set of G is called the **k-independence number** of G and is denoted by $\alpha_k(G)$.

A graph is **(m,k)-colourable** if its vertices can be coloured with m colours such that the subgraph induced on vertices receiving the same colour is k-independent. Note that any (m,k)-colouring of a graph G partitions the vertex set of G into m subsets $V_1, V_2, \ldots, V_m$ such that every V_i is k-independent. These sets V_i are sometimes referred to as the **colour classes**. The **k-defective chromatic number $\chi_k(G)$** of G is the smallest positive integer m for which G is (m,k)-colourable. Note that $\chi_0(G)$ is the usual chromatic number. Clearly $\chi_k(G) \leq \left\lceil \frac{p}{k+1} \right\rceil$, where p is the order of G.

These concepts have been studied by several authors. Hopkins and Staton [11] refer to a k-independent set as a k-small set. Maddox [15,16] and Andrews and Jacobson [3] refer to the same as a k-dependent set. The k-defective chromatic number has been investigated by Frick [7]; Frick and Henning [8]; Maddox [15,16]; Hopkins and Staton [11] under the name k-partition number; Andrews and Jacobson [3] under the name k-chromatic number.

The **Nordhaus-Gaddum (N-G) problem** [18] associated with the parameter χ_k is to find sharp bounds for $\chi_k(G) + \chi_k(\overline{G})$ and $\chi_k(G) \,.\, \chi_k(\overline{G})$ as G ranges over the class of all graphs of order p. Maddox [15,16] investigated the N-G problem for χ_k and proved that if either G or $\overline{G}$ is triangle-free, then $\chi_k(G) + \chi_k(\overline{G}) \leq 5\left\lceil \frac{p}{3k+4} \right\rceil$ where p is the order of G. When k = 1 he improved the above bound to $6\left\lceil \frac{p}{9} \right\rceil$. Achuthan et al. [2] proved that $\chi_1(G) + \chi_1(\overline{G}) \leq \frac{2p+4}{3}$ for any graph G of order p. The k-defective chromatic number of a graph is related to the point partition number $\rho_k(G)$ defined by Lick and White [13]. It is well known that $\chi_k(G) \geq \rho_k(G)$. Lick and White [13] established that

$$\rho_k(G) + \rho_k(\overline{G}) \leq \frac{p-1}{k+1} + 2$$

for a graph G of order p. Maddox [15] suggested the following conjecture for $k \geq 1$:

For a graph G of order p,

$$\chi_k(G) + \chi_k(\overline{G}) \leq \left\lceil \frac{p-1}{k+1} \right\rceil + 2.$$

In [1] we disproved Maddox's conjecture for all $k \geq 1$ by constructing a graph G of order $p \equiv 1(\text{mod } (k+1))$ with $\chi_k(G) + \chi_k(\overline{G}) = \left\lceil \frac{p-1}{k+1} \right\rceil + 3$. These graphs have P_4 as an induced subgraph and hence Maddox's conjecture can be restated when G ranges over the subclass of P_4-free graphs of order p. This restated conjecture is proved for the subclass of P_4-free graphs in [1,19] for k = 1,2. Further, Achuthan et al. [1] established the following weak upper bound :

For a graph G of order p,

$$\chi_k(G) + \chi_k(\overline{G}) \leq \frac{2p+2k+4}{k+2}$$

Furthermore, they established the following sharp lower bound for the product :

For any graph G of order p,

$$\chi_k(G).\chi_k(\overline{G}) \geq \left\lceil \frac{p}{R-1} \right\rceil$$

where R = R(K(1,k+1),K(1,k+1)). In the same paper they settled the associated realizability problem when k = 1 and G ranges over the subclass of P_4-free graphs.

In this paper we will solve the N-G problem for the 1-defective chromatic number over the class of triangle-free graphs. In Section 2 we state some results concerning the 1-defective chromatic number that will be used repeatedly. In Section 3, we prove that if G or $\overline{G}$ is a triangle-free graph of order p ≥ 3 then $\chi_1(G) + \chi_1(\overline{G}) \leq \left\lceil \frac{p-1}{2} \right\rceil + 2$ and that this bound is sharp. This proves Maddox's conjecture for k = 1 over the subclass of triangle-free graphs of order p. Furthermore, we establish a sharp lower bound for $\chi_k(G) \cdot \chi_k(\overline{G})$ as G ranges over the class of triangle-free graphs of order p.

To prove our results we need to investigate the problem of determining the smallest order of a triangle-free graph with respect to the parameter $\chi_k(G)$. Let f(m,k) be the smallest order of a triangle-free graph G such that $\chi_k(G) = m$. The determination of f(m,0) is still an open problem (see Toft [21], Problem 29). However partial results concerning this problem have been obtained by several authors (see Mycielski [17], Chvátal [6], Avis [4], Hanson and MacGillivray [10], Grinstead, Katinsky and Van Stone [9], Jensen and Royle [12]).

For notational convenience the path $u_1,u_2,\ldots,u_n$ and the cycle $u_1,u_2,\ldots,u_n,u_1$ will be denoted by $u_1u_2\ldots u_n$ and $u_1u_2\ldots u_nu_1$ respectively. In all the figures a dotted line between a vertex u and a set A means that all the edges between u and A belong to the complement.

2. Some results concerning the 1-defective chromatic number

The following theorem has been obtained independently by Lovász[14] and Hopkins and Staton [11].

Theorem 1: Let G be a graph with maximum degree Δ. Then

$$\chi_k(G) \leq \left\lceil \frac{\Delta+1}{k+1} \right\rceil. \qquad \square$$

The following theorems have been established by Simanihuruk et al. [20].

Theorem 2 : The smallest order of a triangle-free graph G such that $\chi_1(G) = 3$ is 9, that is, $f(3,1) = 9$. □

Theorem 3 : Let G be a triangle-free graph of order 9. Then $\chi_1(G) = 3$ if and only if G is one of the graphs shown in Figure 1. □

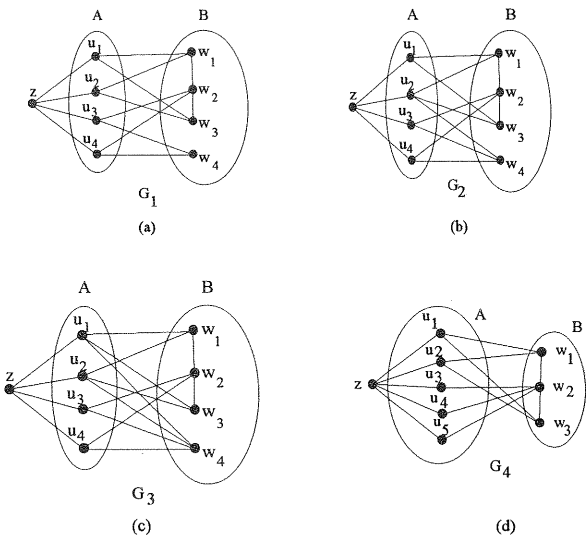


Figure 1

Theorem 4 : For $m \geq 4$, the smallest order of a triangle-free graph G with $\chi_1(G) = m$ is at least $m^2 + 1$, that is, $f(m,1) \geq m^2 + 1$. □

3. Defective colourings of triangle-free graphs and the N-G problem :

In this section we establish Maddox's [15] conjecture that

$$\chi_1(G) + \chi_1(\overline{G}) \leq \left\lceil \frac{p-1}{2} \right\rceil + 2$$

when G ranges over the class of triangle-free graphs of order p. The proof is very technical and makes use of the consequences of the properties tP_3 and $D(1,t-1)$ in a triangle-free graph. These consequences are established in a series of lemmas. The assumptions made in the Lemmas 2 to 4 are closely related. We prove Maddox's conjecture for triangle-free graphs in Theorem 5. Furthermore, we establish a sharp lower bound for the product of $\chi_k(G)$ and $\chi_k(\overline{G})$ when G ranges over the class of triangle-free graphs.

Lemma 1: Let G be a triangle-free graph of order $p \geq 7$. If $\alpha_1(G) \geq p - 3$ then $\chi_1(G) \leq 2$.

Proof : Firstly if $\alpha_1(G) \geq p - 2$ then clearly $\chi_1(G) \leq 2$. Now assume that $\alpha_1(G) = p - 3$. Let U be a 1-independent set of cardinality p-3. If G-U has no P_3 then $\chi_1(G) \leq 2$. Therefore we assume that G-U contains a P_3. Since G is triangle-free it follows that G - U is isomorphic to P_3. Let xyz be the P_3 in G-U. We define sets A, B, and C as follows: $A = N_U(x) \cup N_U(z)$, $B = N_U(y)$ and $C = U-(A \cup B)$. Now assign colour 1 to the elements of $\{x,z\} \cup B \cup C$ and colour 2 to those of $\{y\} \cup A$. Therefore $\chi_1(G) \leq 2$. Hence the lemma. □

Lemma 2 : Let G be a triangle-free graph of order p with property tP_3 and without property D(1,t-1). Let $Q_1,Q_2,\ldots,Q_t$ be a collection of vertex disjoint paths of order 3 each. Let $V(Q_i) = \{u_i,v_i,w_i\}$ where v_i is the middle vertex of Q_i, $1 \leq i \leq t$; $M = \bigcup_{i=1}^{t} V(Q_i)$ and $F = V(G) - M$. The following hold :

(i) If an end vertex of Q_i is adjacent to a vertex of degree one in G[F] then v_i has no neighbours in F.

(ii) The vertices u_i and w_i do not have a common neighbour in F.

Proof : Since G has property tP_3, it follows that F is 1-independent. Now (i) and (ii) follow from the assumptions that G is triangle-free, G has property tP_3 and does not have property D(1,t-1). □

Lemma 3 : Let G be a triangle-free graph of order p satisfying the hypothesis of Lemma 2. In addition, the paths $Q_1,Q_2,\ldots,Q_t$ are chosen such that the number of edges in G[F] is as large as possible. Let $A = \{x : x \in F$ and the degree of x in G[F] is $1\}$ and $B = F - A$. The following hold :

(i) The end vertices u_i and w_i of Q_i have at most one neighbour each in B, for all i, $1 \leq i \leq t$;

(ii) If $\alpha_1(G) \leq p - 3t + 1$, then $t \leq 1$.

Proof : Firstly note that F is 1-independent and thus B is 0-independent. We will present the proof of (i) for i = 1. The proof is identical for $i \geq 2$.

Recall that the paths $Q_1,Q_2,\ldots,Q_t$ have been chosen such that the number of edges in G - M = G[F] is as large as possible. Assume that u_1 has at least two neighbours, say x and y in B. Clearly v_1 and w_1 are not adjacent to any element of F, for otherwise G would have t+1 vertex disjoint P_3's, a contradiction to the assumption that G has property tP_3. Now xu_1y, $Q_2,\ldots,Q_t$ form a set of t vertex disjoint paths of order 3. Thus for $F' = (F - \{x,y\}) \cup \{v_1,w_1\}$, note that $|E(G[F'])| > |E(G[F])|$, a contradiction to the choice of the t paths $Q_1,Q_2,\ldots,Q_t$. Thus u_1 has at most one neighbour in B. Similarly it can be shown that w_1 has at most one neighbour in B. This proves (i).

To prove (ii), let $\alpha_1(G) \leq p - 3t + 1$. Now suppose that u_1 and w_1 are not adjacent to any vertex of A. Then it follows from (ii) of Lemma 2 and part (i) above, that $F \cup \{u_1,w_1\}$ is a 1-independent set of cardinality p - 3t + 2, a contradiction. Thus u_1 or w_1 is adjacent to a vertex of A and hence, by (i) of Lemma 2, v_1 is not adjacent to any vertex of F. If $t \geq 2$, a similar argument will prove that v_2 is not adjacent to any vertex of F. But then $F \cup \{v_1,v_2\}$ is a 1-independent set of cardinality p - 3t + 2. This contradiction proves (ii). □

Lemma 4 : Let G be a triangle-free graph of order p satisfying the hypothesis of Lemma 3. Furthermore, suppose that every subgraph of order at most 9, of G is (2,1)-colourable. Also let t = 2 or 4 and $\chi_1(G) = 3$ or 4 according as t = 2 or 4. Then

(i) $\alpha_1(G) = p - 3t+2$,

(ii) for $1 \leq i \leq t$, either u_i or w_i has no neighbours in B,

(iii) there is an i, $1 \leq i \leq t$, such that the end vertices u_i and w_i of Q_i have no neighbours in A and, for every $j \neq i$, every vertex of Q_j is adjacent to atmost one vertex of Q_i,

(iv) for every i, $1 \leq i \leq t$, the vertices u_i and w_i have no neighbours in A.

Proof : It follows from (ii) of Lemma 3 that $\alpha_1(G) \geq p - 3t + 2$.

If possible let $\alpha_1(G) \geq p - 3t+3$ and S be a 1-independent set of G with $|S| = \alpha_1(G)$. If t = 2 then $\alpha_1(G) \geq p - 3$. By Lemma 1 it follows that $\chi_1(G) \leq 2$, a contradiction to our assumption. On the other hand, if t = 4 then G - S is a graph of order at most 9. By our assumption $\chi_1(G\text{-}S) \leq 2$. Thus $\chi_1(G) \leq \chi_1(G\text{-}S) + \chi_1(G[S]) \leq 3$. Again this is a contradiction to our assumption that $\chi_1(G) = 4$ if t = 4. Thus it follows that $\alpha_1(G) = p - 3t + 2$ and proves (i).

To prove (ii) we suppose that for some i, $1 \leq i \leq t$, both the vertices u_i and w_i have a neighbour in B. Let x be the neighbour of u_i and y be the neighbour of w_i. Clearly $x \neq y$. Now we can easily construct paths $Q_1', Q_2', \ldots, Q_t'$ such that $\left| E(G - \bigcup_{i=1}^{t} V(Q_i')) \right| > \left| E(G[F]) \right|$ a contradiction to the choice of $Q_1, Q_2, \ldots, Q_t$. This proves (ii).

To prove the first part of (iii) assume that for each i, $1 \leq i \leq t$, an end vertex of Q_i has a neighbour in A. Without any loss of generality assume that u_i has a neighbour, say a_i, in A, for each i, $1 \leq i \leq t$. Note that a_i may be equal to a_j for some $i \neq j$. Let b_i be the neighbour of a_i in A. If for some i, w_i has a neighbour in F - $\{b_i\}$ then G would have t + 1 vertex disjoint P_3's, a contradiction to the maximality of t. Thus it follows that w_i has no neighbours in F - $\{b_i\}$, for $1 \leq i \leq t$.

Now we prove the following claim.

Claim : For each i, $1 \le i \le t$, w_i is adjacent to b_i.

Suppose not. Without any loss of generality we assume that w_1 is not adjacent to b_1. Now from (i) of Lemma 2 it follows that $F \cup \{v_1, w_1\}$ is 1-independent. From part (i) of this lemma, it follows that $F \cup \{v_1, w_1\}$ is a maximal 1-independent set. Let $I = F \cup \{v_1, w_1\}$ (see Figure 2.a). Consider the centre vertex v_2 of Q_2. Since v_2 is not adjacent to any vertex of F and I is maximal 1-independent it follows that v_2 is adjacent to one of v_1 and w_1. Since G is triangle- free it follows that v_2 is adjacent to exactly one of v_1 and w_1.

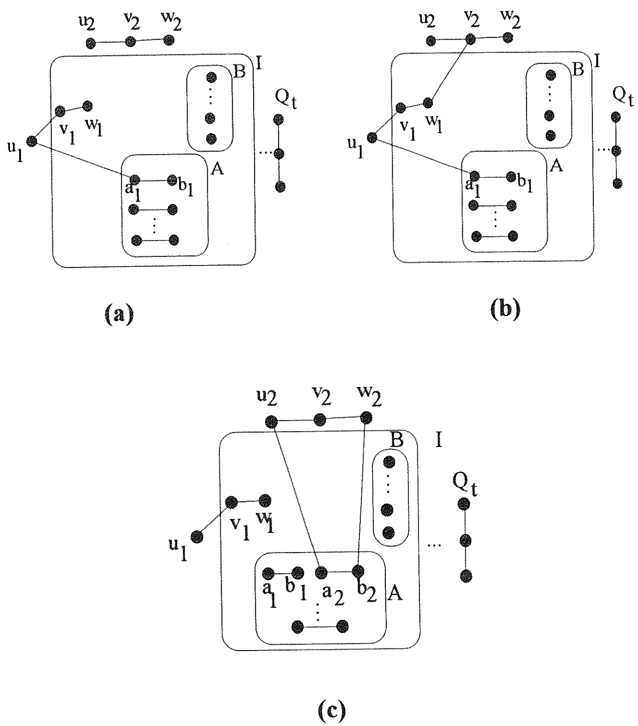


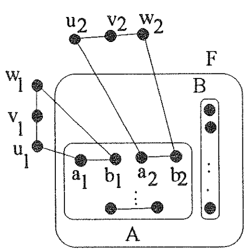
Figure 2

Firstly let v_2 be adjacent to w_1 (see Figure 2.b). Since G is triangle-free and does not possess the property D(1,t-1) it follows that the vertex u_2 is not adjacent to either of w_1 and v_1. For the same reason we can conclude that w_2 is not adjacent to either of v_1 and w_1. Recall that a_2 is the neighbour of u_2 in A. Now if w_2 is not adjacent to b_2 then $I \cup \{w_2\}$ would form a 1-independent set contradicting the maximality of I. Therefore w_2 is adjacent to b_2. Note that the edge (a_2,b_2) may be the same as the edge (a_1,b_1) (see Figure 2.c). Now consider the set $I' = I \cup \{u_2,w_2\} - \{a_2\}$ of size p - 3t + 3. It is easy to see that I' is 1-independent contradicting the fact that $\alpha_1(G) = p - 3t + 2$. Hence the claim is proved in case v_2 is adjacent to w_1. A similar contradiction can be arrived at, if we assume that v_2 is adjacent to v_1. This proves the claim.

To complete the proof of the first part of (iii) we will consider the cases t = 4 and t = 2 separately and arrive at a contradiction in each case.

Firstly let t = 4. Consider the set $F \cup \{v_1,v_2,v_3,v_4\}$. Recall that v_i is the central vertex of the path Q_i and that v_i has no neighbours in F, for $1 \leq i \leq 4$. Now $|F| = p - 12$ and $\alpha_1(G) = p - 10$. Hence for every subset S of size 3 of $\{v_1,v_2,v_3,v_4\}$, G[S] contains a P_3. It can easily be seen that $G[\{v_1,v_2,v_3,v_4\}]$ is isomorphic to a cycle, say $C = v_1v_2v_3v_4v_1$. Now if the edge $(a_1,b_1) \neq$ the edge (a_2,b_2) then G has 5 vertex disjoint P_3's namely $u_1a_1b_1$, $w_1v_1v_2$, $u_2a_2b_2$, Q_3, Q_4, contradicting the property tP_3 i.e. $4P_3$. Thus $(a_1,b_1) = (a_2,b_2)$. Similarly it can be shown that $(a_2,b_2) = (a_3,b_3) = (a_4,b_4)$. Let $W = \bigcup_{i=1}^{4} V(Q_i) \cup \{a_1,b_1\}$. Note that $|W| = 14$. From Theorem 4, it follows that $\chi_1(G[W]) \leq 3$. It is easy to show that there are no edges between W and $F - \{a_1,b_1\}$. Thus $\chi_1(G) = \chi_1(G[W])$ and hence $\chi_1(G) \leq 3$, a contradiction to our assumption that $\chi_1(G) = 4$.

Next let t = 2. We will first assume that $(a_1,b_1) \neq (a_2,b_2)$ (see Figure 3.a).



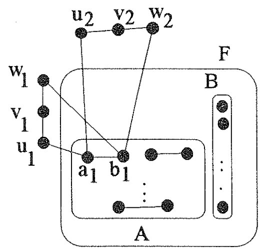


Figure 3

Consider the cycle $C = u_1v_1w_1b_1a_1u_1$. If there is an edge between V(C) and V(G) - V(C), then there are three vertex disjoint P_3's, a contradiction to the property tP_3 with $t = 2$. Thus there are no edges between V(C) and V(G) - V(C). Similarly there are no edges between the vertices of the cycle $u_2v_2w_2b_2a_2u_2$ and the rest of the vertices in G. Thus it follows that every connected component of G is either a C_5 or a K_2 or a K_1. Thus $\chi_1(G) = 2$, a contradiction to our assumption that $\chi_1(G) = 3$.

Now assume that $(a_1,b_1) = (a_2,b_2)$ (see Figure 3.b). Since G is triangle-free, (w_1,w_2), (u_1,u_2), (u_1,w_1), and (u_2,w_2) are not edges of G. Clearly there are no edges between $\{u_1,w_1,u_2,w_2\}$ and $F - \{a_1,b_1\}$. Thus $\{u_1,w_1,u_2,w_2\} \cup F - \{a_1,b_1\}$ is a 1-independent set. Now we assign colour 1 to $\{u_1,w_1,u_2,w_2\} \cup F - \{a_1,b_1\}$ and colour 2 to $\{v_1,v_2,a_1,b_1\}$. Thus $\chi_1(G) \leq 2$, a contradiction. This completes the proof of the first part of (iii).

To prove the second part of (iii), we assume without any loss of generality that u_1 and w_1 of Q_1 have no neighbours in A. Now using part (ii) of Lemma 2 it follows that $F \cup \{u_1,w_1\}$ is 1-independent. Define $J = F \cup \{u_1,w_1\}$. Since $\alpha_1(G) = p - 3t + 2$ it follows that J is a maximal 1-independent set. Note that by (ii), either u_1 or w_1 has no neighbours in B. Without loss of generality we assume that w_1 has no neighbours in B.

Also note that by (i) of Lemma 3, u_1 has at most one neighbour in B. We now consider the two cases separately to establish the second part of (iii).

Case 1 : u_1 has no neighbours in B.

Note that $B \cup \{u_1,w_1\}$ is 0-independent. Since $J = F \cup \{u_1,w_1\}$ is a maximal 1-independent set, it follows that for any vertex z of Q_i, for $i \geq 2$, either z has a neighbour in A or has two neighbours in $B \cup \{u_1,w_1\}$. Suppose z is a vertex of Q_i, $i \geq 2$, such that z is adjacent to u_1 and w_1 (see Figure 4).

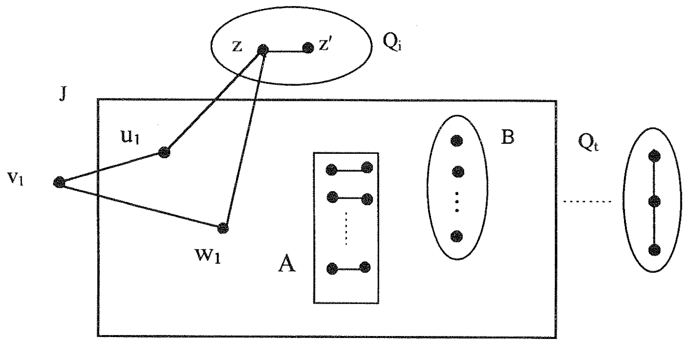


Figure 4

Let z′ be a neighbour of z in Q_i. Since G is triangle-free, z′ is not adjacent to u_1 or w_1. From the maximality of J, either z′ is adjacent to a vertex of A or is adjacent to at least two vertices of B. In either case G has the property D(1,t-1), a contradiction to our assumption. This proves (iii) in Case 1.

Case 2 : u_1 has a neighbour, say x, in B.

Note that $\{w_1\} \cup B - \{x\}$ is 0-independent. Again since $J = F \cup \{u_1,w_1\}$ is a maximal 1-independent set, it follows that for any vertex z of Q_i, $i \geq 2$, either z has a neighbour in $A \cup \{u_1,x\}$ or it has two neighbours in $\{w_1\} \cup B - \{x\}$.

Suppose z is a vertex of Q_i, $i \geq 2$, such that z is adjacent to u_1 and w_1 (see Figure 5). Let z′ be a neighbour of z in Q_i.

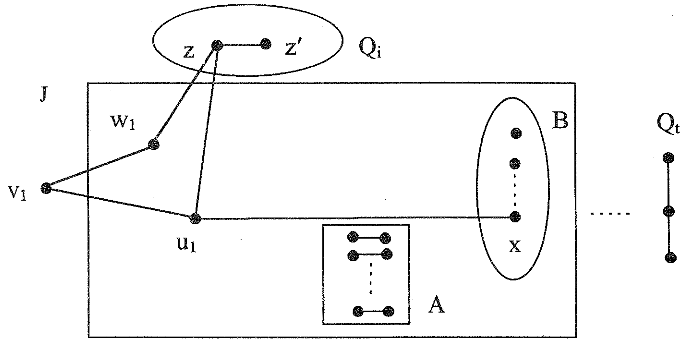


Figure 5

Firstly note that z' is not adjacent to u_1 or w_1. From the maximality of J we have one of the following :

(a) z' is adjacent to a vertex of A;

(b) z' is adjacent to at least two vertices of B - {x};

(c) z' is adjacent to x.

If (a) or (b) is true then G has the property D(1,t-1), a contradiction to our assumption. Thus z' is adjacent to x.

Now suppose that z is an end vertex of Q_i, say $z = u_i$. Then $z' = v_i$. The cycle $u_iw_1v_1u_1u_i$, the paths x v_iw_i and Q_α, $\alpha \neq 1$ and i imply that G has the property D(1,t-1), a contradiction to our assumption. Thus it follows that z is the centre vertex v_i of Q_i (see Figure 6).

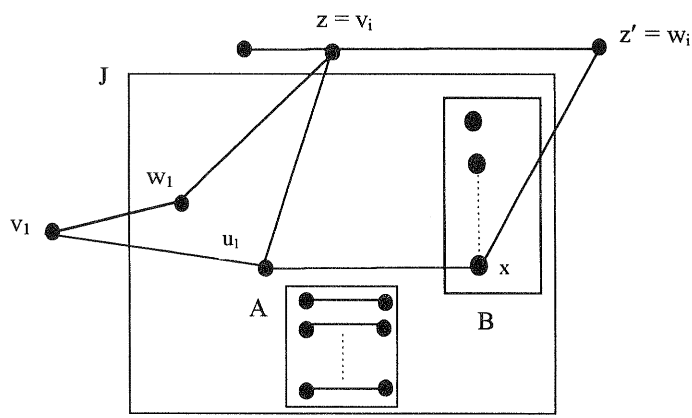


Figure 6

Now consider the vertex u_i. By part (ii), u_i has no neighbours in B. Since G is triangle-free u_i is not adjacent to either u_1 or w_1. The maximality of J implies that u_i is adjacent to a vertex of A. Again it can be shown that G has the property D(1,t-1), a contradiction. This completes the proof of (iii) in Case 2.

We now present the proof of (iv). Since J is 1-independent, clearly (iv) is true for $i = 1$. We will prove (iv) for $i = 2$. The proof for $i \geq 3$ is identical.

Suppose u_2 is adjacent to a vertex a_1 in A. Let b_1 be the neighbour of a_1 in A (see Figure 7.a). From (i) of Lemma 2, it follows that v_2 is not adjacent to any vertex of $A \cup B$. Since $J = \{u_1,w_1\} \cup F$ is maximal 1-independent, the vertex v_2 has to be adjacent to at least one of the vertices u_1 and w_1. Combining this with (iii) of this lemma, we conclude that the vertex v_2 is adjacent to exactly one vertex of $\{u_1,w_1\}$. Without any loss of generality assume that v_2 is adjacent to u_1 (see Figure 7.b). Now consider the set $J \cup \{v_2\}$. It has $p - 3t + 3$ vertices. Since $\alpha_1(G) = p - 3t + 2$ and v_2 is not adjacent to any vertex of F (by Lemma 2) it follows that u_1 is adjacent to a vertex, say d, of B (see Figure 7.c).

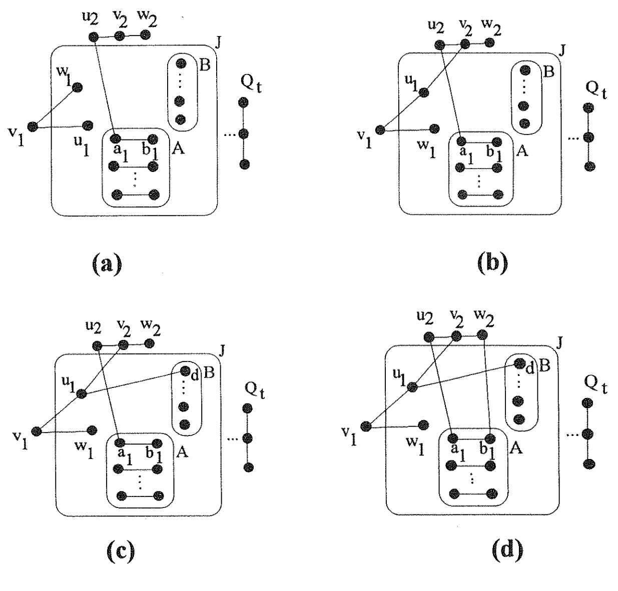


Figure 7

Now if $(w_2,w_1) \in E(G)$ then G has (t+1) P_3's namely $w_2w_1v_1$, v_2u_1d, $u_2a_1b_1$ and the t-2 paths $Q_3,\ldots,Q_t$. This is a contradiction to the assumption that G has the property tP_3. Thus w_2 is not adjacent to w_1. Also since G is triangle-free, w_2 is not adjacent to u_1. Using (ii) of Lemma 2 and the fact that t is the largest number of vertex disjoint 3-paths in G, we conclude that w_2 has no neighbours in $F - \{b_1\}$. Now if w_2 is not adjacent to b_1, then $J \cup \{w_2\}$ forms a 1-independent set contradicting part (i). Thus it follows that $(w_2,b_1) \in E(G)$ (see Figure 7.d). Consider the vertex u_2 in Figure 7.d. Clearly u_2 is not adjacent to w_1, otherwise G has (t+1) P_3's. Now $J \cup \{u_2,w_2\} - \{a_1\}$ is a 1-independent set of cardinality p - 3t + 3, a contradiction to the fact that $\alpha_1(G) = p - 3t + 2$. This proves that u_2 does not have any neighbours in A. Similarly it can be shown that w_2 has no neighbours in A.

This completes the proof of (iv) and hence Lemma 4. □

Theorem 5 : Let G be a triangle-free graph of order p. Then

$$\chi_1(G) + \chi_1(\overline{G}) \leq \left\lceil \frac{p-1}{2} \right\rceil + 2.$$

Moreover this bound is sharp for $p \geq 3$.

Proof : Firstly let $\chi_1(G) \leq 2$. If $\chi_1(G) = 1$ then $\chi_1(\overline{G}) = \left\lceil \frac{p}{2} \right\rceil$. Hence $\chi_1(G) + \chi_1(\overline{G}) \leq \left\lceil \frac{p-1}{2} \right\rceil + 2$. If $\chi_1(G) = 2$ then G has a path P of order 3. The vertices of the path P form a 1-independent set in $\overline{G}$ and consequently $\chi_1(\overline{G}) \leq \left\lceil \frac{p-3}{2} \right\rceil + 1 = \left\lceil \frac{p-1}{2} \right\rceil$. Hence the required inequality.

Henceforth we assume that $\chi_1(G) \geq 3$. From Theorems 2 and 4, it follows that $p \geq 9$. We prove Theorem 5 by induction on p. Let G be a triangle-free graph of order 9. From Theorem 4 we have $\chi_1(G) \leq 3$. Thus $\chi_1(G) = 3$. Now by Theorem 3, G is isomorphic to one of the graphs G_i, $1 \leq i \leq 4$, in Figure 1. It is easy to see that

$u_1w_3u_2w_1u_1$ and $u_3w_4u_4w_2u_3$ form vertex disjoint C_4's in G_i, for $1 \le i \le 3$. Similarly the two 4-cycles $u_1w_3u_2w_1u_1$ and $u_3w_2u_4zu_3$ are vertex disjoint in G_4. Now the vertex sets of these C_4's are 1-independent in the graph $\overline{G}$. Thus $\chi_1(\overline{G}) \le 3$. Hence $\chi_1(G) + \chi_1(\overline{G}) \le 6$. This establishes the basis for induction.

Now let $p \ge 10$. We make the induction hypothesis that the theorem is true for any triangle-free graph of order less than p and then prove it for any triangle-free graph of order p.

Case 1 : There is a subset L of cardinality 9 of V(G) such that $\chi_1(G[L]) \ge 3$.

From Theorem 4 we have $\chi_1(G[L]) \le 3$. Thus $\chi_1(G[L]) = 3$. By Theorem 3, G[L] is isomorphic to one of the graphs shown in Figure 1. As mentioned before, each of these graphs has two vertex disjoint C_4's. Recall that the vertex set of a C_4 in G is a 1-independent set in $\overline{G}$. Now if X is the vertex set of the union of the two C_4's in G[L] then $\chi_1(\overline{G}[X]) \le 2$. From Theorem 2 we have $\chi_1(G[X]) \le 2$ since $|V(G[X])| = 8$. Now using these inequalities and the induction hypothesis we have

$$\chi_1(G) + \chi_1(\overline{G}) \le \chi_1(G\text{-}X) + \chi_1(\overline{G}\text{-}X) + \chi_1(G[X]) + \chi_1(\overline{G}[X])$$

$$\le \left\lceil \frac{p-9}{2} \right\rceil + 2 + 2 + 2 = \left\lceil \frac{p-1}{2} \right\rceil + 2.$$

This proves the theorem in this case.

Case 2 : For every subset L of cardinality 9 of V(G), $\chi_1(G[L]) \le 2$

Since $\chi_1(G) \ge 3$, G contains a P_3. Let t be the largest number of vertex disjoint paths of order 3 in G, i.e. G has the property tP_3. Let $Q_1, Q_2, \ldots, Q_t$ be t vertex disjoint paths of order 3 in G. Let $M = \bigcup_{i=1}^{t} V(Q_i)$ and $V(Q_i) = \{u_i, v_i, w_i\}$ such that u_i and w_i are the end vertices of Q_i for $1 \le i \le t$.

Note that V(G) - M is 1-independent in G. Without any loss of generality we can assume that the paths $Q_1, Q_2, \ldots, Q_t$ have been chosen such that the number of edges in G-M is as large as possible. This means that if $R_1, R_2, \ldots, R_t$ are vertex disjoint paths of order 3 in G and $Y = \bigcup_{i=1}^{t} V(R_i)$ then $|E(G\text{-}M)| \geq |E(G\text{-}Y)|$. Note that the subgraph $\overline{G}[V(Q_i)]$ is P_3-free for each i. Thus

$$\chi_1(\overline{G}[M]) \leq t. \tag{1}$$

Since $\overline{G}$ - M is a graph of order p - 3t, we have $\chi_1(\overline{G} - M) \leq \left\lceil \frac{p-3t}{2} \right\rceil$. Combining this with (1) we have

$$\chi_1(\overline{G}) \leq \chi_1(\overline{G}[M]) + \chi_1(\overline{G} - M) \leq t + \left\lceil \frac{p-3t}{2} \right\rceil = \left\lceil \frac{p-t}{2} \right\rceil. \tag{2}$$

Also

$$\chi_1(G) \leq \chi_1(G[M]) + \chi_1(G - M) = \chi_1(G[M]) + 1, \tag{3}$$

since V(G) - M is 1-independent in G.

First let $t \geq 8$ and let $N = \bigcup_{i=1}^{8} V(Q_i)$. Note that $|N| = 24$. By Theorem 4, $f(5,1) \geq 26$ and thus we have $\chi_1(G[N]) \leq 4$. Since $V(Q_i)$ is a 1-independent set in $\overline{G}$ for each i, $1 \leq i \leq t$, it follows that $\chi_1(\overline{G}[N]) \leq 8$. Now $\chi_1(G) \leq \chi_1(G[N]) + \chi_1(G\text{-}N)$ and $\chi_1(\overline{G}) \leq \chi_1(\overline{G}[N]) + \chi_1(\overline{G}\text{-}N)$. Thus

$$\chi_1(G) + \chi_1(\overline{G}) \leq \chi_1(G[N]) + \chi_1(\overline{G}[N]) + \chi_1(G\text{-}N) + \chi_1(\overline{G}\text{-}N)$$
$$\leq 12 + \chi_1(G\text{-}N) + \chi_1(\overline{G}\text{-}N).$$

By the induction hypothesis

$$\chi_1(G\text{-}N) + \chi_1(\overline{G}\text{-}N) \leq \left\lceil \frac{p-25}{2} \right\rceil + 2.$$

Therefore,

$$\chi_1(G) + \chi_1(\overline{G}) \leq \left\lceil \frac{p-1}{2} \right\rceil + 2.$$

Thus the theorem is proved when $t \geq 8$.

Henceforth let us assume that $t \leq 7$. From (2) and (3) we have

$$\chi_1(G) + \chi_1(\overline{G}) \leq \chi_1(G[M]) + 1 + \left\lceil \frac{p-t}{2} \right\rceil. \quad (4)$$

Note that $t \geq 2$, for otherwise, $\alpha_1(G) \geq p - 3$ and thus by Lemma 1, we have $\chi_1(G) \leq 2$, contradicting our assumption that $\chi_1(G) \geq 3$.

Subcase 2.1 : t is odd, $2 \leq t \leq 7$.

Firstly let $t = 3$. Since every subgraph of order 9 can be coloured with 2 colours and $|M| = 9$, it follows that $\chi_1(G[M]) = 2$. Incorporating this in (4) we have

$$\chi_1(G) + \chi_1(\overline{G}) \leq 2 + 1 + \left\lceil \frac{p-3}{2} \right\rceil = \left\lceil \frac{p-1}{2} \right\rceil + 2.$$

Hence the theorem is proved in this case when $t = 3$.

Finally let $t = 5$ or 7. Accordingly G[M] has order 15 or 21. From Theorems 2 and 4 we have $\chi_1(G[M]) \leq 3$ or 4 according as $t = 5$ or 7. Incorporating this in (4) we have the required inequality. This proves the theorem in Subcase 2.1.

Subcase 2.2: t is even, $2 \leq t \leq 6$.

We will first show that if G has the property D(1,t-1) then $\chi_1(G) + \chi_1(\overline{G}) \leq 2 + \left\lceil \frac{p-1}{2} \right\rceil$. Assume that G has the property D(1,t-1). Let $R_1, R_2, \ldots, R_{t-1}$ be t-1 vertex disjoint paths of order 3 and C a cycle of order 4 which is vertex disjoint from each R_i. Let $Z = \left\{ \bigcup_{i=1}^{t-1} V(R_i) \right\} \cup V(C)$. Clearly $|Z| = 3t + 1$. It is easy to see that $\chi_1(\overline{G}[Z]) \leq t$ and $\chi_1(\overline{G}\text{-}Z) \leq \left\lceil \frac{p-3t-1}{2} \right\rceil$. Therefore

$$\chi_1(\overline{G}) \leq t + \left\lceil \frac{p-3t-1}{2} \right\rceil = \left\lceil \frac{p-t-1}{2} \right\rceil. \quad (5)$$

We will now prove that $\chi_1(G) \leq 3$ or 4 or 5 according as $t = 2$ or 4 or 6. Note that G[Z] has order 7 or 13 or 19 according as $t = 2$ or 4 or 6. From Theorems 2 and 4 we have $\chi_1(G[Z]) \leq 2$ or 3 or 4 according as $t = 2$ or 4 or 6. From the maximality of t, it follows that V(G) - Z is 1-independent in G and hence $\chi_1(G - Z) = 1$. Thus $\chi_1(G) \leq \chi_1(G[Z]) + \chi_1(G\text{-}Z) \leq 3$ or 4 or 5 according as $t = 2$ or 4 or 6. Now combining this inequality with (5) we have the required inequality. This proves the theorem when G has the property D(1,t-1).

From now onwards we will assume that G does not possess the property D(1,t-1). We will first introduce the following notation. Let F = V(G) - M. Clearly F is 1-independent in G. Now Let $A = \{x: x \in F$ and the degree of x in G[F] is $1\}$ and B = F - A. Clearly B is 0-independent in G. Recall that u_i and w_i are the end vertices and v_i is the centre vertex of the path Q_i, $1 \leq i \leq t$. We divide the rest of the proof into two more subcases based on the value of t.

Subcase 2.2.1 : t = 6.

From (ii) of Lemma 3, it follows that $\alpha_1(G) \geq p-3t+2 = p-16$. Thus there is a 1-independent set R of size at least p - 16. Since $f(4,1) \geq 17$, the subgraph G-R is (3,1)-colourable. Hence $\chi_1(G) \leq 4$. Combining this with inequality (2) we have $\chi_1(G) + \chi_1(\overline{G}) \leq 4 + \left\lceil \frac{p-6}{2} \right\rceil \leq \left\lceil \frac{p-1}{2} \right\rceil + 2$, which proves the theorem in this subcase.

Subcase 2.2.2 : t = 2 or 4.

Recall that $\chi_1(G) \geq 3$ and G does not possess the property D(1,t-1). Note that G[M] is a graph of order 6 or 12 according as t = 2 or 4. Thus from Theorems 2 and 4 it follows that $\chi_1(G[M]) \leq 2$ or 3 according as t = 2 or 4. Incorporating this in inequality (3) we have $\chi_1(G) \leq 3$ or 4 according as t = 2 or 4. Thus we have

$$\chi_1(G) = \begin{cases} 3, & \text{if } t = 2, \\ 3 \text{ or } 4, & \text{if } t = 4. \end{cases}$$

Firstly let t = 4 and $\chi_1(G) = 3$. From (2), $\chi_1(\overline{G}) \leq \left\lceil \frac{p-4}{2} \right\rceil$. Thus

$$\chi_1(G) + \chi_1(\overline{G}) \leq 3 + \left\lceil \frac{p-4}{2} \right\rceil \leq 2 + \left\lceil \frac{p-1}{2} \right\rceil.$$

This proves the theorem when t = 4 and $\chi_1(G) = 3$. Henceforth we will assume that

$$\chi_1(G) = \begin{cases} 3, & \text{if } t = 2 \\ 4, & \text{if } t = 4. \end{cases} \qquad (6)$$

We will show that this will lead to a contradiction. By (ii) of Lemma 2 and (iii) of Lemma 4 we may assume that $F \cup \{u_1, w_1\}$ is a 1-independent set and that each vertex of Q_2 is adjacent to at most one vertex of Q_1. Define $J = F \cup \{u_1, w_1\}$. Note that J is maximal 1-independent by (i) of Lemma 4.

Now we arrive at the final contradiction. Using Lemma 4 we can assume without any loss of generality that, the vertices $w_1, w_2, \ldots, w_t$ do not have any neighbours in F (see Figure 8.a).

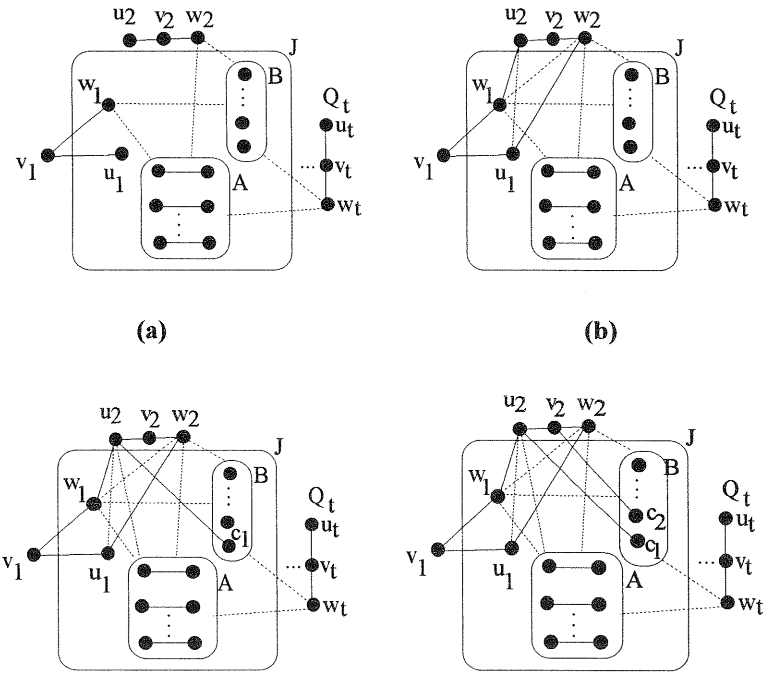


Figure 8

If w_1 is not adjacent to u_2 then $F \cup \{w_1, u_2, w_2\}$ is a 1-independent set of cardinality $p - 3t + 3$, a contradiction to the fact $\alpha_1(G) = p - 3t + 2$. Similarly if u_1 is not adjacent to w_2 then $J \cup \{w_2\}$ forms a 1-independent set of cardinality $p - 3t + 3$, a contradiction. Thus (u_2,w_1) and (u_1,w_2) are edges of G. This implies that (u_2,u_1) and (w_2,w_1) are not edges of G (see Figure 8.b). Consider the vertex u_2 of Figure 8.b. By Lemma 4, u_2 is not adjacent to any vertex of A. Since J is maximal it follows that u_2 is adjacent to a vertex ,say c_1, of B (see Figure 8.c).

Now consider the vertex v_2 in Figure 8.c. Since G is triangle-free, v_2 is adjacent to neither u_1 nor w_1. Since J is a maximal 1-independent set, v_2 is adjacent to at least one vertex of $A \cup B$. Now if v_2 has a neighbour in A, it is easy to show that G has the property $(t + 1)P_3$, a contradiction. Hence v_2 does not have neighbours in A and thus it has a neighbour in B. If v_2 has at least two neighbours in B, again we can show that G has the property $(t + 1)P_3$. Thus it follows that v_2 has exactly one neighbour, say c_2, in B. Since G is triangle-free, $c_2 \neq c_1$ (see Figure 8.d)). Clearly c_2 is adjacent to u_1, otherwise $J \cup \{v_2\}$ is a 1-independent set of cardinality $p- 3t + 3$, a contradiction to $\alpha_1(G) = p - 3t + 2$. Now the paths $Q_1' = v_1w_1u_2,Q_3,\ldots,Q_t$ and the cycle $C_4 = c_2u_1w_2v_2c_2$, imply that G has the property D(1,t-1), a contradiction. This forms the final contradiction for the Subcase 2.2.2.

Thus we have shown that $\chi_1(G) + \chi_1(\overline{G}) \leq \left\lceil \frac{p-1}{2} \right\rceil + 2$. The graph $G \cong K(1,p-1)$ shows that the above inequality is sharp for $p \geq 3$. This completes the proof of Theorem 5. □

We now determine the Ramsey number R'(K(1,k+1),K(1,k+1)), for every positive integer k. Consider a triangle-free graph G of order R(K(1,k+1),K(1,k+1)). By the definition of the generalized Ramsey number R(K(1,k+1),K(1,k+1)), it follows that either G or $\overline{G}$ contains K(1,k+1). Thus we have the inequality

$$R'(K(1,k+1),K(1,k+1)) \leq R(K(1,k+1),K(1,k+1)) \quad (7)$$

The following theorem is useful to determine the exact value of R'(K(1,k+1),K(1,k+1)).

Theorem 6 (Chartrand and Lesniak [5]) : For a positive integer k,

$$R(K(1,k+1),K(1,k+1)) = \begin{cases} 2k+1, & \text{if k is odd,} \\ 2k+2, & \text{otherwise.} \end{cases} \qquad \square$$

Lemma 5 : For a positive integer k,

$$R'(K(1,k+1),K(1,k+1)) = \begin{cases} 2k+1, & \text{if } k \neq 2, \\ 6, & \text{if } k = 2. \end{cases}$$

Proof : Consider the graph $H \cong K(k,k)$. Clearly H is triangle-free, $\Delta(H) = k$ and $\Delta(\overline{H})=k-1$. Thus $R'(K(1,k+1),K(1,k+1)) \geq 2k+1$, for every positive integer k. Combining this with inequality (7), we have $R'(K(1,k+1),K(1,k+1)) = 2k+1$, whenever k is an odd positive integer. Similarly the graph C_5 in conjunction with (7) implies that $R'(K(1,3),K(1,3)) = 6$.

Henceforth we will assume that $k \geq 4$ and is even. We now prove that $R'(K(1,k+1),K(1,k+1)) \leq 2k+1$. Consider a traingle-free graph G of order 2k+1 such that $\Delta(G) \leq k$. We will show that $\overline{G}$ contains K(1,k+1) as a subgraph. Suppose not, that is, $\Delta(\overline{G}) \leq k$. This implies that G is k-regular.

Let u be a vertex of G, A = N(u) and B = V(G) - N[u]. Since G is triangle-free, A is 0-independent. Thus every vertex of A has exactly k-1 neighbours in B and hence the number of edges between A and B is k(k-1). Thus $|E(G[B])| = \frac{k}{2}$. Firstly assume that $\Delta(G[B]) \geq 2$ and let $v \in B$ such that v has at least two neighbours in B. This implies that a neighbour v' of v such that $v' \in A$ has at most k-2 neighbours in B, a contradiction. Thus $\Delta(G[B]) \leq 1$. Since $\varepsilon(G[B]) = \frac{k}{2}$, it follows that G[B] is isomorphic to a matching of size $\frac{k}{2}(\geq 2)$. Again this implies that every vertex of A has at most $\frac{k}{2}$ neighbours in B. This is a contradiction since $\frac{k}{2} < k-1$. This contradiction implies that $\overline{G}$ contains K(1,k+1) as a subgraph. Hence $R'(K(1,k+1),K(1,k+1)) \geq 2k+1$, for all even integers $k \geq 4$. The graph K(k,k) establishes the sharpness of the above inequality. This completes the proof of the lemma. □

For notational convenience we denote $R'(K(1,k+1),K(1,k+1))$, by R'. From the definition of R' it follows that for any positive integer $t \leq R - 1$, there exists a graph H of order t such that neither H nor $\overline{H}$ contains a vertex of degree at least k+1. We refer to such a graph as a Ramsey graph and denote it by H[t]. The following lemma is easy and can be proved along the same lines as Lemma 6 in Achuthan et al. [1].

Lemma 6 : Let G be a triangle-free graph of order p with $\chi_k(G) = 1$. Then

$$\chi_k(\overline{G}) \geq \frac{p}{R'-1}.$$ □

We now present a sharp lower bound for $\chi_k(G).\chi_k(\overline{G})$, where G is a triangle-free graph.

Theorem 7 : Let G be a triangle-free graph of order p. Then

$$\chi_k(G).\chi_k(\overline{G}) \geq \left\lceil \frac{p}{R'-1} \right\rceil.$$

Moreover this bound is sharp.

Proof : Let $\chi_k(G) = m$ and consider a partition of V(G) into m k-independent sets $V_1, V_2, \ldots, V_m$ such that $|V_1| = \max_i |V_i|$. Since $\chi_k(\overline{G}) \geq \chi_k(\overline{G}[V_1])$, it follows from Lemma 6 that

$$\chi_k(\overline{G}) \geq \frac{|V_1|}{R'-1} \geq \frac{p}{m(R'-1)}.$$

Thus

$$\chi_k(G).\chi_k(\overline{G}) \geq \left\lceil \frac{p}{R'-1} \right\rceil = \lambda, \text{ say}.$$

To establish the sharpness we define a graph G, of order p, to be the disjoint union of λ Ramsey graphs $H_1, H_2, \ldots, H_\lambda$ where each H_i has at most R-1 vertices. This completes the proof of the theorem. □

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