Strong Subtournaments of Multipartite Tournaments

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Abstract

An orientation of a complete graph is a tournament, and an orientation of a complete n-partite graph is an n-partite tournament. For each $n \geq 4$, there exist examples of strongly connected n-partite tournament without any strongly connected subtournaments of order $p \geq 4$. If D is a digraph, then let $d^+(x)$ be the outdegree and $d^-(x)$ the indegree of the vertex x in D. The minimum (maximum) outdegree and the minimum (maximum) indegree of D are denoted by δ^+ (Δ^+) and δ^- (Δ^-), respectively. Furthermore, we define $\delta = \min\{\delta^+, \delta^-\}$ and $\Delta = \max\{\Delta^+, \Delta^-\}$. A digraph D is almost regular, if $\Delta - \delta \leq 1$. If $V_1, V_2, \ldots, V_n$ are the partite sets of an n-partite tournament D, then we define $\gamma(D) = \min_{1 \leq i \leq n}\{|V_i|\}$. In this paper we prove that every almost regular n-partite tournament with $n \geq 4$ contains a strongly connected subtournament of order p for each $p \in \{3, 4, \ldots, n-1\}$. Examples show that this result is best possible for $n = 4$. If in addition, $\gamma(D) < 3n/2 - 6$, for an almost regular n-partite tournament D with $n \geq 5$, then D even contains a strong subtournament of order n.

1. Terminology and Introduction

An n-partite or multipartite tournament is an orientation of a complete n-partite graph, and a tournament is an n-partite tournament with exactly n vertices. The vertex set of a digraph D is denoted by $V(D)$ and the arc set by $A(D)$. The number $|V(D)|$ is called the order of the digraph D. If there is an arc from x to y in a digraph D, then we say that x dominates y, denoted by $x \rightarrow y$. Let X and Y be two disjoint subsets of $V(D)$. We use $X \rightarrow Y$ to denote the fact that $x \rightarrow y$ for all vertices $x \in X$ and all $y \in Y$. Furthermore, if $x \rightarrow y$ for all $x \in X$ and $y \in Y$, which are in different partite sets of a multipartite tournament, then we write $X \leadsto Y$. By $d(X, Y)$ we denote the number of arcs from the set X to the set Y, i.e., $d(X,Y) = |\{xy \in A(D) : x \in X, y \in Y\}|$. The vertex x is a neighbor of the vertex y, if $x \to y$ or $y \to x$. The outset $N^+(x,D) = N^+(x)$ of a vertex x in D is the set of vertices dominated by x, and the inset $N^-(x,D) = N^-(x)$ is the set of vertices dominating x. We denote by $d^+(x,D) = d^+(x) = |N^+(x)|$ the outdegree and by $d^-(x,D) = d^-(x) = |N^-(x)|$ the indegree of the vertex $x \in V(D)$. The minimum (maximum) outdegree and the minimum (maximum) indegree of D are denoted by $\delta^+(D)$ ($\Delta^+(D)$) and $\delta^-(D)$ ($\Delta^-(D)$), respectively. In addition, we define $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ and $\Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}$. A digraph D is regular, if $\delta(D) = \Delta(D)$ and almost regular, if $\Delta(D) - \delta(D) \leq 1$. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. By a cycle (path) we mean a directed cycle (directed path). A cycle (path) of a digraph D is Hamiltonian if it includes all the vertices of D. A digraph D is said to be strongly connected or just strong, if for every pair x, y of vertices in D, there is a path from x to y. A strong component of D is a maximal induced strong subdigraph of D. If D is an n-partite tournament with the partite sets $V_1, V_2, \ldots, V_n$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_n|$, then $|V_n| = \alpha(D)$ is the independence number of D, and we define by $\gamma(D) = |V_1|$.

In 1976, Bondy [2] has proved that a strongly connected n-partite tournament contains an m-cycle for all m between 3 and n. If one could find in such an n-partite tournament a strong subtournament of order n, then by the well-known theorem of Moon (see Theorem 2.1 below), Bondy's result would be a direct consequence. But the next example will show that this way is not practicable in general.

Example 1.1 Let $V_1, V_2, \ldots, V_n$ be the partite sets of an n-partite tournament with $n \geq 4$ and $V_n = U_1 \cup U_2 \cup \ldots \cup U_{n-1}$ such that $V_i \to V_j$ for $1 \leq i < j \leq n-1$, $\{V_1, V_2, \ldots, V_{t-1}, V_{t+1}\} \to U_t$, and $U_t \to \{V_t, V_{t+2}, V_{t+3}, \ldots, V_{n-1}\}$ for $1 \leq t \leq n-1$. Then it is a simple matter to verify that the resulting n-partite tournament D is strongly connected. But the largest strong subtournament of D only consists of three vertices.

There is extensive literature on cycles and paths in multipartite tournaments, see e.g., Bang-Jensen and Gutin [1], Guo [3], Gutin [4], Volkmann [7], and Yeo [8]. In view of this, it is somewhat surprising that the closely-related question for strongly connected subtournaments in multipartite tournaments have, as yet, received no attention. In this paper we will develop the first contributions to this interesting problem.

We prove that every almost regular n-partite tournament with $n \geq 4$ contains a strongly connected subtournament of order p for each $p \in \{3, 4, \ldots, n-1\}$. An infinite family of regular 4-partite tournaments without a strong subtournament of order 4 shows that this result is best possible for $n = 4$. If in addition, $\gamma(D) < 3n/2 - 6$, for an almost regular n-partite tournament D with $n \geq 5$, then we are able to show that D even contains a strong subtournament of order n. In regular n-partite tournaments one can weaken the last condition slightly to $\gamma(D) < 3n/2 - 2$. But since we are quite sure that it is possible to extend the last two results, we omit the proofs.

2. Preliminary Results

The following results play an important role in our investigations.

Theorem 2.1 (Moon [5] 1966) Let T be a strongly connected tournament. Then every vertex of T is contained in a cycle of order m for all $3 \leq m \leq |V(T)|$.

Theorem 2.2 (Bondy [2] 1976) Each strongly connected n-partite tournament contains a cycle of order m for each $m \in \{3, 4, \ldots, n\}$.

The next two lemmas can be found in a very recent article of Tewes, Volkmann, and Yeo [6].

Lemma 2.3 If $V_1, V_2, \ldots, V_n$ are the partite sets of an an almost regular n-partite tournament D, then $||V_i| - |V_j|| \leq 2$ for $1 \leq i \leq j \leq n$.

Lemma 2.4 If D is an almost regular multipartite tournament, then for every vertex x of D we have

$$\frac{|V(D)| - \alpha(D) - 1}{2} \leq d^+(x), d^-(x) \leq \frac{|V(D)| - \gamma(D) + 1}{2}$$

Lemma 2.5 If X is a vertex set of an almost regular digraph D, then

$$|X| \geq |d(X, V(D) - X) - d(V(D) - X, X)|.$$

Proof. We consider the following sum $S = \sum_{x \in X}(d^+(x) - d^-(x))$. Every arc with both ends in X is added once and subtracted once. Furthermore, every arc going out of X is added once, and every arc going into X is subtracted once. Therefore, we obtain $S = d(X, V(D) - X) - d(V(D) - X, X)$. Since D is almost regular, each term in the sum is between minus and plus one, and hence the desired estimation $|X| \geq |S| = |d(X, V(D) - X) - d(V(D) - X, X)|$ follows. □

Lemma 2.6 Let T be a strongly connected tournament of order $|V(T)| \geq 4$. Then there exists a vertex $u \in V(T)$ of maximum outdegree such that for all $x \in V(T) - \{u\}$, the subtournament $T - x$ has a Hamiltonian path with the initial vertex u.

Proof. If $x_1, x_2, \ldots, x_t$ are the vertices of maximum outdegree in the tournament T, then we choose $u \in \{x_1, x_2, \ldots, x_t\}$ as a vertex of maximum outdegree in the subtournament $T[\{x_1, x_2, \ldots, x_t\}]$. Now let x be an arbitrary vertex of $T - u$. If $T - x$ is strong, then by Theorem 2.1, $T - x$ has a Hamiltonian cycle, and thus also a Hamiltonian path with the initial vertex u. If $T - x$ is not strong, then let $T_1, T_2, \ldots, T_r$ be the strong components of $T - x$ such that $V(T_i) \to V(T_j)$ whenever

$1 \le i < j \le r$. If $|V(T_1)| = 1$, then, by the definition of u, and because of $|V(T)| \ge 4$, it follows that $V(T_1) = \{u\}$. If $|V(T_1)| \ge 3$, then for every $w \in V(T_1)$ and every $v \in V(T_2) \cup V(T_3) \cup \ldots \cup V(T_r)$, we deduce that

$$d^+(v) \le |V(T)| - |V(T_1)| - 1 < |V(T)| - |V(T_1)| \le d^+(w).$$

Hence, also in this case, we see that $u \in V(T_1)$. Since, in view of Theorem 2.1, each strong component of $T - x$ has a Hamiltonian cycle or consists of a single vertex, it follows from $V(T_i) \to V(T_j)$ for $1 \le i < j \le r$ that $T - x$ has a Hamiltonian path with the initial vertex u. □

3. Main Results

Theorem 3.1 Let D be an almost regular n-partite tournament with $n \ge 4$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, n-1\}$.

Proof. Let $V_1, V_2, \ldots, V_n$ be the partite sets of D and let $k = \gamma(D)$. Since, in view of Lemma 2.3, $||V_i| - |V_j|| \le 2$ for $1 \le i, j \le n$, we deduce that $1 \le k \le |V_i| \le k + 2$ for $i \in \{1, 2, \ldots, n\}$. Thus, $|V(D)| = nk + r$ with $0 \le r \le 2(n-1)$. We proceed the proof by induction on the order p of the strongly connected subtournaments. By the hypothesis, it is a simple matter to show that D is strongly connected. Hence, according to Theorem 2.2, there exists a 3-cycle in D, which is a strong subtournament of order 3.
Now let $n \ge 5$ and let T_p be a strong subtournament of order p with $3 \le p \le n-2$. We assume without loss of generality that $T_p = D[\{v_1, v_2, \ldots, v_p\}]$ such that $v_i \in V_i$ for $i = 1, 2, \ldots, p$. If there is a vertex $z \in V_{p+1} \cup V_{p+2} \cup \ldots \cup V_n$ such that z has a positive neighbor as well as a negative neighbor in T_p, then is straightforward to verify that $D[\{z, v_1, v_2, \ldots, v_p\}]$ is a strong subtournament of order $p+1$. If such a vertex does not exist, then let $V_i' \subseteq V_i$ and $V_i'' = V_i - V_i'$ such that $V(T_p) \to V_i'$ when $V_i' \ne \emptyset$ and $V_i'' \to V(T_p)$ when $V_i'' \ne \emptyset$ for $i = p+1, p+2, \ldots, n$. In addition, we define $V' = V'_{p+1} \cup V'_{p+2} \cup \ldots \cup V'_n$ and $V'' = V''_{p+1} \cup V''_{p+2} \cup \ldots \cup V''_n$. According to Lemma 2.4, we obtain for every vertex x of D

$$\frac{nk + r - \alpha(D) - 1}{2} \le d^+(x), d^-(x) \le \frac{nk + r - k + 1}{2} \tag{1}$$

with $k \le \alpha(D) \le k + 2$. Now we distinguish two cases.
Case 1. Let $V' \ne \emptyset$ and $V'' \ne \emptyset$. If there exists an arc xy with $x \in V'$ and $y \in V''$, then $D[\{x, y, v_1, v_2, \ldots, v_p\}]$ is a strong subtournament of order $p+2$. As a consequence of Theorem 2.1, we see immediately that there also exists a strong subtournament of order $p+1$. Therefore, we assume in the following that $V'' \leadsto V'$. Furthermore, let $R = V(D) - (V' \cup V'' \cup V(T_p))$, $|V_i'| = t_i$ for $p+1 \le i \le n$, and suppose without loss of generality that $t_{p+1} \ge t_{p+2} \ge \ldots \ge t_n$. If $\alpha(D) = k+2$, then the hypothesis $\Delta(D) - \delta(D) \le 1$ implies that $|V(D)| - k$ is even, and hence it follows from (1)

$$\frac{(n-1)k + r - 2}{2} \le d^+(x), d^-(x). \tag{2}$$

Clearly, in view of (1), this inequality is also valid in the case that $\alpha(D) \leq k+1$ and therefore, in every case.

Subcase 1.1 Let $V_n'' \neq \emptyset$. The estimation (2) yields for an arbitrary vertex $v \in V_{p+1}'$

$$|N^+(v) \cap R| \geq \max\Big\{0, \frac{(n-1)k+r-2}{2} - (t_{p+2} + t_{p+3} + \ldots + t_n)\Big\}, \tag{3}$$

and for an arbitrary vertex $w \in V_n''$

$$|N^-(w) \cap R| \geq \max\Big\{0, \frac{(n-1)k+r-2}{2} - (k-t_{p+1}) - (k-t_{p+2}) - \ldots - (k-t_{n-1}) - s_1\Big\}, \tag{4}$$

where $0 \leq s_1 \leq \min\{r, 2(n-p-1)\}$ such that $|(V' \cup V'') - V_n| = (n-p-1)k + s_1$. If $|R| = pk - p + s_2$, then we observe that $0 \leq s_2 \leq \min\{r, 2p\}$ and $s_1 + s_2 \leq r$. Clearly, the cases $|N^+(v) \cap R| > |R|$ or $|N^-(w) \cap R| > |R|$ lead to a contradiction. Otherwise, because of $t_{p+1} \geq t_n$, $s_1 + s_2 \leq r$ and $p \geq 3$, we deduce from (3) and (4)

$$\begin{aligned} |N^+(v) \cap R| + |N^-(w) \cap R| &\geq (n-1)k + r - 2 - s_1 - (n-p-1)k + t_{p+1} - t_n \\ &\geq pk + r - s_1 - 2 \geq pk + s_2 - 2 \\ &\geq pk + s_2 - p + 1 = |R| + 1. \end{aligned}$$

Hence, there exists a vertex $x \in ((N^+(v) \cap R) \cap (N^-(w) \cap R))$. If, without loss of generality, $x \in V_1$, then, since $V(T_p) \to v$ and $w \to V(T_p)$, and since v and w are in different partite sets, $D[\{v, x, w, v_3, v_4, \ldots, v_p\}]$ is a strongly connected subtournament of order $p+1$.

Subcase 1.2 Let $V_n'' = \emptyset$. This implies $V_n' = V_n$ and $t_{p+1} \geq t_{p+2} \geq \ldots \geq t_n = |V_n| \geq k$. If $|V'| = (n-p)k + l_1$ and $|V''| = l_2$, then $1 \leq l_1 + l_2 \leq \min\{r, 2(n-p)\}$. According to (2), we obtain for an arbitrary vertex $v \in V_n'$

$$|N^+(v) \cap R| \geq \max\Big\{0, \frac{(n-1)k+r-2}{2} - (n-p-1)k - l_1\Big\}, \tag{5}$$

and for an arbitrary vertex $w \in V''$

$$|N^-(w) \cap R| \geq \max\Big\{0, \frac{(n-1)k+r-2}{2} - l_2 + 1\Big\}, \tag{6}$$

If $|R| = pk - p + s_2$, then $0 \leq s_2 \leq \min\{r, 2p\}$ and $s_2 + l_1 + l_2 \leq r$. Analogously to Subcase 1.1, it follows from (5) and (6) that $|N^+(v) \cap R| + |N^-(w) \cap R| > |R|$. Hence, there exists again a vertex $x \in ((N^+(v) \cap R) \cap (N^-(w) \cap R))$. If, without loss of generality, $x \in V_1$, then, since v and w are in different partite sets, $D[\{v, x, w, v_3, v_4, \ldots, v_p\}]$ is a desired strong subtournament.

Case 2. Let $V' = \emptyset$ or $V'' = \emptyset$. Without loss of generality, we discuss the case $V'' = \emptyset$. Then $V_i' = V_i$ for $p+1 \leq i \leq n$, and we write V instead of V'. Let U contain all the vertices of $V(D) - (V \cup V(T_p))$ which are dominated by a vertex from V, and let W be the set of vertices from $V(D) - (V \cup V(T_p))$ which are not dominated by any vertex from V. Thus, $W \to V$, and hence it follows that $d(V, V(D) - V) \leq |V||U|$

and $d(V(D)-V,V) \geq |V||V(D)-(U\cup V)|$. Consequently, Lemma 2.5 implies $|V| \geq d(V(D)-V,V) - d(V,V(D)-V) \geq |V|(|V(D)-|V|-2|U|)$, and this yields

$$|U| \geq \frac{|V(D)|-|V|-1}{2}. \tag{7}$$

We now consider the following two subcases.

Subcase 2.1 Let $p=3$. If there exists any vertex $u \in U$ such that u dominates two vertices of T_p, then u, a vertex from V which dominates u, and the two vertices from T_p, which are not in the same partite set as u, induce a strong subtournament of order 4. If such a vertex does not exist, then, since every vertex of U has exactly two neighbors in T_p, we deduce that $d(U,V(T_p)) \leq d(V(T_p),U)$. If $w \in W$, then w also has exactly two neighbors in T_p, and hence it follows that $d(W,V(T_p)) \leq 2|W|$. In view of Lemma 2.5, we now obtain

$$\begin{aligned}
3 = |V(T_p)| &\geq d(V(T_p),V(D)-V(T_p)) - d(V(D)-V(T_p),V(T_p))\\
&= d(V(T_p),V) + d(V(T_p),U) + d(V(T_p),W) - d(V,V(T_p))\\
&\qquad - d(U,V(T_p)) - d(W,V(T_p))\\
&\geq |V(T_p)||V| + d(V(T_p),U) - d(U,V(T_p)) - d(W,V(T_p))\\
&\geq 3|V| - 2|W|\\
&= 3|V| - 2(|V(D)| - |V| - |U| - |V(T_p)|)\\
&= 5|V| - 2|V(D)| + 2|U| + 6.
\end{aligned}$$

Combining this estimation with (7), we find

$$|V(D)| - |V| - 1 \leq 2|U| \leq 2|V(D)| - 5|V| - 3, \tag{8}$$

and this implies $4|V|+2 \leq |V(D)|$. As $|V(D)| = |V|+|V_1|+|V_2|+|V_3| \leq |V|+3(k+2)$, we deduce that $3|V|+2 \leq 3(k+2)$. Because of $|V| \geq 2k$, this can only be valid for $k=1$, $|V|=2$, and $\alpha(D) = k+2 = 3$. However, in this case let $V = \{v_{n-1}, v_n\}$ and assume without loss of generality that $v_{n-1} \to v_n$. Because of $|V|=2$ and $\alpha(D) = k+2 = 3$, we note that $d^+(v_n) = (|V(D)|-1)/2$, and consequently, we obtain the inequality $|U| \geq d^+(v_n) = (|V(D)|-1)/2 = (|V(D)|-|V|+1)/2$. Analogously to (8), this leads to $|V(D)|-|V|+1 \leq 2|U| \leq 2|V(D)|-5|V|-3$. It follows that $12 \leq |V(D)|$, a contradiction.

Subcase 2.2 Let $p \geq 4$. According to Lemma 2.6, there exists a vertex $v \in V(T_p)$ such that for all $y \in V(T_p)-\{v\}$, the subtournament $T_p - y$ has a Hamiltonian path with the initial vertex v. If there is a vertex $u \in U$ with $u \to v$, then let $w \in V$ such that $w \to u$. If $u \in V_t$, then the vertices w, u, and v_j with $1 \leq j \leq p$ and $j \neq t$ induce a strongly connected subtournament of order $p+1$. If otherwise, there is no such vertex u, then clearly, $v \leadsto U$. By Lemma 2.6, the vertex v has maximum outdegree in T_p, and thus, $d^+(v,T_p) \geq 2$. If $v \in V_i$, then, because of $|V| \geq 2k > k \geq |V_i|-2$, it follows from (7)

$$d^+(v) \geq |V| + |U-(V_i-\{v\})| + d^+(v,T_p)$$

$$
\begin{aligned}
&\geq |V| + \frac{|V(D)| - |V| - 1}{2} - (|V_i| - 1) + 2 \\
&= \frac{|V(D)| - |V_i| + 3}{2} + \frac{|V| - |V_i| + 2}{2} \\
&> \frac{|V(D)| - |V_i| + 3}{2} \\
&\geq \frac{nk + r - k - 2 + 3}{2} = \frac{nk + r - k + 1}{2},
\end{aligned}
$$

a contradiction to (1), and the proof is complete. □

Corollary 3.2 If D is a regular n-partite tournament with $n \geq 4$, then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, n-1\}$.

The next example will show that Theorem 3.1 as well as Corollary 3.2 are best possible for $n = 4$.

Example 3.3 Let $V_i = V_i' \cup V_i''$ with $|V_i'| = |V_i''| = t$ for $i = 1, 2, 3, 4$ be the partite sets of a 4-partite tournament such that $V_1' \to V_2' \to V_3' \to V_1'$, $V_1'' \to V_2'' \to V_3'' \to V_1''$,

$$(V_1' \cup V_2' \cup V_3') \to V_4' \to (V_1'' \cup V_2'' \cup V_3'') \to V_4'' \to (V_1' \cup V_2' \cup V_3'),$$

$$V_1' \to V_3'' \to V_2' \to V_1'' \to V_3' \to V_2'' \to V_1'.$$

Now it is a simple matter to check that the resulting 4-partite tournament is $3t$-regular without a strongly connected subtournament of order 4.

4. Concluding Remarks and Open Problems

With similar methods we are able to prove the following results.

Theorem 4.1 Let D be an almost regular n-partite tournament such that $n \geq 5$. If $\gamma(D) < 3n/2 - 6$, then D contains a strongly connected subtournament of order n.

Theorem 4.2 Let D be a regular n-partite tournament with $n \geq 5$. If $\gamma(D) < 3n/2 - 2$, then D contains a strongly connected subtournament of order n.

Example 3.3, Theorem 4.1, and Theorem 4.2 leads us to the following conjectures, where, clearly, the second one is stronger than the first one.

Conjecture 4.3 Let D be a regular n-partite tournament with $n \geq 5$. Then D contains a strongly connected subtournament of order n.

Conjecture 4.4 Let D be an almost regular n-partite tournament with $n \geq 5$. Then D contains a strongly connected subtournament of order n.

In connection with our results and these conjectures one may ask the following question.

Problem 4.5 How close to regular must an n-partite tournament be, to secure a strongly connected subtournament of order n?

Problem 4.6 Does there exist a polynomial algorithm for finding the largest strongly connected subtournament in a multipartite tournament?

Problem 4.7 Determine other sufficient conditions for (strongly connected) n-partite tournaments to contain strong subtournaments of order p for some $4 \leq p \leq n$.

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