Factorizations of Complete Multigraphs

B. J. Battersby*, D. E. Bryant†

Centre for Combinatorics
Department of Mathematics, The University of Queensland
Queensland 4072, Australia

C.A. Rodger‡

Department of Discrete and Statistical Sciences
120 Math Annex
Auburn University
Auburn, Alabama 36849-5307, U.S.A

Dedicated to the memory of Derrick Breach, 1933–1996

Abstract

In this paper, several general results are obtained on the Oberwolfach problem that provide isomorphic 2-factorizations of $2K_n$. One consequence of these results is that the existence of a 2-factorization in which each 2-factor of $2K_n$ consists of one cycle of length x and one of length $n-x$ is completely settled. The techniques used to obtain these results are novel, using for example the Lindner-Rodger generalizations of Marshall Hall's classic embedding theorem for incomplete latin squares.

1 Introduction

Let λK_n denote the multigraph on n vertices in which each pair of vertices is joined by exactly λ edges. An m-cycle $(v_0, v_1, \ldots, v_{m-1})$ is a graph with vertex set $\{v_0, \ldots, v_{m-1}\}$ and edge set $\{v_iv_{i+1} \mid i \in \mathbb{Z}_m\}$, reducing the subscript modulo m. An *m-cycle system* or *order* n and *index* λ is an ordered pair (V, C), where C is a collection of m-cycles whose edges partition the edges of λK_n defined on the vertex set V (so $|V| = n$). There have been many results obtained concerning the existence

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of m-cycle systems of λK_n (and of other graphs). Such results date back to 1847 when Kirkman [8] settled the existence problem for 3-cycle systems of K_n (Steiner triple systems), but most papers on this subject have been written over the past thirty years. A survey of these results can be found in [10].

It has also been of interest to study m-cycle systems that have additional properties. A *parallel class* of an m-cycle system (V, C) of λK_n is a set of cycles in C that form a 2-factor of λK_n. An m-cycle system (V, C) of λK_n is said to be *resolvable* if C can be partitioned into parallel classes. The existence problem for resolvable 3-cycle systems of K_n (Kirkman triple systems) was first posed and solved in the case $n = 15$ by Kirkman in 1850 [9]; it was solved for all n by Ray-Chadhuri and Wilson in 1971 [12]. A different generalization of their result was then obtained: resolvable m-cycle systems of K_n were shown to exist iff m is odd and $n \equiv m \pmod{2m}$ in [1, 7] after various people had contributed preliminary results (see [10] for a survey).

However a much more general conjecture still remains unsolved (and is likely to do so for a long time): in 1967 Ringel asked whether for any integers $m_1, \dots, m_t$ with $n = \sum_{i=1}^{t} m_i$, there exists a 2-factorization of K_n in which each two factor consists of t cycles, one of each length $m_1, m_2, \dots, m_t$; so each 2-factor is isomorphic to each other 2-factor in this 2-factorization. Of course, if $m_1 = m_2 = \cdots = m_t$ then Ringel is asking for a resolvable m_1-cycle system of K_n. It is known that solutions to Ringel's question (also known as the Oberwolfach Problem) do not exist when $m_1 = 4$, $m_2 = 5$ and $n = 9$, and when $m_1 = 3$, $m_2 = 3$, $m_3 = 5$ and $n = 11$, but it is widely believed that these are the only exceptional cases.

Of course, a solution to the Oberwolfach problem requires n to be odd, because every vertex must have even degree. This restriction on n is avoided by considering 2-factorizations of $2K_n$. In this paper we obtain several general results concerning the existence of isomorphic 2-factorizations of $2K_n$. In particular, we completely settle the Oberwolfach problem in the case where $t = 2$ (see Theorem 3.2). The techniques we use to prove these results are also of interest, being novel approaches to this problem. For example, the Lindner-Rodger generalizations (Theorem 2.2, see [11]) of Hall's classic theorem [5] on embedding incomplete latin rectangles is the main tool required to prove Theorem 2.3.

Define an $\{m_1, m_2, \dots, m_t\}$-2-factor of a graph G to be a 2-factor of G in which there are t cycles, one of each length $m_1, m_2, \dots, m_t$; so G has $\sum_{i=1}^{t} m_i$ vertices. An $\{m_1, m_2, \dots, m_t\}$-2-factorization of G is a 2-factorization of G in which each 2-factor is a $\{m_1, m_2, \dots, m_t\}$-2-factor. Any graph theoretical terms not defined here can be found in [3].

2 General Results

Proposition 2.1. *Let G be a $2s$-regular spanning subgraph of $2K_\nu$. There exists an edge-disjoint decomposition of G into ν s-stars such that:*

(1) each vertex is the center of exactly one s-star; and

(2) *there exists an s-colouring of $E(G)$ such that for each $v \in V(G)$, each colour appears on exactly two edges incident with v, one in the star centered on v, and one not.*

Proof. Let $V(G) = \mathbb{Z}_\nu$. Form a simple bipartite graph B with bipartition $V(G)$ and $E(G)$ by joining $i \in V(G)$ to $e \in E(G)$ if and only if i is incident with e in G. Then $d_B(i) = 2s$ and $d_B(e) = 2$. Give B an equitable 2-edge-colouring with colours α and β. For each $i \in V(G)$, let $S(i)$ be the vertices in B joined to i by an edge coloured α. Since the edge-colouring of B is equitable, i is incident with s edges coloured α, so $|S(i)| = s$, and $e \in E(G)$ is incident with one edge coloured α, so $\{S(i) \mid i \in V(G)\}$ is a partition of $E(G)$. Since $S(i)$ induces an s-star in G centered at i, we have proved (1).

To prove (2), let B' be a bipartite graph with bipartition $V = \{v_i \mid i \in \mathbb{Z}_\nu\}$ and $W = \{w_j \mid j \in \mathbb{Z}_\nu\}$ formed by joining v_i to w_j if and only if $\{i, j\}$ is an edge in $S(i)$. Then since each vertex is the center of one s-star and therefore is a pendant vertex in s s-stars, $d_{B'}(v_i) = s = d_{B'}(w_j)$. Give B' a proper s-edge-colouring, then give G an s-edge-colouring by colouring $\{i, j\} \in S(i)$ with the same colour as the edge $\{v_i, w_j\}$ in B'. Since v_i in B' is incident with one edge of each colour, the edges in $S(i)$ are all coloured differently, and since w_j is incident with one edge of each colour, the edges incident with j in G that are not in $S(j)$ are all coloured differently. So (2) has been proved. $\square$

For any graph G and any $X \subseteq V(G)$, let $N_G(X)$ be the neighbourhood of X in G. Philip Hall proved the following result.

Theorem 2.1 ([6]). *Let B be a bipartite graph with bipartition V and W of the vertex set. B contains a 1-factor if and only if $|X| \leq |N_B(X)|$ for all $X \subset V$.*

Proposition 2.2. *Let G be a $2s$-regular multigraph on ν vertices where $\nu \geq 8(s-1)$. Let $\{F_1, \ldots, F_s\}$ be any 2-factorization of G. Then there exists an edge-disjoint decomposition of G into ν matchings, each of size s, such that each matching contains one edge in F_i for $1 \leq i \leq s$.*

Proof. The proof is by induction on s. The result is clearly true when $s = 1$, so assume that $s > 1$. Let $G(s)$ be any $2s$-regular graph on ν vertices with $\nu \geq 8(s-1)$ and let $\{F_1, \ldots, F_s\}$ be a 2-factorization of $G(s)$. Let $G(s-1) = G(s) - E(F_s)$. Then $\{F_1, \ldots, F_{s-1}\}$ is a 2-factorization of $G(s-1)$, and obviously $\nu \geq 8(s-2)$, so by the induction hypothesis there exist matchings $M(j) = \{e_1(j), \ldots, e_{s-1}(j)\}$ for $1 \leq j \leq \nu$ such that $\{M(1), \ldots, M(\nu)\}$ is a partition of $E(G(s-1))$.

Form a simple bipartite graph B with bipartition $M = \{M_1, \ldots, M_\nu\}$ and $E = E(F_s)$ by joining $M_j \in M$ to $e \in E$ if and only if $M(j) \cup \{e\}$ is an independent set (of size s) in $G(s)$. Note that since each edge in F_s is incident with at most 4 edges in F_i for $1 \leq i \leq s-1$, $d_B(e) \geq \nu - 4(s-1)$ for each $e \in E$. Similarly, each of the $s-1$ edges in $M(j)$ for $1 \leq j \leq \nu$ is incident with at most 4 edges in F_s, so $d_B(M_j) \geq \nu - 4(s-1)$. So the minimum degree $\delta(B)$ of B satisfies $\delta(B) \geq \nu - 4(s-1)$.

We now show that B contains a 1-factor, by applying Hall's Theorem. Let $X \subseteq M$. Since $\delta(B) \geq \nu - 4(s-1)$ and B is simple, $|N_B(X)| \geq \nu - 4(s-1)$. Clearly if

$|N_B(X)| = \nu$ then $|X| \leq |N_B(X)|$, so we can assume that $e \in E - N_B(X)$. Then since $d_B(e) \geq \nu - 4(s-1)$, we have that $\nu - |X| = |M - X| \geq |N_B(e)| \geq \nu - 4(s-1)$, so $|X| \leq 4(s-1) \leq \nu - 4(s-1) \leq |N_B(X)|$. Therefore B has a 1-factor, F.

The result now follows by adding e to $M(j)$ if and only if e is joined to M_j by an edge in F. □

Proposition 2.3. *Let G be a $2s$-regular multigraph on ν vertices and let $\{F_1, F_2, ..., F_s\}$ be any 2-factorization of G. Let $\{G_1, G_2, \ldots, G_\nu\}$ be an edge disjoint decomposition of G into ν 1-regular subgraphs, each containing s edges, such that $|E(F_i) \cap E(G_j)| = 1$ for any i and j with $1 \leq i \leq s$ and $1 \leq j \leq \nu$. Then there exist injective functions $f_j : V(G_1) \to \{1, 2, \ldots, \nu\}$ for $1 \leq j \leq \nu$ such that:*

(1) If $\{v, w\} \in E(G_1) \cap E(F_i)$ then $\{f_j(v), f_j(w)\} \in E(G_j) \cap E(F_i)$; and

(2) for all $v \in V(G_1)$, $\{f_1(v), f_2(v), \ldots, f_\nu(v)\} = \{1, 2, \ldots, \nu\}$.

Proof. For $i = 1, 2, \ldots, s$, form a simple bipartite graph B_i with bipartition $V(F_i)$ and $E(F_i)$ by joining $v \in V(F_i)$ to $e \in E(F_i)$ if and only if v is incident with e in F_i. For $i = 1, 2, \ldots, s$, B_i is 2-regular and so we can give B_i a 2-edge-colouring with colours α and β such that each vertex of B_i is incident with one edge coloured α and one edge coloured β.

For each $v \in V(G_1)$, let $\{v, w\}$ be the unique edge in G_1 incident with v; then if F_i is the 2-factor containing $\{v, w\}$ then define $f_j(v) = v'$ and $f_j(w) = w'$ where $\{v', w'\} \in E(G_j) \cap E(F_i)$ and the edge joining v to $\{v, w\}$ is the same colour as the edge joining v' to $\{v', w'\}$ in B_i. For $1 \leq j \leq \nu$, f_j is a function since G_1 is 1-regular, and f_j is injective since G_j is 1-regular. □

The following is a generalization of Marshall Hall's theorem proved in [5]. The Lindner-Rodger generalization was proved in [11] (Theorem 3.1), where it is described in terms of patterned holes. Here we change the notation to fit the proof of Theorem 2.3. A row or column ℓ of an array L is said to be *latin* if each symbol occurs in at most one cell of ℓ in L.

Theorem 2.2. *Let L' be an $s \times \nu$ array on the symbols $1, \ldots, \nu$ in which each cell contains exactly one symbol, each column is latin, and each symbol occurs in exactly s cells of L'. Then L' can be embedded in a $\nu \times \nu$ array in which each column is latin and each of rows $s+1, \ldots, \nu$ is latin.*

If G and H are two graphs with $V(G) \cap V(H) = \emptyset$, then let $G \vee_t H$ be the graph with $V(G \vee_t H) = V(G) \cup V(H)$ and where $E(G \vee_t H)$ consists of the edges in $E(G) \cup E(H)$ together with t edges joining each vertex in $V(G)$ to each vertex in $V(H)$.

Theorem 2.3. *Let s, t be positive integers such that t divides $2s$. Let G be a $2s$-regular spanning subgraph of tK_ν. Let H be a subgraph of G with maximum degree $\Delta \leq t$ that contains exactly s edges, and in which for $1 \leq i \leq \Delta$, H contains $t_i \geq 0$ vertices of degree i. Suppose there exist injective functions $f_j : V(H) \to \{1, 2, \ldots, \nu\}$ for $1 \leq j \leq \nu$ such that:*

(1) $f_1(H),\dots,f_\nu(H)$ form an edge-disjoint decomposition of G; and

(2) for all $v \in V(H)$, if $d_H(v) < t$ then $\{f_1(v),\dots,f_\nu(v)\} = \{1,2,\dots,\nu\}$.

Finally, let F be any t-factor of $K^c_{\nu-(2s/t)} \vee_t H^*$ which contains all the edges in H^*, where H^* is the spanning subgraph of G in which $E(H^*) = E(H)$. Then there exists a t-factorization of $K^c_{\nu-(2s/t)} \vee_t G$ in which each t-factor is isomorphic to F.

Remark. As the proof of this Theorem shows, one consequence of these conditions is that if $t = \Delta$ then t must divide $2s - \sum_{i=1}^{\Delta-1}(it_i)$.

Proof. Let $V(G) = \{1,2,\dots,\nu\}$, $V(H) = \{1,2,\dots,h\}$, and let $K^c_{\nu-(2s/t)}$ have vertex set W (so $W \cap \{1,\dots,\nu\} = \phi$). Clearly we can assume that $f_1(v) = v$ for all $v \in V(H)$, and that $d_H(v) < t$ or $d_H(v) = t$ for $1 \le v \le h'$ or $h' < v \le h$ respectively (so h' is the number vertices of degree less than t in H).

Form an $h \times \nu$ array L' as follows: for $1 \le i \le h$ and $1 \le j \le \nu$ let cell (i,j) of L' contain $f_j(i)$. Clearly L' is column latin since f_j is injective, and by property (2) if $d_H(v) < t$ then each of the symbols, $1,\dots,\nu$ appears exactly once in row v of L'. If $\Delta = t$ then note that the remaining rows $h'+1,\dots,h$ of L' need not be latin, but we do have the following property. By (1) $E(G)$ is partitioned by the edges in $f_1(H),\dots,f_\nu(H)$, and for $1 \le v \le \nu$ symbol v occurs exactly once in row i for $1 \le i \le h'$, thus accounting for $T = \sum_{i=1}^{\Delta-1}(it_i)$ edges incident with v in G; so v must occur in exactly $(2s-T)/t$ cells in rows $h'+1,\dots,h$ of L'.

Therefore by Theorem 2.2, L' can be embedded in a $\nu \times \nu$ column latin array that is also row latin in row i for $1 \le i \le h'$ and $h < i \le \nu$. Clearly we can assume cell $(i,1)$ of L contains symbol i for $1 \le i \le \nu$. Let $L(i,j)$ be the symbol in cell (i,j) of L. Finally, for $1 \le j \le \nu$, let ℓ_j: $V(G) \cup W \to V(G) \cup W$ be the automorphism $F_j = \ell_j(F)$ of F defined by $\ell_j(i) = L(i,j)$ for $1 \le i \le \nu$, and $\ell_j(w) = w$ for all $w \in W$. Then $F_1,\dots,F_\nu$ is a t-factorization of $K^c_{\nu-(2s/t)} \vee_t G$, as the following argument shows.

First note that each t-factor of $K^c_{\nu-(2s/t)} \vee_t G$ contains $t(\nu - (2s/t) + \nu)/2 = t(\nu - s/t)$ edges, so $F_1,\dots,F_\nu$ contain $\nu^2 t - \nu s$ edges altogether; and $|E(K^c_{\nu-(2s/t)} \vee_t G)| = (\nu - (2s/t))\nu t + 2s\nu/2 = \nu^2 t - \nu s$, so it remains to show that each edge of $K^c_{\nu-(2s/t)} \vee_t G$ occurs in at least one of $F_1,\dots,F_\nu$.

Each edge $e \in E(G)$ is one of $F_1,\dots,F_\nu$ because by (1) e occurs in one of $H_1,\dots,H_\nu$, say H_j, and F_j contains $E(H_j)$.

For each $w \in W$ and each $v \in V$ we find t edges joining w to v as follows. Since F is a t-factor of $K^c_{\nu-(s/t)} \vee_t G$, F contains t edges incident with w; suppose that for $1 \le i \le x$ w is joined to $v_i \in V$ by y_i edges. Then clearly $d_H(v_i) < t$, so each of $1,\dots,\nu$ occurs in a cell in row v_i of L. In particular, suppose that v occurs in cell (v_i,j); then v is joined to w with y_i edges in F_j. Therefore v is joined to w by at least $\sum_{i=1}^{x} y_i = t$ edges in $F_1,\dots,F_\nu$, so the result is proved. □

Corollary 2.1. *Let G be a $2s$-regular spanning subgraph of $2K_\nu$. Let F be any t-factor of $K^c_{\nu-(s/t)} \vee_t G$ such that the edges in the subgraph of F induced by the vertices in G induce an s-star. Then there exists a t-factorization of $K^c_{\nu-(2s/t)} \vee_t G$ in which each t-factor is isomorphic to F.*

Proof. Let $V(G) = \{1, 2, \ldots, \nu\}$ and let H be an s-star with $V(H) = \{1, \ldots, s+1\}$, where $d_H(s+1) = s$. By Proposition 2.1 (1), there exists a decomposition of G into s-stars $H(1), \ldots, H(\nu)$ such that the center of $H(i)$ is the vertex i. Furthermore, by Proposition 2.1 (2), for $1 \le j \le \nu$ and for $1 \le v \le s$ there exists an s-edge-colouring of G such that j is incident with exactly one edge e coloured v that is in a star $H(i)$ for some $i \ne j$; so we can define $f_j(v) = i$. Then also by Proposition 2.1 (2) $\{f_1(v), \ldots, f_\nu(v)\} = \{1, \ldots, \nu\}$ for $1 \le v \le s$ (since each vertex i in $V(G)$ is incident with exactly one edge coloured v that is in the s-star with center i).

The proof now follows from Theorem 2.3. □

3 2-factorizations of $2K_\nu$

In this section we first prove, using the results of the previous section, some general results concerning 2-factorizations of graphs. We then prove that for any two integers μ and ν (each at least 2) there is a uniform 2-factorization of $2K_{\mu+\nu}$ in which each 2-factor consists of a cycle of length μ and a cycle of length ν (see Theorem 3.2).

Corollary 3.1. *Let G be a 4-regular spanning subgraph of $2K_\nu$. Let $n \ge 1$, and let $m_1, \ldots, m_n$ satisfy $m_i \ge 1$ for $1 \le j \le n-1$, $m_n \ge 2$, and $\sum_{j=1}^{n} m_j = \nu - 1$. Then there exists a $\{2m_1, \ldots, 2m_n\}$-2-factorization of $K_{\nu-2}^c \vee_2 G$.*

Proof. Let $V(K_{\nu-2}^c) = \{1, \ldots, \nu-2\} \times \{0\}$ and $V(G) = \{1, \ldots, \nu\} \times \{1\}$. We may assume that G contains the edges $\{\nu-2, \nu-1\}$ and $\{\nu-1, \nu\}$. Let $M_i = \sum_{j=1}^{i-1} m_j$ (with $M_1 = 0$), and let F consist of the cycles $c_1, \ldots, c_n$ defined as follows. For $1 \le i \le n-1$

$$c_i = ((M_i+1, 0), (M_i+1, 1), (M_i+2, 0), (M_i+2, 1), \ldots, (M_{i+1}, 0), (M_{i+1}, 1))$$

and

$$c_n = ((M_n+1, 0), (M_n+1, 1), (M_n+2, 0), (M_n+2, 1), \ldots,$$
$$(\nu-2, 0), (\nu-2, 1), (\nu-1, 1), (\nu, 1)).$$

Then c_i has length $2m_i$ for $1 \le i \le n$, F is 2-factor of $K_{\nu-2}^c \vee_2 G$. Since the edges in the subgraph of F induced by the vertices in G induce a 2-star, the result follows from Corollary 2.1. □

Theorem 3.1. *Let μ and ν be integers with $2 \le \mu < \nu \le 8(\mu+1)/7$. Suppose there is an $\{m_1, m_2, \ldots, m_\alpha\}$-2-factorization of $2K_\mu$, an $\{m_{\alpha+1}, m_{\alpha+2}, \ldots, m_\beta\}$-2-factorization of $2K_\nu$ and suppose there exist non-negative integers $x_1, x_2, \ldots, x_\beta$ and $y_1, y_2, \ldots, y_\beta$ such that $x_i \ge y_i$ and $2x_i + y_i = m_i$ for $1 \le i \le \beta$ and $\sum_{i=1}^{\beta} y_i = \nu - \mu$. Then there exists an $\{m_1, m_2, \ldots, m_\beta\}$-2-factorization of $2K_{\mu+\nu}$.*

Proof. Let $2K_{\mu+\nu} = A \vee_2 B$ where $A \cong 2K_\mu$, $B \cong 2K_\nu$ and let $s = \nu - \mu$. Pair off each of the $\mu - 1$ 2-factors in an $\{m_1, m_2, \ldots, m_\alpha\}$-2-factorization of A with an $\{m_{\alpha+1}, m_{\alpha+2}, \ldots, m_\beta\}$-2-factor of B to obtain $\{\mu - 1\ m_1, m_2, \ldots, m_\beta\}$-2-factors

$F_1', F_2', \ldots, F_{\mu-1}'$ of $A \vee_2 B$. Then, $A \vee_2 B \setminus \{F_1' \cup F_2' \cup \cdots \cup F_{\mu-1}'\} \cong K_\mu^c \vee_2 G$ where G is $2s$-regular on ν vertices. We now apply Proposition 2.2. Note that since $\nu \leq 8(\mu+1)/7$ we have $\nu \geq 8(s-1)$ and the remaining s 2-factors $F_1, \ldots, F_s$ in the $\{m_{\alpha+1}, m_{\alpha+2}, \ldots, m_\beta\}$-2-factorization of B form a 2-factorization of G.

By Proposition 2.2, there is an edge-disjoint decomposition of G into ν matchings $M_1, M_2, \ldots, M_\nu$, each of size s, such that for $1 \leq j \leq \nu$ and $1 \leq i \leq s$, $|E(M_j) \cap E(F_i)| = 1$. Hence, by Proposition 2.3, there exist injective functions $f_j : V(M_1) \to \{1, 2, \ldots, \nu\}$ for $1 \leq j \leq \nu$ such that:

(1) $f_1(M_1), \ldots, f_\nu(M_1)$ form an edge-disjoint decomposition of G (since by Proposition 2.3 (1), $f_j(M_1) = M_j$); and

(2) for all $v \in V(M_1)$, $\{f_1(v), \ldots, f_\nu(v)\} = \{1, 2, \ldots, \nu\}$.

Hence the result follows by Theorem 2.3 (with $t = 2$ and $H = M_1$) if we can find an $\{m_1, m_2, \ldots, m_\beta\}$-2-factor F of $K_\mu^c \vee_2 M^*$, containing all the edges of M^*, where M^* is a matching of size s and $V(M^*) = \{1, 2, \ldots, \nu\}$.

Write M^* as the union of vertex-disjoint graphs $G_1, \ldots, G_\beta$ where G_i is a matching of size y_i on $(m_i + y_i)/2$ vertices (note that $\sum_{i=1}^{\beta} y_i = s$ and $\sum_{i=1}^{\beta}(m_i + y_i)/2 = (\mu + \nu + \nu - \mu)/2 = \nu$). Also, partition the vertex set of K_μ^c into sets $X_1, \ldots, X_\beta$ of sizes $x_1, \ldots, x_\beta$ (note that $\sum_{i=1}^{\beta} x_i = \sum_{i=1}^{\beta}(m_i - y_i)/2 = \mu$). Finally, for $1 \leq i \leq \beta$ let C_i be an m_i-cycle with vertex set $X_i \cup V(G_i)$ and with $E(G_i) \subseteq E(C_i)$ (we can do this since $x_i \geq y_i$ for $1 \leq i \leq \beta$) and let $F = C_1 \cup \cdots \cup C_\beta$. □

Theorem 3.2. *For all integers μ and ν, $2 \leq \mu, \nu$, there exists a $\{\mu, \nu\}$-2-factorization of $2K_{\mu+\nu}$, except that there is no $\{3, 3\}$-2-factorization of $2K_6$.*

Proof. It is shown in [4] that if $\mu = \nu$ then there exists a $\{\mu, \nu\}$-2-factorization of $2K_{\mu+\nu}$, except that there is no $\{3, 3\}$-2-factorization of $2K_6$. Hence we may assume that $2 \leq \mu < \nu$. For any given μ and ν we show that there is a set $S_{\mu,\nu} \subseteq \mathbb{Z}_\nu \setminus \{0\}$ which satisfies

(1) $|S_{\mu,\nu}| = \mu - 1$;

(2) $H_\nu(S_{\mu,\nu})$ has a hamilton decomposition; and

(3) if μ is odd then either $1 \notin S_{\mu,\nu}$ and $2 \in S_{\mu,\nu}$ or $\mu = \nu - 1$ and $S_{\mu,\nu} = \mathbb{Z}_\nu \setminus \{0, \nu/2\}$.

First, consider the pairs of differences $\{d_1, d_2\} = \{2, 3\}, \{4, 5\}, \ldots$ with $\{d_1, d_2\} \subseteq \mathbb{Z}_\nu \mid \{10\}$. For each such pair $\{d_1, d_2\}$, except the pair containing $\nu/2$ when ν is even, $H_\nu(\{d_1, d_2\})$ has a hamilton decomposition by [2].

If μ is odd and $\mu < \nu - 1$ then we can choose $S_{\mu,\nu}$ to consist of the elements of $\mu - 1/2$ of these pairs and ensure that $2 \in S_{\mu,\nu}$ and, when ν is even, that the both elements of the pair containing $\nu/2$ are not in $S_{\mu,\nu}$. If μ is even then we let $S_{2,\nu} = \{1\}$ and for $\mu \geq 4$, we can choose $S_{\mu,\nu} = S_{\mu-1,\nu} \cup \{1\}$.

Finally, if ν is even and $\mu = \nu - 1$ then we choose $S_{\mu,\nu} = \mathbb{Z}_\nu \setminus \{0, \nu/2\}$. Note that in this case, we pair up the differences as $\{d_1, d_2\}, \ldots, \{d_{\nu-3}, d_{\nu-2}\}$ where $d_1 =$

$1, d_2 = 2, \ldots, d_{\nu/2-1} = \nu/2 - 1, d_{\nu/2} = \nu/2 + 1, d_{\nu/2+1} = \nu/2 + 2, \ldots, d_{\nu-2} = \nu - 1$ to see that $H_\nu(S_{\mu,\nu})$ has a hamilton decomposition by [2].

Let $\mathbb{Z}_\nu$ be the vertex set of $2K_\nu$ and let $D = \{d_1, d_2, \ldots, d_{\nu-\mu}\} = (\mathbb{Z}_\nu \setminus \{0\}) \setminus S_{\nu,\mu}$ where $d_1 < d_2 < \cdots < d_{\nu-\mu}$. Then define the path $P(D) = v_0, v_1, \ldots, v_{\nu-\mu}$ where $v_0 = 0$ and for $i = 1, 2, \ldots, \nu - \mu$, $v_i = \sum_{j=1}^{i} (-1)^{j+1} d_j$.

If μ is even, we apply Theorem 3 with $s = \nu - \mu, t = 2, G = H_\nu(D), H = P(D), f_j(v) = v + j - 1$ (for $j = 1, 2, \ldots, v$ and $v \in V(H)$). Hence, we require a 2-factor F, consisting of a μ-cycle and a ν-cycle, of $K_\mu^c \vee_2 H^*$ where H^* consists of a path of length $\nu - \mu$ and $\mu - 1$ isolated vertices. We form the μ-cycle of F by using $\mu/2$ vertices from K_μ^c and $\mu/2$ isolated vertices from H^*. It is then straight-forward to form a ν-cycle using the remaining $\mu/2$ vertices from K_μ^c,the remaining $\mu/2 - 1$ isolated vertices in H^* and the path of length $\nu - \mu$. Hence by Theorem 3 there is a $\{\mu, \nu\}$-2-factorization of $K_\mu^c \vee_2 G$.

The $\{\mu, \nu\}$-2-factorization of $2K_{\mu+\nu}$ can now be completed since $2K_{\mu+\nu} \setminus (K_\mu^c \vee_2 G) \cong 2K_\mu + H_\nu(S_{\mu,\nu})$ has a $\{\mu, \nu\}$-2-factorization: we can pair the μ-cycles in a μ-cycle system of $2K_\mu$ with the $\mu - 1$ ν-cycles in a hamilton decomposition of $H(S_{\mu,\nu})$.

If μ is odd, we again apply Theorem 3 with $s = \nu - \mu, t = 2, G = H_\nu(D), f_j(v) = v + j - 1$ (for $j = 1, 2, \ldots, v$ and $v \in V(H)$) but this time when $\mu < \nu - 1$ we let H be the graph formed by removing the edge in $H_\nu(\{2\})$ from $P(D \cup \{2\})$, so that H contains an isolated edge (namely the edge of H that is also in $H_\nu(\{1\})$). When $\mu = \nu - 1, D = \{\nu/2\}$ and we let $H = P(D)$ so that H consists of a single edge. Hence, we require a $\{\mu, \nu\}$-2-factor F of $K_\mu^c \vee_2 H^*$ where H^* consists of a path containing $\nu - \mu - 1$ edges, an isolated edge and $\mu - 2$ isolated vertices (in the case $\mu = \nu - 1$ H^* consists of an isolated edge and $\mu - 1$ isolated vertices). We form the μ-cycle of F by using $(\mu - 1)/2$ vertices from K_μ^c, the isolated edge of H^*, and $(\mu - 3)/2$ isolated vertices of H^*. It is then straight-forward to form a ν-cycle using the remaining $(\mu + 1)/2$ vertices from K_μ^c and the remaining isolated vertices and paths in H^*. Hence by Theorem 3 there is a $\{\mu, \nu\}$-2-factorization of $K_\mu^c \vee_2 G$.

As in the case μ even, we complete the 2-factorization of $2K_{\mu+\nu}$ by pairing the $\mu - 1$ μ-cycles in a μ-cycle system of $2K_\mu$ with the $\mu - 1$ ν-cycles in a hamilton decomposition of $H(S_{\mu,\nu})$. □

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