Towards the spectrum of critical sets

Diane Donovan* and Adelle Howse

Centre for Discrete Mathematics and Computing,
Department of Mathematics,
The University of Queensland,
Queensland, 4072, Australia

Abstract

This paper provides constructions that prove that critical sets exist of all sizes between $\lfloor \frac{n^2}{4} \rfloor$ and $\frac{n^2-n}{2}$, with the exception of $\frac{n^2}{4}+1$ for even n, in a latin square of order n.

1 Introduction

A *latin square* of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs precisely once in each row and column.
This paper establishes critical set constructions in latin squares of any order n, thus providing a basis for the spectrum of critical sets. Theorem 4 in Section 3 provides the basis for the main results of the paper, which are given in Theorems 17 and 30. The necessary background information is provided in the following section.

2 Definitions

In what follows the set N is assumed to be $\{0, 1, \ldots, n-1\}$. Further, the representation of a latin square with a set of ordered triples $\{(i,\ j;\ k) \mid$ element k occurs in position $(i,\ j)\ \}$ will be used often. In particular a *back circulant* latin square, denoted by BC_n, is given by the set $\{(i, j; i+j (\text{mod } n)) \mid 0 \leq i, j \leq n-1\}$.

A *partial latin square* P, of order n, is an $n \times n$ array where the entries in non-empty positions are chosen from a set N, in such a way that each element of N occurs at most once in each row and at most once in each column of the array. Let P be a partial latin square of order n. Then $|P|$ is said to be the *size* of the partial latin square and the set of positions $\mathcal{S}_P = \{(i,j) \mid (i,j;k) \in P, \exists k \in N\}$ is said to determine the *shape* of P. Let P and P' be two partial latin squares of the same order, with the same size and shape. Then P and P' are said to be *mutually balanced* if the entries in each row (and column) of P are the same as those in the

*Research Supported by Australian Research Council Grant A99/ARC105G.

corresponding row (and column) of P'. They are said to be *disjoint* if no position in P' contains the same entry as the corresponding position in P. A *latin interchange* I is a partial latin square for which there exists another partial latin square I', of the same order, size and shape with the property that I and I' are disjoint and mutually balanced. The partial latin square I' is said to be a *disjoint mate* of I. See Table 1 for an example. An *intercalate* is an example of a latin interchange of size four, and this is the smallest possible size for a latin interchange.

A partial latin square C, of order n is said to be *uniquely completable* (UC) (or to have *unique completion*) if there is precisely one latin square L of order n that has element k in position $(i,\, j)$ for each $(i,\, j;\, k) \in C$. A *critical set* is a partial latin square which is UC, and for which the removal of any entry destroys this property. An example is presented in Table 1.

0	*	2
2	3	4
3	4	0

2	*	0
3	4	2
0	3	4

0	1	2	*	*	*	*
1	2	*	*	*	*	*
2	*	*	*	*	*	*
*	*	*	*	*	*	*
*	*	*	*	*	*	3
*	*	*	*	*	3	4
*	*	*	*	3	4	5

Table 1: A latin interchange (on the left) of size 8 with its disjoint mate (in the centre), and a critical set (on the right) of order 7 and size 12.

LEMMA 1 [2] *A partial latin square $C \subset L$, of size s and order n, is a critical set for a latin square L if and only if the following hold:*

1. *C contains an element of every latin interchange that occurs in L;*
2. *for each $(i,\, j;\, k) \in C$, there exists a latin interchange I in L so that $I \cap C = \{(i,\, j;\, k)\}$.*

LEMMA 2 [2] *For any latin interchange I, (with disjoint mate I') and any critical set C in a latin square L, the set $(C \setminus I) \cup I'$ has unique completion to $(L \setminus I) \cup I'$.*

In 1978, Curran and van Rees [5] produced a UC set for BC_n of size $\frac{n^2-1}{4}$ for odd values of n. For even n, they produced a critical set of size $\frac{n^2}{4}$. These findings have been expanded on to provide the following general result proved by Donovan and Cooper [1]. This result, and the partial latin square denoted by S, will form an integral part of the main results in this paper.

LEMMA 3 *The partial latin square*

$$\begin{aligned} S = &\{(i, j; i + j (\text{mod } n)) \mid 0 \le i \le a, 0 \le j \le a - i\} \cup \\ &\{(i, j; i + j (\text{mod } n)) \mid a + 2 \le i \le n - 1, n + 1 + a - i \le j \le n - 1\}, \end{aligned}$$

where a is an integer such that $\frac{n-3}{2} \le a \le n - 2$, is a critical set for BC_n.

3 Interesting and useful latin interchanges

This section focuses on constructing latin interchanges which will be used in the proof of the main results of this paper.

For $r \geq c$, let $\mathcal{A}$ denote an $r \times c$ array, with rows indexed by x for $0 \leq x \leq r-1$, and columns indexed by z for $0 \leq z \leq c-1$, that satisfies the following:

1. the symbol occurring in the position corresponding to the first row and first column, also occurs in the last column; and

2. the entry occurring in row x and column z of $\mathcal{A}$ is the symbol i_{x+z}.

Thus, for $0 \leq x \leq r-1$, row x of $\mathcal{A}$ contains the symbols $i_{x+0}, i_{x+1}, i_{x+2}, \ldots, i_{x+c-1}$, in the given order. Let y denote the row that contains symbol i_0 in column $c-1$. Hence, $i_{y+c-1} = i_0$. Note that if $\mathcal{A}$ is embedded in BC_n, then these properties are satisfied if $r + c > n + 1$. Further, $y = n + 1 - c$.

We define the sequence of numbers $\alpha_1, \alpha_2, \ldots$ to be integers where

$$\alpha_1 = c - 1 (\text{mod } y), \text{ and for } i \geq 2,$$
$$\alpha_i = \alpha_{i-1} (\text{mod } y - \alpha_1 - \ldots - \alpha_{i-1}).$$

That is, α_1 is the remainder when $c-1$ is divided by y and α_i the remainder when α_{i-1} is divided by $y - \alpha_1 - \ldots - \alpha_{i-1}$. Clearly, there exists a value P where $\alpha_P \neq 0$ and $\alpha_{P+i} = 0$ for all $i > 0$. We are thus interested in the sequence $\alpha_1, \alpha_2, \ldots, \alpha_P$. For $i = 1, 2, \ldots, P$, let $\delta_i = \alpha_1 + \alpha_2 + \ldots + \alpha_i$. We use these integers to define sets A_0, B_0, A_1, B_1, $\ldots$, A_P, B_P, each occurring in $\mathcal{A}$ as follows. Note however, that if $\alpha_1 = c - 1$ then the set B_0 is empty, and for $1 \leq i \leq P-1$, if $\alpha_i = \alpha_{i+1}$ then the set B_i is empty.

$$A_0 = \{(0,\ 0;\ i_0),\ (0,\ c-1;\ i_{c-1})\} \text{ and if } \alpha_1 \neq c-1 \text{ define}$$

$$B_0 = \{(c-1-ay,\ ay;\ i_{c-1}),\ (c-1-ay,\ (a+1)y;\ i_0) \mid 0 \leq a \leq \frac{c-1-\alpha_1}{y} - 1\}.$$

If $\alpha_1 \neq 0$, define

$$A_1 = \{(y,\ c-1-\alpha_1;\ i_{c-1+y-\alpha_1}), (y,\ c-1\ ;\ i_0)\}, \text{ and if } \alpha_1 \neq \alpha_2 \text{ define}$$

$$\begin{aligned} B_1 = \{&(\alpha_1 - a(y-\alpha_1),\ c-1-\alpha_1+a(y-\alpha_1)\ ;\ i_{c-1}),\\ &(\alpha_1 - a(y-\alpha_1),\ c-1-\alpha_1+(a+1)(y-\alpha_1)\ ;\ i_{c-1+y-\alpha_1})\\ &\mid 0 \leq a \leq \frac{\alpha_1-\alpha_2}{y-\alpha_1} - 1\}. \end{aligned}$$

If $P \geq 2$, for $2 \leq i \leq P$, define

$$A_i = \{(y-\delta_{i-1},\ c-1-\alpha_i\ ;\ i_{c-1+y-\delta_i}), (y-\delta_{i-1},\ c-1\ ;\ i_{c-1+y-\delta_{i-1}})\},$$

and if $\alpha_i \neq \alpha_{i+1}$, define

$$\begin{aligned} B_i = \{&(\alpha_i - (y-\delta_i)a,\ c-1-\alpha_i + a(y-\delta_i)\ ;\ i_{c-1}), \\ &(\alpha_i - a(y-\delta_i),\ c-1-\alpha_i + (a+1)(y-\delta_i)\ ;\ i_{c-1+y-\delta_i}), \\ &\mid 0 \le a \le \frac{\alpha_i - \alpha_{i+1}}{y-\delta_i} - 1\}. \end{aligned}$$

THEOREM 4 *Let the sequence of integers $\alpha_1, \alpha_2, \ldots, \alpha_P$ be defined as above. Then the set $I = A_0 \cup B_0 \cup A_1 \cup B_1 \cup \ldots \cup A_P \cup B_P$, is a latin interchange in the array $\mathcal{A}$.*

Proof It is easy to check that the set $I = A_0 \cup B_0 \cup A_1 \cup B_1 \cup \ldots \cup A_P \cup B_P$, is a partial latin square contained in $\mathcal{A}$. Note that each row of I contains precisely two entries. Construct a partial latin square I' which has the same positions in $\mathcal{A}$ as I does, but with the entries in each corresponding row of I' interchanged. The result is a partial latin square I' of the same size and shape and which is disjoint from I. Note also that column j, for $j \notin \{c-1-\alpha_1, c-1-\alpha_2, \ldots, c-1-\alpha_P, c-1\}$, contains precisely two entries and these entries are the same in both I and I'. Therefore to complete the proof that I is a latin interchange, it is necessary to prove that columns $c-1-\alpha_1$, $c-1-\alpha_2$, $\ldots$, $c-1-\alpha_P$, $c-1$, of I and I' are balanced.

If $\alpha_1 > \alpha_2 > \ldots > \alpha_P$ then column $c-1-\alpha_1$ of I contains the symbols i_0, $i_{c-1+y-\alpha_1}$ and i_{c-1}, in the rows $y+\alpha_1$, y and α_1, respectively. Columns $c-1-\alpha_1$ of I' contains the symbols i_{c-1}, i_0 and $i_{c-1+y-\alpha_1}$, in the rows $y+\alpha_1$, y and α_1, respectively. And for $2 \le k \le P$, column $c-1-\alpha_k$ of I contains the symbols $i_{c-1+y-\delta_{k-1}}$, $i_{c-1+y-\delta_k}$ and i_{c-1}, in the rows $y-\delta_{k-1}+\alpha_k$, $y-\delta_{k-1}$ and α_k, respectively. Columns $c-1-\alpha_k$ of I' contains the symbols i_{c-1}, $i_{c-1+y-\delta_{k-1}}$ and $i_{c-1+y-\delta_k}$, in the rows $y-\delta_{k-1}+\alpha_k$, $y-\delta_{k-1}$ and α_k, respectively.

If $\alpha_g = \alpha_{g+1} = \ldots = \alpha_{g+h}$, where $g \in \{1, \ldots, P\}$ and $1 \le h \le P$. Then the column $c-1-\alpha_g$ contains symbols $i_{c-1}+y-\delta_{g-1}, i_{c-1}+y-\delta_g, \ldots i_{c-1}+y-\delta_{g+h}$ and i_{c-1} as does I'.

Column $c-1$ of I contains the symbols i_0 in row y, $i_{c-1+y-\alpha_1}$ in row $y-\alpha_1$, and for $2 \le k \le P$, $i_{c-1+y-\delta_k}$ in row $y-\delta_k$, and finally i_{c-1} in row 0. Column $c-1$ of I' contains the symbols $i_{c-1+y-\alpha_1}$ in row y, and for $2 \le k \le P$, $i_{c-1+y-\delta_k}$ in row $y-\delta_{k-1}$, and then i_{c-1} in row $y-\delta_P$, and i_0 in row 0.

Consequently I and I' are column balanced and I is a latin interchange with disjoint mate I'. □

It should be noted that each element $(i, j; k)$ of S, (see Lemma 3), occurs in a latin interchange in BC_n, of the form given in Theorem 4. Further, this latin interchange intersects S in $(i, j; k)$ alone. This observation, together with the fact that BC_n is symmetric, can be used to prove that S is a critical set in BC_n, and drastically reduces the complexity of the proof given by Donovan and Cooper [1].

An example of the latin interchange, with $r = c = 13$, corresponding to Theorem 4 is displayed in Table 2.

0	*	*	*	*	*	*	*	*	*	*	*	12
*	*	*	*	*	*	*	*	*	*	*	12	13
*	*	*	*	*	*	*	*	*	*	*	13	14
*	*	*	*	*	*	*	*	*	12	*	14	*
*	*	*	*	*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	12	*	14	*	*	*
*	*	*	*	*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	14	*	*	*	*	0
*	*	*	*	*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*	*	*	*	*
12	*	*	*	*	*	*	0	*	*	*	*	*

Table 2: A latin interchange, as per Theorem 4.

COROLLARY 5 *Assume that $r < c$, and take an $r \times c$ array, $\mathcal{A}^T$ where symbol i_0 occurs in row $r-1$ of $\mathcal{A}^T$, in say, column y. Further, column x, for $0 \leq x \leq c-1$, contains the integers*

$$i_{0+x}, i_{1+x}, i_{2+x}, \ldots, i_{r-1+x},$$

in the given order. In particular, $i_{r-1+y} = i_0$. It follows that $\mathcal{A}^T$ contains a latin interchange of the form $I^T = A_0^T \cup B_0^T \cup A_1^T \cup B_1^T \cup \ldots \cup A_P^T \cup B_P^T$, where A_i^T and B_i^T are, respectively, the transpose of the sets A_i and B_i given above. □

COROLLARY 6 *For $r < c$ suppose an $r \times c$ array $\mathcal{B}$ satisfies:*

1. *the symbol in position $(r-1, c-1)$ also occurs in row 0; and*
2. *the symbol in position $(0, c-1)$, also occurs in row $r-1$; and*
3. *the symbol in row x and column z is i_{x+z}.*

Then a latin interchange I exists in $\mathcal{B}$ that is isotopic to the latin interchange of Corollary 5.

Proof Apply the permutation $\alpha(x) = r - 1 - x$ to row x of $\mathcal{B}$ for $0 \leq x \leq r-1$. Then, apply the permutation $\beta(z) = c - 1 - z$ to column z of $\mathcal{B}$ for $0 \leq z \leq c-1$. Let $\mathcal{A}$ denote the transpose of $\mathcal{B}$ after the action of α and β. Then $\mathcal{A}$ satisfies the requirements to be used in Theorem 4. □

Note in the array $\mathcal{A}$, with $r \geq c$, that row y may be either less than, equal to, or greater than $c-1$. In particular, if $\mathcal{A}$ is embedded in BC_n, and if $c > \frac{n+2}{2}$, then $y < c-1$; if $c < \frac{n+2}{2}$, then $y > c-1$; and if $c = \frac{n+2}{2}$, then $c - 1 = y = \frac{n}{2}$. The latin interchange I previously defined is suitable for all these cases. Note that, in the event that $y = c-1$, I is simply an intercalate. Later in this paper it will be necessary to construct latin interchanges which do not intersect specific rows or columns. To meet

this need we now consider another two constructions. The first deals specifically with the case where $y < c-1$, and the second the case where $y > c-1$.
Let $\mathcal{A}$ be an $r \times c$ array where $r > c$ and $y < c-1$. Then, define β_i as follows:

$$\beta_1 = c(\text{mod } y),$$
$$\beta_i = \beta_{i-1}(\text{mod } y - \beta_1 - \ldots - \beta_{i-1}).$$

It is clear that there exists a Q such that $\beta_Q \neq 0$, and for all $i > Q$, $\beta_i = 0$. For $1 \leq i \leq Q$, let $\gamma_i = \beta_1 + \ldots + \beta_i$. Define the sets JA_0, $JB_0, \ldots,$ JA_Q, JB_Q as follows, noting that if $y > \frac{c}{2}$ then the set JB_0 will be empty and if $\beta_i = \beta_{i+1}$, then the set JB_i will be empty.

$$JA_0 = \{(c-1,\, b;\, i_{c-1+b}), (c,\, b;\, i_{c+b}) \mid 0 \leq b \leq y-1\} \text{ and if } y \leq \frac{c}{2} \text{ define}$$
$$JB_0 = \{(c-ay,\, ay-1;\, i_{c-1}),\ (c-ay,\, (a+1)y-1;\, i_0) \mid 1 \leq a \leq \frac{c-\beta_1}{y} - 1\}.$$

If $\beta_1 \neq 0$ define

$$JA_1 = \{(y, c-1-\beta_1\ ;\ i_{c-1+y-\beta_1}), (y, c-1\ ;\ i_0)\}, \text{ and if } \beta_1 \neq \beta_2$$
$$JB_1 = \{(\beta_1 - a(y-\beta_1), c-1-\beta_1 + a(y-\beta_1)\ ;\ i_{c-1}),$$
$$(\beta_1 - a(y-\beta_1), c-1-\beta_1 + (a+1)(y-\beta_1)\ ;\ i_{c-1+y-\beta_1})$$
$$\mid 0 \leq a \leq \frac{\beta_1-\beta_2}{y-\beta_1} - 1\}.$$

If $Q \geq 2$, for $2 \leq i \leq Q$, define

$$JA_i = \{(y-\gamma_{i-1}, c-1-\beta_i\ ;\ i_{c-1+y-\gamma_i}), (y-\gamma_{i-1}, c-1\ ;\ i_{c-1+y-\gamma_{i-1}})\},$$

and if $\beta_i \neq \beta_{i+1}$ define

$$JB_i = \{(\beta_i - (y-\gamma_i)a,\ c-1-\beta_i + a(y-\gamma_i)\ ;\ i_{c-1}),$$
$$(\beta_i - a(y-\gamma_i),\ c-1-\beta_i + (a+1)(y-\gamma_i)\ ;\ i_{c-1+y-\gamma_i}),$$
$$\mid 0 \leq a \leq \frac{\beta_i - \beta_{i+1}}{y-\gamma_i} - 1\}.$$

THEOREM 7 *Let the sequence of integers $\beta_1, \beta_2, \ldots, \beta_Q$ be defined as above. Then the set $J = A_0 \cup JA_0 \cup JB_0 \cup JA_1 \cup JB_1 \cup \ldots \cup JA_Q \cup JB_Q$, is a latin interchange in the array $\mathcal{A}$, when $y < c-1$.*

Proof Let J' be a partial latin square with the same size and shape as J. In J all the nonempty rows, with the exception of row $c-1$ and c, contain precisely two entries. The corresponding rows of J' will contain the same symbols chosen from N but with their positions interchanged. Now fill the remaining positions of J' with the entries $(c-1, 0; i_c)$, $(c-1, b; i_{c+b})$, $(c-1, y-1; i_{c-1})$, $(c, 0; i_0)$, $(c, b; i_{c-1+b})$, $(c, y-1; i_{c-2+y})$ for $1 \leq b \leq y-2$. The proof of Theorem 4 can then be used to prove that the columns of J and J' are mutually balanced and thus that J is a latin interchange. □

COROLLARY 8 *Let $r > c$ and $1 < y < c-1$, where y is the row that contains symbol i_0 in column $c-1$. Consider the latin interchanges $I = A_0 \cup B_0 \cup A_1 \cup B_1 \cup \ldots \cup A_P \cup B_P$, and $J = A_0 \cup JA_0 \cup JB_0 \cup JA_1 \cup JB_1 \cup \ldots \cup JA_Q \cup JB_Q$. Both of these are contained in $\mathcal{A}$. Let $\sigma = \{j \mid y \leq j \leq c-1-y\}$ denote a subset of the columns of $\mathcal{A}$. Then J and I do not intersect in any column j of $\mathcal{A}$ if $j \in \sigma$. Hence, if I intersects column j of $\mathcal{A}$, where $j \in \sigma$, then J does not.*

Proof First note that $\alpha_1 \neq \beta_1$. To see this, note there are two cases: either $\alpha_1 = y-1$, $\alpha_2 = 0$ and $\beta_1 = 0$; or, $\alpha_1 < y-1$ and $\beta_1 = \alpha_1 + 1$. In either case, we have $0 \leq \alpha_1 < y$ and $0 \leq \beta_1 < y$. Hence, $c-1-y < c-1-\alpha_1$ and $c-1-y < c-1-\beta_1$. Thus, the entries in column j of I for $y \leq j \leq c-1-y$, are taken from the set B_0, and each column contains precisely two symbols i_{c-1} and i_0. The columns used by B_0 in this range are $y, 2y, 3y, \ldots, c-1-y-\alpha_1$.

Similarly, in this range, entries for J are taken from the set JB_0. Again each column contains precisely two entries, i_{c-1} and i_0, and the columns used are $2y-1, 3y-1, \ldots, c-1-y-\beta_1$.

As $y > 1$, these two sets of columns are disjoint, and the result follows. □

COROLLARY 9 *Let $r > c$, $y < c-1$ and suppose that the partial latin square I as defined above intersects row j of $\mathcal{A}$ where $y < j < c-1$. Then the partial latin square J as defined above does not intersect I in row j.* □

We now consider the case where $y > c-1$. Note that $\alpha_1 = c-1$ and so the set B_0 is empty. For this case, we set

$$\beta_1 = c-1 (\text{mod } y-1);$$
$$\beta_i = \beta_{i-1} (\text{mod } y-1-\beta_1-\ldots-\beta_{i-1});$$

and

$$\begin{aligned} JA_1 &= \{(y, b; i_{y+b}), (y-1, b; i_{y-1+b}) \mid 0 \leq b \leq c-1\}; \text{ and if } \beta_1 \neq 0, \\ JB_1 &= \{(c-1-a(y-1-c+1), a(y-1-c+1); i_{c-1}), \\ &\quad (c-1-a(y-1-c+1), (a+1)(y-1-c+1); i_{y-1}), \\ &\quad \mid 0 \leq a \leq \frac{c-1-\beta_2}{y-1-c+1} - 1\}. \end{aligned}$$

If $Q \geq 2$, for $2 \leq i \leq Q$, define

$$\begin{aligned} JA_i &= \{(y-1-\gamma_{i-1}, c-1-\beta_i \; ; \; i_{c-1+y-1-\gamma_i}); \\ &\quad (y-1-\gamma_{i-1}, c-1 \; ; \; i_{c-1+y-1-\gamma_{i-1}})\}, \text{ and if } \beta_i \neq \beta_{i+1}, \\ JB_i &= \{(\beta_i - (y-1-\gamma_i)a, c-1-\beta_i + a(y-1-\gamma_i) \; ; \; i_{c-1}), \\ &\quad (\beta_i - a(y-1-\gamma_i), c-1-\beta_i + (a+1)(y-1-\gamma_i) \; ; \; i_{c-1+y-1-\gamma_i}) \\ &\quad \mid 0 \leq a \leq \frac{\beta_i - \beta_{i+1}}{y-1-\gamma_i} - 1\}, \end{aligned}$$

where $\gamma_i = \beta_1 + \ldots + \beta_i$. Again note that if $\beta_i = \beta_{i+1}$ then the set JB_i is empty.

THEOREM 10 *Let the sequence of integers $\beta_1, \beta_2, \ldots, \beta_Q$ be defined as above. Then the set $J = A_0 \cup JA_1 \cup JB_1 \cup JA_2 \cup JB_2 \cup \ldots \cup JA_Q \cup JB_Q$, is a latin interchange in the array $\mathcal{A}$, when $y > c - 1$.*

Proof The proof follows that of the previous theorems. □

COROLLARY 11 *Let $r > c$, $y > c - 1$ and consider the latin interchanges $I = A_0 \cup B_0 \cup A_1 \cup B_1 \cup \ldots \cup A_P \cup B_P$, and $J = A_0 \cup JA_0 \cup JB_0 \cup JA_1 \cup JB_1 \cup \ldots \cup JA_Q \cup JB_Q$. Both of these are contained in $\mathcal{A}$. Let $\sigma = \{j \mid c-1 < j < y-1\}$ denote a subset of the rows of $\mathcal{A}$. Then if I intersects row j of $\mathcal{A}$, where $j \in \sigma$, then J does not.*

Proof The row j is between $y-1$ and $c-1$ and any such row of J is empty. Hence if I intersects row j, $c-1 < j < y-1$, J will not. □

4 The size of a critical set

From Lemma 3, we note that for every order n and each integer a such that $\frac{n-3}{2} \leq a \leq n-2$, the set

$$\begin{aligned} S = &\{(i, j; i+j (\bmod n)) \mid 0 \leq i \leq a, 0 \leq j \leq a - i\} \cup \\ &\{(i, j; i+j (\bmod n)) \mid a+2 \leq i \leq n-1, n+1+a-i \leq j \leq n-1\}, \end{aligned}$$

provides us with an example of a critical set of size $a^2 + 3a + 2 + \frac{1}{2}n^2 - \frac{3}{2}n - na$.

For the remainder of this paper S will denote the set given above.

When $a = n-2$, the critical set S is of size $\frac{n^2-n}{2}$. If n is even and $a = \frac{n}{2} - 1$ then the critical set is of size $\frac{n^2}{4}$ and if n is odd and $a = \frac{n-3}{2}$ the critical set is of size $\frac{n^2-1}{4}$. With one exception, it will be shown that for each n there exists a critical set of every size r where $\lfloor \frac{n^2}{4} \rfloor \leq r \leq \frac{n^2-n}{2}$. The exception refers to the case where n is even and the size of the possible critical set is $\frac{n^2}{4} + 1$. To date, a general construction is not known for a critical set of this size. For $n = 4$, a critical set of size 5, is known for $C_2 \times C_2$ and for BC_4. For $n = 6$, many critical sets exist of size 10 in latin squares other than BC_6 (see [4]). However, we observe the following new result.

LEMMA 12 *No critical set of size 10 exists for BC_6.*

Proof Utilization of the method by one of the authors in [4] produces all UC sets of size 10. Each of these is found to contain a critical set of size 9, and hence no set of size 10 can be a critical set. □

To achieve the main results of this paper, we consider the cases n even and n odd separately.

4.1 Even n

In a back circulant latin square of even order the set of elements

$$I_2 = \{(1, \frac{n}{2} - 1; \frac{n}{2}), (1, n-1; 0), (\frac{n}{2} + 1, \frac{n}{2} - 1; 0), (\frac{n}{2} + 1, n-1; \frac{n}{2})\}$$

forms an intercalate. Let I'_2 denote the disjoint mate of this intercalate.

Using Lemma 2 we see that $(S \setminus I_2) \cup I'_2$ has unique completion. This fact can be used, initially (see Lemma 13), to establish the existence of a critical set of size $|S|+2$ and then ultimately to establish critical sets of size $\frac{n^2}{4} + 2$ to $\frac{n^2-n}{2} - 1$ (see Theorem 17). These critical sets will define latin squares that differ from BC_n in columns $\frac{n}{2}-1$ and $n-1$. Therefore, in the proof of the various results, care will be taken to find latin interchanges which, as necessary, do not intersect columns $\frac{n}{2} - 1$ and $n-1$. In the proof of Lemma 13 this requirement is not strictly necessary, however it is adhered to as it will simplify the proof of subsequent results. For clarity, the information necessary to validate that each of the elements of the relevant partial latin square is necessary for unique completion has been summarised in a table. Each line of the table lists details which can be used to find a latin interchange which intersects the partial latin square in the element $(i, j; k)$ alone, where i and j are in the range given, respectively, in columns 1 and 2 of the table. In most cases $k = i + j (\mathrm{mod}\ n)$ but if this is not so the specific value of k will be given in column 3. To minimize space we have used the notation $\{x, y\}$ to mean the $\min\{x, y\}$.

LEMMA 13 *If n is even, $n > 4$ and $\frac{n}{2} \le a \le n-3$, the set $I_2 = \{(1, \frac{n}{2} - 1; \frac{n}{2}), (1, n-1; 0), (\frac{n}{2} + 1, \frac{n}{2} - 1; 0), (\frac{n}{2} + 1, n-1; \frac{n}{2})\}$ forms an intercalate in BC_n. Denote the disjoint mate of I_2 by I'_2. Then the partial latin square*

$$C_2 = (S \setminus I_2) \cup \{(1, \frac{n}{2} - 1; 0), (\frac{n}{2} + 1, \frac{n}{2} - 1; \frac{n}{2}), (\frac{n}{2} + 1, n-1; 0)\}$$

is a critical set in $(BC_n \setminus I_2) \cup I'_2$. The size of C_2 is $|S| + 2$.

Proof We first show that C_2 has UC. Consider row 0. For $k = a+1, a+2, \ldots, n-1$, element k must occur in position $(0, k)$. Next, in row 1, element $a+1$ is forced to occur in $(1, a)$. Then, for $k = a+1, a+2, \ldots, n-2$, element $k+1$ must occur in position $(1, k)$. Then, element $\frac{n}{2}$ is forced to occur in cell $(1, n-1)$ and so the element $(1, n-1; \frac{n}{2})$ is not necessary for unique completion and so by Lemma 2 the given set has unique completion.

It will be shown that all remaining elements form a critical set of size $|S| + 2$.

RANGE OF i	RANGE OF j	COMMENTS
$a+2$ to $n-1$	$n+1+a-i$ to $n-1$	Use an intercalate on rows $i - \frac{n}{2}$ to i, columns $j - \frac{n}{2}$ to j.

Range of i	Range of j	Comments
0 to j	0 to $\lfloor\frac{n-6}{4}\rfloor$	Use Theorem 4 on columns j to $n-2$, rows i to $n-2-j+i$. As $i+j$ occurs in row $i+j+2$ of column $n-2$, $y=j+2$ giving $j+y \leq \frac{n-1}{2}$, and $c-1=n-2-j$ giving $y < c-1$ and $n-2-y \geq \frac{n}{2}-1$. Thus use Corollary 8 to avoid column $\frac{n}{2}-1$, if necessary.
0 to $\{j,\ a-j\}$	$\lfloor\frac{n-6}{4}\rfloor+1$ to $\frac{n}{2}-2$	Use Theorem 4 on columns j to $n-2$, rows i to $n-2+i-j$. Note column y of $\mathcal{A}$ is column $2j+2 \geq \frac{n}{2}+1$ in BC_n, so column $\frac{n}{2}-1$ is not intersected.
1 to $\frac{n}{2}-3$	0 to $\{i-1, \frac{n}{2}-3-i\}$	Consider the transpose and use Theorem 4 on columns i to $n-1$, and rows j to $n-1+j-i$ in the transpose. Since $y=i+1$, $c-1=n-1-i$, $y<c-1$, $j+y<\frac{n}{2}-1<j+c-1$. Thus use Corollary 9 to avoid row $\frac{n}{2}-1$, and so column $\frac{n}{2}-1$ in the original array.
$\lfloor\frac{n}{4}\rfloor+1$ to $\frac{n}{2}-1$	$\frac{n}{2}-2-i$ to $\{i-1, a-i\}$	Use intercalates on rows i to $i+\frac{n}{2}$, columns j to $j+\frac{n}{2}$.
$\frac{n}{2}$ to a	0 to $\{i-\frac{n}{2}, a-i\}$	Consider the transpose and use Theorem 4 on columns i to $n-2$, rows j to $i+j+2$, or since $y=i+2, c-1=n-2-i$, $y>c-1, j+c-1<\frac{n}{2}-1<y-1+j$. Thus use Corollary 11 to avoid row $\frac{n}{2}-1$, and so column $\frac{n}{2}-1$ in the transpose.
$\frac{n}{2}$ to $\{\frac{n}{2}-1+j, a-j\}$	1 to $\lfloor\frac{n-6}{4}\rfloor$	Consider the transpose and use Theorem 4 on columns i to $n-2-j+(i-\frac{n}{2})$ and rows j to $\frac{n}{2}+2j+2$. Note that $c-1=\frac{n}{2}-2-j$ and $y=\frac{n}{2}+2+j$ and that $j+c-1<\frac{n}{2}-1<j+y$, so use Corollary 11 to avoid row $\frac{n}{2}-1$.

Range of i	Range of j	Comments
$\frac{n}{2}$ to $a-j$	$\lfloor\frac{n-6}{4}\rfloor+1$ to $\frac{n}{2}-3$	Use Corollary 6 on rows i to $i+j+2$ and columns j to $n-2$, since there are no entries in columns $j+1$ to $2j$ column $\frac{n}{2}-1$ is avoided.
0 to $a-j$	$\frac{n}{2}$ to a	Use Theorem 4 on columns j to $n-2$, rows i to $i+j+2$.
0 and 2 to $a-\frac{n}{2}-1$	$\frac{n}{2}-1$	Use intercalates on rows i and $\frac{n}{2}+i$ with columns $\frac{n}{2}-1$ and $n-1$.
1	$\frac{n}{2}-1$	For symbol 0, use Theorem 4 on columns $\frac{n}{2}-1$ to $n-2$, rows 1 to $\frac{n}{2}$.
$\frac{n}{2}+1$	$\frac{n}{2}-1$	For symbol $\frac{n}{2}$, use Theorem 4 on columns $\frac{n}{2}-1$ to $n-3$, rows $\frac{n}{2}+1$ to $n-1$. It intersects column $\frac{n}{2}-1$ in rows $\frac{n}{2}+1$ and $n-1$ only. Note for $n=6$ use column $\frac{n}{2}-1$ to $n-2$.
$\frac{n}{2}+1$	$n-1$	For symbol 0, use Theorem 4 on columns $\frac{n}{2}$ to $n-1$, rows $\frac{n}{2}$ to $\frac{n}{2}+1$. □

In what follows, let $T_k[I_i] = I_1 \cup I_2 \cup \ldots \cup I_k$ and let $T'_k[I_i]$ denote the disjoint mate.

COROLLARY 14 *If n is even and $\frac{n}{2} \leq a \leq n-3$, the set $I_i = \{(i, \frac{n}{2}-1; \frac{n}{2}-1+i), (i, n-1; i-1), (\frac{n}{2}+i, \frac{n}{2}-1; i-1), (\frac{n}{2}+i, n-1; \frac{n}{2}+i-1)\}$, for $i = 1, \ldots, a-\frac{n}{2}+1$, forms an intercalate in BC_n. Denote the disjoint mate of I_i by I'_i, Then for $1 \leq k \leq a-\frac{n}{2}+1$, the partial latin square*

$$C_{2k} = (S \setminus T_k[I_i]) \cup T'_k[I_i \setminus (i, n-1; i-1)]$$

is a critical set in $(BC_n \setminus T_k[I_i]) \cup T'_k[I_i]$. The size of C_{2k} is $|S| + 2k$.

Proof The proof of UC follows the argument given in the proof of Lemma 13. We now show that all elements are necessary.

The necessity of the elements in $S \setminus T_k[I_i]$ follows from Lemma 13.

Range of i	Range of j	Comments
1 to $a-\frac{n}{2}+1$	$\frac{n}{2}-1$	For symbol $i-1$, use Theorem 4, (or Corollary 5, if $i > \frac{3n}{4}$) on columns $\frac{n}{2}-1$ to $n-1-i$ ($\frac{n}{2}-1+i+j+1$) rows i to $\frac{n}{2}$.
$\frac{n}{2}+1$ to $a+1$	$\frac{n}{2}-1$	For symbol $i-1$, use Theorem 4 (or Corollary 5, if $i \geq \frac{n}{4}-2$) on columns $\frac{n}{2}-1$ to $\frac{3n}{2}-i-2$, and rows i to $n-1$.
$\frac{n}{2}+1$ to $a+1$	$n-1$	For symbol $i-\frac{n}{2}-1$, use Corollary 6 on rows $\frac{n}{2}$ to i, and columns $\frac{n}{2}$ to $n-1$. □

Similar arguments can be used to verify the following theorem.

LEMMA 15 *If n is even the set $J_3 = \{(a+2-\frac{n}{2},\frac{n}{2}-1;a+1),(a+2-\frac{n}{2},n-1;a-\frac{n}{2}+1),(a+2,\frac{n}{2}-1;a-\frac{n}{2}+1),(a+2,n-1;a+1)\}$ forms an intercalate in BC_n. Denote the disjoint mate of J_3 by J_3'. Then partial latin square*

$$C_3 = (S \setminus J_3) \cup J_3'$$

is a critical set in $(BC_n \setminus J_3) \cup J_3'$. The size of C_3 is $|S|+3$.

Proof Unique completion follows directly from Lemma 2. A similar argument to that used in the proof of Lemma 13 can be used to verify that each of the elements of the set $S \setminus J_3$ is necessary for UC. (For full details see [3].) □

COROLLARY 16 *If n is even the set $I_i = \{(i,\frac{n}{2}-1;\frac{n}{2}-1+i),(i,n-1;i-1),(\frac{n}{2}+i,\frac{n}{2}-1;i-1),(\frac{n}{2}+i,n-1;\frac{n}{2}-1+i)\}$, for $i=1,\ldots,a-\frac{n}{2}+1$, forms an intercalate in BC_n, as does the set $J_3 = \{(a+2-\frac{n}{2},\frac{n}{2}-1;a+1),(a+2-\frac{n}{2},n-1;a-\frac{n}{2}+1),(a+2,\frac{n}{2}-1;a-\frac{n}{2}+1),(a+2,n-1;a+1)\}$. Denote the disjoint mate of I_i by I_i' and the disjoint mate of J_3 by J_3'. Then for $k=1,\ldots,a-\frac{n}{2}+1$, the partial latin square*

$$C_{2k+3} = (S \setminus (J_3 \cup T_k[I_i])) \cup T_k'[\,I_i \setminus (i,n-1;i-1)\,] \cup J_3'$$

is a critical set in $(BC_n \setminus (J_3 \cup T_k[I_i]\,) \cup T_k[I_i'\,] \cup J_3'$. The size of C_{2k+3} is $|S|+2k+3$. (Note, for $a=n-3$, the range for k is restricted to $1 \leq k \leq a-\frac{n}{2}$).

THEOREM 17 *When n is even, $n>4$, there exists a critical set of order n and size $s \in \{\frac{n^2}{4},\frac{n^2}{4}+2,\frac{n^2}{4}+3,\ldots,\frac{n^2-n}{2}-1,\frac{n^2-n}{2}\}$.*

Proof Let n be an even number, and fix a such that $\frac{n}{2}-1 \leq a \leq n-2$. From Lemma 3 there exists a critical set of size $a^2+3a+2+\frac{1}{2}n^2-\frac{3}{2}n-na$. If a is restricted to the range $\frac{n}{2}-1 \leq a \leq n-3$. Corollary 14 provides examples of critical sets of sizes

$$a^2+3a+2+\frac{1}{2}n^2-\frac{3}{2}n-na+\alpha,$$

where $\alpha \in \{2,4,\ldots,2(a-\frac{n}{2}+1)\}$. Note that $a^2+3a+2+\frac{1}{2}n^2-\frac{3}{2}n-na+2(a-\frac{n}{2}+1) = (a+1)^2+3(a+1)+2+\frac{1}{2}n^2-\frac{3}{2}n-n(a+1)-2$. Likewise if we restrict a to the range $\frac{n}{2}-1 \leq a \leq n-3$, Lemma 15 provides a critical set of size $a^2+3a+2+\frac{1}{2}n^2-\frac{3}{2}n-na+3$. Similarly, when a satisfies $\frac{n}{2} \leq a \leq n-4$, Corollary 16 proves the existence of critical sets of sizes

$$a^2+3a+2+\frac{1}{2}n^2-\frac{3}{2}n-na+\alpha,$$

where $\alpha \in \{5,7,\ldots,2(a-\frac{n}{2})+3, 2(a-\frac{n}{2}+1)+3\}$. Note that $a^2+3a+2+\frac{1}{2}n^2-\frac{3}{2}n-na+2(a-\frac{n}{2})+3 = (a+1)^2+3(a+1)+2+\frac{1}{2}n^2-\frac{3}{2}n-n(a+1)-1$ and $a^2+3a+2+\frac{1}{2}n^2-\frac{3}{2}n-na+2(a-\frac{n}{2}+1)+3 = (a+1)^2+3(a+1)+2+\frac{1}{2}n^2-\frac{3}{2}n-n(a+1)+1$.

Finally, for $a=n-3$, there are critical sets of size $\frac{n^2-3n}{2}+2+5$, $\frac{n^2-3n}{2}+2+7$, $\ldots$, $\frac{n^2-n}{2}-1$. Thus, all values in the range $\frac{n^2}{4},\ldots,\frac{n^2-n}{2}$ with the exception of $\frac{n^2}{4}+1$ are covered. □

4.2 Odd n

In a back circulant latin square of odd order the set of elements

$$\begin{aligned} I_2 = \{&(0,\frac{n-3}{2};\frac{n-3}{2}),(0,n-1;n-1),(\frac{n+1}{2},\frac{n-3}{2};n-1),\\ &(\frac{n+1}{2},n-2;\frac{n-3}{2}),(i,n-2;n-2+i),(i,n-1;n-1+i),\\ &\mid i=1,\ldots,\frac{n-1}{2}\} \end{aligned}$$

forms a latin interchange. The partial latin square $I_2' = \{(0,\frac{n-3}{2};n-1),(0,n-1;\frac{n-3}{2}),(i,n-2;n-1+i),(i,n-1;n-2+i),(\frac{n+1}{2},\frac{n-3}{2};\frac{n-3}{2}),(\frac{n+1}{2},n-2;n-1) \mid i=1,\ldots,\frac{n-1}{2}\}$ forms a disjoint mate of I_2.
Using Lemma 2 we see that $(S \setminus I_2) \cup I_2'$ has unique completion and this fact can be used initially (see Lemma 18), to establish the existence of a critical set of size $|S|+2$ and then ultimately to establish critical sets of size $\frac{n^2-1}{4}+1$ to $\frac{n^2-n}{2}-1$ when $n \geq 11$ (see Theorem 30). (Note that the existence of critical sets of this size, for $n \leq 10$, has been established by Donovan in [6].) These critical sets will define latin squares which differ from BC_n in columns $\frac{n-3}{2}$, $n-2$ and $n-1$. Therefore, in the proof of the various results, care will be taken to find latin interchanges which, as necessary, do not intersect columns $\frac{n-3}{2}$, $n-2$ and $n-1$. In the proof of Lemma 18 this requirement is not strictly necessary, however it is adhered to as it will simplify the proof of subsequent results. Once again for clarity, the information necessary to validate that each of the elements of the relevant partial latin square is necessary

for unique completion, has been summarised in a table. To minimize space we have used the notation $\{x, y\}$ to mean the $\min\{x, y\}$.

LEMMA 18 *If $n \geq 11$ is odd and $\frac{n-3}{2} \leq a \leq n-4$, the set*

$$I_2 = \{(0, \frac{n-3}{2}; \frac{n-3}{2}), (0, n-1; n-1), (\frac{n+1}{2}, \frac{n-3}{2}; n-1),$$
$$(\frac{n+1}{2}, n-2; \frac{n-3}{2}), (i, n-2; n-2+i), (i, n-1; n-1+i),$$
$$\mid i = 1, \ldots, \frac{n-1}{2}\}$$

forms a latin interchange in BC_n. Denote the disjoint mate of I_2 by I_2'. Then the partial latin square

$$C_2 = (S \setminus I_2) \cup \{(0, \frac{n-3}{2}; n-1), (\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2}), (\frac{n+1}{2}, n-2; n-1)\}$$

is a critical set in $(BC_n \setminus I_2) \cup I_2'$. The size of C_2 is $|S| + 2$.

Proof We first show that C_2 has UC. The partial latin square C_2, is a subset of $(S \setminus I_2) \cup I_2'$, therefore if it is shown that C_2 has UC to $(S \setminus I_2) \cup I_2'$, then UC to $(BC_n \setminus I_2) \cup I_2'$ follows from Lemma 2. For $i = a+1$ to $n-2$ symbol i must occur in cell $(0, i)$ and thus $\frac{n-3}{2}$ must occur in column $n-1$ of row 0. For $i = a+1$ down to $\frac{n+1}{2}$ symbol $i-1$ must occur in cell $(i, n-1)$. Similarly for $i = a+2$ down to $\frac{n+3}{2}$ symbol $i-2$ must occur in cell $(i, n-2)$. Cell $(\frac{n+1}{2}, n-2)$ already contains the symbol $n-1$ and so for $i = \frac{n-1}{2}$ down to 1 symbol $i-1$ must occur in cell $(i, n-2)$. This completes column $n-2$. Next for $i = \frac{n-1}{2}$ down to 2 symbol $i-2$ must occur in cell $(i, n-1)$. It is then clear that symbol $n-1$ must occur in cell $(1, n-1)$. Thus, any latin square which contains the partial latin square C_2 must also contain $(S \setminus I_2) \cup I_2'$. Finally since $(S \setminus I_2) \cup I_2'$ has UC, C_2 has UC.

Next it will be shown that each of the entries of C_2 are necessary for unique completion and hence that C_2 is a critical set of size $|S| + 2$.

RANGE OF i	RANGE OF j	COMMENTS
$\frac{n+1}{2}$	$n-1$	If $a = \frac{n-3}{2}$ use Theorem 4 on columns $\frac{n-5}{2}$ to $n-1$, rows 2 to $\frac{n+1}{2}$.
$a+2$ to $n-1$	$n-1$	Use an intercalate on rows $i - \frac{n-1}{2}$ and i, columns $\frac{n-3}{2}$ and $n-1$.

Range of i	Range of j	Comments
$a+3$ to $n-1$	$n-2$	If $n \equiv 1 \pmod 4$ use the latin interchange K_1 listed in the Appendix and if $n \equiv 3 \pmod 4$, use K_2 given in the Appendix.
$a+4$ to $n-1$	$n+1+a-i$ to $n-3$	Use Theorem 4 or Corollary 6 on rows 0 to i, columns $a+1$ to j.
0 to j	0 to $\lfloor \frac{n-9}{4} \rfloor$	Use Theorem 4 on columns j to $n-3$, rows i to $n-3-j+i$. Since $c-1 = n-3-j$, $y = j+3$ and $y < c-1$ it follows that $y+j \leq \frac{n-3}{2}$ and $j+c-1-y \geq \frac{n-3}{2}$, for $n \geq 9$, so Corollary 8 can be used to avoid column $\frac{n-3}{2}$ if necessary.
1 to $\lfloor \frac{n-9}{4} \rfloor$	0 to $i-1$	Take the transpose and use Theorem 4 on columns i to $n-3$ and rows j to $i+j+3$ in the transpose. Since $i \leq \lfloor \frac{n-9}{4} \rfloor$, $j+y < \frac{n-3}{2}$ and Corollary 9 will avoid row $\frac{n-3}{2}$ and hence column $\frac{n-3}{2}$ if necessary.
0 to $\frac{n-7}{2} - j$.	$\lfloor \frac{n-9}{4} \rfloor + 1$ to $\frac{n-7}{2}$	Use Theorem 4 on columns j to $n-3$, rows i to $n-3-j+i$. Since $y < c-1$ and $2j+3 > \frac{n-3}{2}$, column $\frac{n-3}{2}$ is not intersected.
$\lfloor \frac{n-9}{4} \rfloor + 1$ to $\frac{n-7}{2}$	0 to $\frac{n-7}{2} - i$	Take the transpose and use Theorem 4 on columns i to $n-1$, rows j to $n+j-i-1$, or since $j+y < \frac{n-3}{2} < j+c-1$ use Corollary 9 to avoid row $\frac{n-3}{2}$; and thus column $\frac{n-3}{2}$ in the original array.
1 to $\frac{n-5}{2}$	$\frac{n-5}{2} - i$ to $\{\frac{n-7}{2}, a-i\}$	Use a latin interchange isotopic to I_2.
0 to $\{\frac{n-5}{2}, a - \frac{n-5}{2}\}$	$\frac{n-5}{2}$	If $n \equiv 3 \pmod 4$, use the latin interchange K_3 given in the Appendix and if $n \equiv 1 \pmod 4$ use the latin interchange K_4 given in the Appendix.

RANGE OF i	RANGE OF j	COMMENTS
$\frac{n-3}{2}$	0 to $\{\frac{n-9}{2}, a-\frac{n-9}{2}\}$	Take latin interchanges isotopic to K_3 and K_4 (shifted right by one column) thus using rows $\frac{n-3}{2}, n-3, n-2$, and take the transpose.
$\frac{n-1}{2}$	0 to $\{\frac{n-9}{2}, a-\frac{n-9}{2}\}$	Take latin interchanges isotopic to K_3 and K_4 on columns $\frac{n-1}{2}, n-2, n-1$, and take the transpose as per the previous row of this table.
$\frac{n-3}{2}$	$\frac{n-7}{2}$	If $a \geq n-5$, use a latin interchange isotopic to I_2.
$\frac{n-1}{2}$	$\frac{n-7}{2}$	Use K_5 given in the Appendix.
$\frac{n-3}{2}$	$\frac{n-5}{2}$	If $a = n-4$ and $n \equiv 3(\text{mod } 4)$, use K_6 given in the Appendix, and if $a = n-4$ and $n \equiv 1(\text{mod } 4)$, use K_7 given in the Appendix.
$\frac{n+1}{2}$ to a	0 to $\{i-\frac{n+1}{2}, a-i\}$	Consider the transpose and use Theorem 4 on columns i to $n-2$, rows j to $i+j+2$, or since $y = i+2$, $c-1 = n-2-i$, $y > c-1$, $j+c-1 < \frac{n-3}{2}$, and $y-1+j > \frac{n-3}{2}$, use Corollary 11 to avoid row $\frac{n-3}{2}$, and so column $\frac{n-3}{2}$ in the original array.
$\frac{n+1}{2}$ to $\{\frac{n+1}{2}+j-1, a-j\}$	1 to $\lfloor\frac{n-9}{4}\rfloor$.	Take the transpose and use Theorem 4 or Corollary 11 on columns i to $n-2-j$, rows j to $i+2j+2$.
$\frac{n+1}{2}$ to $a-j$	$\lfloor\frac{n-9}{4}\rfloor+1$ to $a-\frac{n+1}{2}$	Use Corollary 6 on rows i to $i+j+3$, columns j to $n-3$. Note $c-1 = j+3$ and $y+c-1 = 2j+3 > \frac{n-3}{2}$ so there is no intersection with column $\frac{n-3}{2}$.

RANGE OF i	RANGE OF j	COMMENTS
1 to $a - \frac{n-3}{2}$	$\frac{n-3}{2}$	Use an intercalate on columns $\frac{n-3}{2}$ and $n-2$, rows i and $i + \frac{n+1}{2}$.
0 to $a - j$	$\frac{n-1}{2}$ to a	Use Theorem 4 on columns j to $n-3$.
0	$\frac{n-3}{2}$	For symbol $n-1$, use Theorem 4 on rows 0 to $\frac{n-3}{2}$, columns $\frac{n-3}{2}$ to $n-3$. This latin interchange intersects column $\frac{n-3}{2}$ in rows 0 and $\frac{n-3}{2}$ only.
$\frac{n+1}{2}$	$\frac{n-3}{2}$	For symbol $\frac{n-3}{2}$, use Theorem 4 on rows $\frac{n+1}{2}$ to $n-2$, columns $\frac{n-3}{2}$ to $n-4$. This latin interchange intersects column $\frac{n-3}{2}$ in rows $\frac{n+1}{2}$ and $n-2$ only.
$\frac{n+1}{2}$	$n-2$	For symbol $n-1$, use Corollary 6 on rows 2 to $\frac{n+1}{2}$, and columns $\frac{n-1}{2}$ to $n-2$. □

LEMMA 19 *Take I_2 and C_2 as in Lemma 18. Then if $n \geq 11$ is odd and $a = n-3$, C_2 is a critical set in $(BC_n \setminus I_2) \cup I_2'$. The size of C_2 is $|S| + 2$.*

Proof The proof follows that of Lemma 18 with a few small exceptions. (For full details see [**3**].) □

LEMMA 20 *When n, $n \geq 11$, is odd, and $\frac{n+1}{2} \leq a \leq n-4$, critical sets exist of order n and sizes $|S| + 4$.*

Proof Let

$$J_2 = \{(a - \frac{n-3}{2}, \frac{n-3}{2}; a), (a - \frac{n-3}{2}, n-1; a - \frac{n-3}{2} - 2),$$
$$(a+1, \frac{n-3}{2}; a - \frac{n-3}{2} - 2), (a+1, n-1; a)\}.$$

It will be shown that the partial latin square

$$C_4 = (S \setminus I_2) \setminus (J_2) \ \cup$$
$$\{(0, \frac{n-3}{2}; n-1), (\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2}), (\frac{n+1}{2}, n-2; n-1)\} \ \cup$$
$$\{(a - \frac{n-3}{2}, \frac{n-3}{2}; a - \frac{n-3}{2} - 2), (a+1, \frac{n-3}{2}; a),$$
$$(a+1, n-1; a - \frac{n-3}{2} - 2)\}$$

is a critical set.

Considering column $n-1$. In the completion of C_4 the symbol a must be placed in row $a - \frac{n-3}{2}$. It is now immediate from Theorem 18 and the UC of C_2 that this partial latin square has UC.

Proof that element $(x, y; z) \in (S \backslash I_2) \backslash J_2$, is necessary for UC follows from Theorem 18, as in each relevant case, a latin interchange was produced that did not intersect column $\frac{n-3}{2}$, $n-2$ or $n-1$.

The necessity of elements $(0, \frac{n-3}{2}; n-1), (\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2}), (\frac{n+1}{2}, n-2; n-1)$ follows as in the proof that C_2 is a critical set.

The elements $(a - \frac{n-3}{2}, \frac{n-3}{2}; a - \frac{n-3}{2} - 2), (a+1, \frac{n-3}{2}; a), (a+1, n-1; a - \frac{n-3}{2} - 2)\}$ can be dealt with individually. For details see **[3]**. □

LEMMA 21 *For $n \geq 11$ odd, and $a = n-3$, there exists a critical set of size $|S|+4$.*

Proof Take the partial latin square given below:

$$
\begin{aligned}
C_4 = &(S \setminus I_2) \setminus \{(n-1, n-1; n-2)\} \\
&\cup \{(\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2}), (\frac{n+1}{2}, n-2; n-1)\} \\
&\cup \{(\frac{n-1}{2}, \frac{n-3}{2}; \frac{n-5}{2}), (\frac{n-1}{2}, n-1; n-2), (n-1, \frac{n-3}{2}; n-2), \\
&(n-1, n-1; \frac{n-5}{2})\}
\end{aligned}
$$

For UC, note that for row 0 symbol $n-2$ must occur in column $n-2$, symbol $\frac{n-3}{2}$ must occur in column $n-1$ and hence symbol $n-1$ must occur in column $\frac{n-3}{2}$. Thus UC follows from Lemma 2.

The necessity of most elements follows that of the Lemma 19. For details of a few exceptions see **[3]**. □

We note that for $2 \leq i \leq a - \frac{n-3}{2}$ the position $(i, \frac{n-3}{2})$ occurs in the following intercalate:

$$
\begin{aligned}
H_i = \{&(i, \frac{n-3}{2}; i + \frac{n-3}{2}), (i, n-1; i-2), \\
&(i + \frac{n-1}{2}, \frac{n-3}{2}; i-2), (i + \frac{n-1}{2}, n-1; i + \frac{n-3}{2})\}.
\end{aligned}
$$

Further, let $T_k[H_i] = H_2 \cup H_3 \cup \ldots \cup H_k$, where $2 \leq k \leq a - \frac{n-3}{2}$. Denote the disjoint mate of this latin interchange by $T'_k[H_i]$.

COROLLARY 22 *For n, $n \geq 11$, odd and $\frac{n+1}{2} \leq a \leq n-4$, the partial latin square*

$$
\begin{aligned}
C_{2k} = &(S \setminus I_2) \setminus T_k[H_i] \ \cup T'_k[H_i \setminus \{(i, n-1; i-2)\}] \ \cup \\
&\{(0, \frac{n-3}{2}; n-1), (\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2}), (\frac{n+1}{2}, n-2; n-1)\}
\end{aligned}
$$

is a critical set of size $|S| + 2k$, for $2 \leq k \leq a - \frac{n-3}{2}$.

Proof For UC, consider row 0, for $e = a+1$ up to $n-2$ symbol e must occur in column e, and hence symbol $\frac{n-3}{2}$ must occur in column $n-1$. In column $n-1$, $k \le a - \frac{n-1}{2}$, for $e = a$ down to $\frac{n-3}{2}+k+1$, symbol e must occur in row $e+1$, and symbol $\frac{n-1}{2}$ in row $\frac{n+1}{2}$. In addition, for e equal to $\frac{n-3}{2}+k$ down to $\frac{n+1}{2}$, symbol e must occur in row $e - \frac{n-3}{2}$ and UC follows from Lemma 2.

For elements in $(S\setminus I_2)\setminus T_k[H_i]$, and $\{(0,\frac{n-3}{2};n-1),(\frac{n+1}{2},\frac{n-3}{2};\frac{n-3}{2}),(\frac{n+1}{2},n-2;n-1)\}$ the proof for Lemma 18 provides a latin interchange which meets the necessary requirements.

RANGE OF i	RANGE OF j	COMMENTS
2 to k	$\frac{n-3}{2}$	For symbol $i-2$, use either Theorem 4 or Corollary 5 on rows i to $\frac{n-3}{2}$, columns $\frac{n-3}{2}$ to $n-3$.
$\frac{n-1}{2}+2$ to $\frac{n-1}{2}+k$	$\frac{n-3}{2}$	For symbol $i+1$, use either Theorem 4 or Corollary 5 on rows i to $n-2$, columns $\frac{n-3}{2}$ to $n-2$.
$\frac{n-1}{2}+2$ to $\frac{n-1}{2}+k$	$n-1$	For symbol $i-\frac{n+3}{2}$, Corollary 6 on rows $\frac{n-1}{2}$ to i (or rows $\frac{n-3}{2}$ to i) and columns $\frac{n+1}{2}$to $n-1$. □

LEMMA 23 *When n, $n \ge 11$, is odd and $a = n-3$ there exists a critical set of size $|S|+1+2(k-2)$ for $3 \le k \le \frac{n-3}{2}$.*

Proof For $3 \le k \le \frac{n-3}{2}$, and $a = n-3$ it will be shown that the partial latin square

$$(S\setminus I_2)\setminus T_k[H_i] \cup \{(\frac{n+1}{2},\frac{n-3}{2};\frac{n-3}{2}),(\frac{n+1}{2},n-2;n-1\}\cup$$
$$T'_k[H_i\setminus\{(i,n-1;i-2)\}]$$

is a critical set.

For UC, observe the following. If the completion of this partial latin square does not contain entry $(0,\frac{n-3}{2};n-1)$ then it must contain the entries $(0,\frac{n-3}{2};n-2)$, $(0,n-2;\frac{n-3}{2})$, $(0,n-1;n-1)$, $(1,n-3;n-1)$, $(1,n-2;n-2)$, $(1,n-1;0)$, $(2,n-4;n-1)$, $(2,n-3;n-2)$, $(2,n-2;0)$, $(2,n-1;1)$. However symbol 1 already occurs in cell $(\frac{n+3}{2}+1,n-1)$ and so we have a contradiction. Thus any completion of this partial latin square must contain the entry $(0,\frac{n-3}{2};n-1)$.

Now consider column $n-1$. For $e = n-3$ down to $\frac{n-1}{2}+k$, symbol e must occur in row $e+1$. Then $\frac{n+1}{2}$ must occur in row $\frac{n+3}{2}$ and $\frac{n-1}{2}$ in row $\frac{n+1}{2}$. Then for $e = k+\frac{n-3}{2}$ down to $\frac{n+3}{2}$ symbol e must occur in row $e-\frac{n-3}{2}$. Now UC follows as in previous results.

The necessity of the elements follows as in previous results except in a small number of cases and details of these cases can be found in [3]. □

COROLLARY 24 *For n, $n \geq 11$, odd and $a = n-3$ there exists critical sets of sizes $|S| + 4 + 2(k-2)$ for $3 \leq k \leq \frac{n-5}{2}$.*

Proof For $a = n-3$ and $3 \leq k \leq \frac{n-5}{2}$, the partial latin square:

$$\begin{aligned} C_{2k} = ((S \setminus I_2) \setminus T_k[H_i]) \setminus \{(n-1, n-1; n-2)\} \ &\cup \{(\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2}), \\ (\frac{n+1}{2}, n-2; n-1) \ \cup \ T'_k[H_i \setminus \{(i, n-1; i-2)\}] \ &\cup \{(\frac{n-1}{2}, \frac{n-3}{2}; \frac{n-5}{2}), \\ (\frac{n-1}{2}, n-1; n-2), (n-1, \frac{n-3}{2}; n-2), &(n-1, n-1; \frac{n-5}{2})\} \end{aligned}$$

is a critical set of the required size. □

LEMMA 25 *For n, $n \geq 11$, odd and $\frac{n-3}{2} \leq a \leq n-4$, there exists a critical set of order n and size $|S| + 5$ and when $\frac{n-1}{2} \leq a \leq n-5$ there exists a critical set of order n and size $|S| + 8$.*

Proof Let $\frac{n-3}{2} \leq a \leq n-4$ and

$$\begin{aligned} I_3 = \{(\frac{n-1}{2}, \frac{n-3}{2}; n-2), (\frac{n-1}{2}, n-1; \frac{n-5}{2}), \\ (n-1, \frac{n-3}{2}; \frac{n-5}{2}), (n-1, n-1; n-2)\}. \end{aligned}$$

It will be shown that the partial latin square

$$\begin{aligned} D_5 = (S \setminus I_2) \setminus I_3 \ \cup \ &\{(0, \frac{n-3}{2}; n-1), \\ &(\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2}), (\frac{n+1}{2}, n-2; n-1)\} \ \cup \{(\frac{n-1}{2}, \frac{n-3}{2}; \frac{n-5}{2}), \\ &(\frac{n-1}{2}, n-1; n-2), (n-1, \frac{n-3}{2}; n-2), (n-1, n-1; \frac{n-5}{2})\} \end{aligned}$$

is a critical set.

Lemma 2 and the proof of Lemma 18 can be used to show that this partial latin square has UC.

The necessity of elements of $(S \setminus I_2) \setminus I_3$ and $(0, \frac{n-3}{2}; n-1)$, $(\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2})$, $(\frac{n+1}{2}, n-2; n-1)$ follows as in the proof of Lemma 18 and for all remaining elements see [3]. □

Let $\frac{n-1}{2} \leq a \leq n-5$ and consider the partial latin square:

$$\begin{aligned} J_3 = \{(a - \frac{n-3}{2} + 1, \frac{n-3}{2}; a+1), (a - \frac{n-3}{2} + 1, n-1; a - \frac{n-3}{2} - 1), \\ (a+2, \frac{n-3}{2}; a - \frac{n-3}{2} - 1), (a+2, n-1; a+1)\} \end{aligned}$$

It will be shown that the partial latin square

$$\begin{aligned}C_5 = {} & ((S \setminus I_2) \setminus I_3) \setminus J_3 \ \cup \\ & \{(0, \frac{n-3}{2}; n-1), (\frac{n+1}{2}, \frac{n-3}{2}; \frac{n-3}{2}), (\frac{n+1}{2}, n-2; n-1)\} \ \cup \\ & \{(\frac{n-1}{2}, \frac{n-3}{2}; \frac{n-5}{2}), (\frac{n-1}{2}, n-1; n-2), \\ & (n-1, \frac{n-3}{2}; n-2), (n-1, n-1; \frac{n-5}{2})\} \ \cup \\ & \{(a - \frac{n-3}{2} + 1, \frac{n-3}{2}; a - \frac{n-3}{2} - 1), (a - \frac{n-3}{2} + 1, n-1; a+1), \\ & (a+2, \frac{n-3}{2}; a+1), (a+2, n-1; a - \frac{n-3}{2} - 1)\}\end{aligned}$$

is a critical set.

The proof that C_5 is a critical set follows as above. (For full details see **[3]**.) □

COROLLARY 26 *For $n \geq 11$ odd and $\frac{n-3}{2} \leq a \leq n-4$, there exists a critical set of order n and size $|S|+5+2(k-1)$, where $2 \leq k \leq a-\frac{n-3}{2}$ and when $\frac{n-1}{2} \leq a \leq n-5$ there exists a critical set of order n and size $|S|+8+2(k-1)$, where $2 \leq k \leq a-\frac{n-3}{2}$.*

The following corollary is necessary to prove the existence of a critical set of size $|S|+1$ when $a = n-3$ and can be proved by using latin interchanges I_2, J_3, D_5 and $H_3 \cup H_{\frac{n-7}{2}}$.

COROLLARY 27 *For $n \geq 11$ odd and $a = n-4$, there exists a critical set of order n and size $|S| + 8 + 2(\frac{n-9}{2} - 1)$.*

LEMMA 28 *For n, $n \geq 11$, odd there exists a critical set of size $\frac{n^2-n}{2} - 2$.*

Proof It will be verified that the partial latin square

$$\begin{aligned}S = {} & (\{(i, j; i+j (\mathrm{mod}\ n)) \mid 0 \leq i \leq n-2, 0 \leq j \leq n-2-i\} \setminus \{(0, n-2; n-2), \\ & (1, n-3; n-2), (n-5, 1; n-4), (n-4, 0; n-4), (n-3, 0; n-3)\}) \cup \\ & \{(n-2, n-2, n-4), (n-1, n-3, n-4), (n-1, n-2, n-3)\}\end{aligned}$$

is a critical set.

To verify UC we note that if symbol $n-2$ is to occur in row $n-1$ it must occur in column $n-1$. Similarly if symbol $n-3$ is to occur in row $n-2$ it must occur in column $n-1$. Now symbol $n-3$ occurs $n-1$ times in the partial latin square hence a completion must have symbol $n-3$ in cell $(n-3, 0)$ and it follows that if symbol $n-4$ is to occur in row $n-2$ it must occur in column $n-1$. Now UC can be obtained from the UC of S when $a = n-5$.

The necessity, for UC, of each of the elements of the above partial latin square follows as in previous arguments. See **[3]** for full details. □

LEMMA 29 *For n, $n \geq 11$, odd there exists a critical set of size $\frac{n^2-n}{2} - 1$.*

Proof It will be verified that the partial latin square

$$S = (\{(i,j; i+j (\text{mod } n)) \mid 0 \leq i \leq n-2, 0 \leq j \leq n-2-i\}$$
$$\setminus \{(0, n-2; n-2), (n-3, 0; n-3)\} \cup \{(n-1, n-2, n-3)\}$$

is a critical set.

The proof of UC is straight forward and the proof that each of the elements is necessary for UC follows that of previous results, with the exception of entry $((n-2, 0; n-2)$. However the following latin interchanges verifies the necessity of this element. $\{(i, n-2; n-2+i), (i, n-1; n-1+i), (n-2, i; n-2+i), (n-1, i; n-1+i), (n-2, n-2; n-4), (n-1, n-1; n-2) \mid i = 0, \ldots, n-3\}$. □

THEOREM 30 *Critical sets exist of all sizes between $\frac{n^2-1}{4}$ and $\frac{n^2-n}{2}$ for odd values of n, $n \geq 11$.*

Proof From Lemma 3 there exists a critical set of size $a^2 + 3a + 2 + \frac{1}{2}n^2 - \frac{3}{2}n - na$ which we will denote by $|S_a|$. Note that $|S_{a+1}| = |S_a| = 2a+4-n$. We now show that critical sets exist for every size between $|S_a|$ and $|S_{a+1}|$ for $\frac{n-3}{2} \leq a \leq n-3$. First, for $a = \frac{n-3}{2}$, $|S_a| = \frac{n^2-1}{4}$ and $|S_a| + 1 = |S_{a+1}|$. Thus the result is true for $a = \frac{n-3}{2}$. For $a = \frac{n-1}{2}$, Lemma 18 with $a = \frac{n-3}{2}$ provides a critical set of size $\frac{n^2-1}{4} + 2 = |S_{\frac{n-1}{2}}| + 1$. Then using Lemma 18 with $a = \frac{n-1}{2}$ provides a critical set of size $\frac{n^2-1}{2} + 3$. Then $|S_{\frac{n+1}{2}}| = \frac{n^2-1}{4} + 4$ so the result holds for $a = \frac{n-1}{2}$. A critical set of size $\frac{n^2-1}{4} + 5$ is provided by Lemma 25, using $a = \frac{n-3}{2}$; and a critical set of size $\frac{n^2-1}{4} + 6$ is provided by Lemma 25 using $a = \frac{n-1}{2}$. A critical set of size $\frac{n^2-1}{4} + 7 = |S_{\frac{n+1}{2}} + 3$ is provided in the Appendix. Lemma 20 with $a = \frac{n+1}{2}$ provides a critical set of size $\frac{n^2-1}{4} + 8 = |S_{\frac{n+1}{2}}| + 4$. Finally $|S_{\frac{n+3}{2}}| = \frac{n^2-1}{4} + 9$. A critical set of size $|S_{\frac{n+3}{2}}| + 1$ is provided in the Appendix, and one of size $|S_{\frac{n+3}{2}}| + 3$ by Lemma 25 with $a = \frac{n+1}{2}$.

We will now generalise the proof of existence for $\frac{n+3}{2} \leq a \leq n-4$ using the results provided in Lemma 18 through to Corollary 26.

For $\frac{n+3}{2} \leq a \leq n-4$, Lemmas 18 and 20 and Corollary 22 provide critical sets of sizes $\{|S_a| + 2, |S_a| + 4, \ldots, |S_{a+1}| - 3, |S_{a+1}| - 1\}$. For $\frac{n+3}{2} \leq a \leq n-4$, Lemma 25 and Corollary 26 and Corollary 22 provide critical sets of sizes $\{|S_a| + 5, |S_a| + 7, \ldots, |S_{a+1}| - 4, |S_{a+1}| - 2\}$. Then for $\frac{n+3}{2} \leq a \leq n-5$, Lemma 25 and and Corollary 26 provide the additional sizes of $\{|S_{a+1}| + 1, |S_{a+1}| + 3\}$ and these then provide critical sets of sizes $|S_a| + 1$ and $|S_a| + 3$ for $\frac{n+5}{2} \leq a \leq n-4$.

We have now produced critical sets of all sizes between $|S_{\frac{n-3}{2}}|$ and $|S_{n-3}|$. It remains to fill in those os sizes between $|S_{n-3}|$ and $|S_{n-2}|$.

Corollary 27 provides us with a critical set of size $|S_{n-3}| + 1 = \frac{n^2-n}{2} - n + 3$. Lemmas 19, 21 and Corollary 24 provided critical sets of sizes $\{|S_{n-3}| + 2, |S_{n-3}| + 4, \ldots, |S_{n-3}| + n - 5, |S_{n-2}| - 3\}$. Lemma 23 provides critical sets of sizes $\{|S_{n-3}| + 3, |S_{n-3}| + 5, \ldots, |S_{n-3}| + n - 6, |S_{n-2}| - 4\}$. Lemmas 28, 29 provided critical sets of sizes $\{|S_{n-3}| - 2, |S_{n-2}| - 1\}$ respectively.

At this point all cases have been verified.

□

References

[1] Donovan Diane and Cooper Joan, "Critical sets in back circulant latin squares", Aequationes Math, **52**, 1996, pp. 157–179.

[2] Donovan Diane and Howse Adelle, "Critical sets for latin squares of order 7", JCMCC, **28**, 1998, pp. 113–123.

[3] Donovan Diane and Howse Adelle, "Towards the spectrum of critical set in full detail", http://www.maths.uq.edu.au/~dmd/research/spec.ps.

[4] Howse Adelle, "Minimal critical sets for some small latin squares", Australas. J. Combin., **17**, 1998, pp. 275–288.

[5] Curran D. and van Rees GHJ., "Critical sets in latin squares", In Proc. 8th Manitoba Conference on Numerical Mathematics and Computing, Winnipeg, 1978, Utilitas Math. Pub. pp. 165–168.

[6] Donovan Diane, "Critical sets in latin squares of order less than 11", JCMCC, (to appear).

[7] Nelder J., " Critical sets in latin squares", In CSIRO Division of Math. and Stats., Newsletter 38, 1977.

Appendix

$K_1 = \{(i-\frac{n-1}{2}, \frac{n-5}{2}; i-2), (i-\frac{n-1}{2}, n-3; i+\frac{n-5}{2}), (i-\frac{n-1}{2}, n-2; i+\frac{n-1}{2})\} \cup \{(i-\frac{n-1}{2}+2k, n-3-2k; i+\frac{n-5}{2}), (i-\frac{n-1}{2}+2k, n-3-2(k-1); i+\frac{n-1}{2}) \mid 1 \leq k \leq \frac{n-1}{4}\}$

$K_2 = \{(i-\frac{n-1}{2}, \frac{n-5}{2}; i-2), (i-\frac{n-1}{2}, n-3; i+\frac{n-5}{2}), (i-\frac{n-1}{2}, n-2; i+\frac{n-1}{2}), (i-2, \frac{n-5}{2}+2; i+\frac{n-5}{2}), (i-2, \frac{n-5}{2}+3; i+\frac{n-3}{2}), (i-1, \frac{n-5}{2}+2; i+\frac{n-3}{2}), (i-1, \frac{n-5}{2}+3; i+\frac{n-1}{2}), (i, \frac{n-5}{2}; i+\frac{n-5}{2}), (i, \frac{n-5}{2}+2; i+\frac{n-1}{2})\} \cup \{(i-\frac{n-1}{2}+2k, n-3-2k; i+\frac{n-5}{2}), (i-\frac{n-1}{2}+2k, n-3-2(k-1); i+\frac{n-1}{2}) \mid 1 \leq k \leq \frac{n-3}{4}-1\}$

$K_3 = \{(i, \frac{n-5}{2}; \frac{n-5}{2}+i), (i, n-3; n-3+i)\} \cup \{(x, \frac{n-5}{2}+2z; \frac{n-5}{2}+x+2z) \mid x = \frac{n-1}{2}+i, \frac{n+3}{2}+i, 0 \leq z \leq \frac{n-3}{4}\} \cup \{(\frac{n-1}{2}+i, n-3; \frac{n-7}{2}+i), (\frac{n+1}{2}+i, n-4; \frac{n-7}{2}+i), (\frac{n+1}{2}+i, n-3; \frac{n-5}{2}+i)\}$

$K_4 = \{(i, \frac{n-5}{2}; \frac{n-5}{2}+i), (i, n-3; n-3+i)\} \cup \{(x, \frac{n-5}{2}+2z; \frac{n-5}{2}+x+2z) \mid x = \frac{n-1}{2}+i, \frac{n+3}{2}+i, 0 \leq z \leq \frac{n-5}{4}\} \cup \{(\frac{n-1}{2}+i, n-3; \frac{n-7}{2}+i), (\frac{n+1}{2}+i, n-4; \frac{n-7}{2}+i), (\frac{n+1}{2}+i, n-3; \frac{n-5}{2}+i), (\frac{n+3}{2}+i, n-4; \frac{n-5}{2}+i)\}$

$K_5 = \{(i, j; i+j), (n-1, j; j-1), (a, b; a+b) \mid \frac{n-1}{2} \leq a \leq n-1, b = n-4, n-3)\}$.

$K_6 = \{(\frac{n-3}{2}, \frac{n-5}{2}; n-4), (\frac{n-3}{2}, n-4; \frac{n-11}{2}), (\frac{n-3}{2}, n-3; \frac{n-9}{2}), (\frac{n-1}{2}, n-4; \frac{n-9}{2}), (\frac{n-1}{2}, n-3; \frac{n-7}{2}), (n-1, \frac{n-5}{2}; \frac{n-7}{2}), (n-1, n-3; n-4), (\frac{n+1}{2}+2i, n-6-2i; \frac{n-11}{2}), (\frac{n+1}{2}+2i, n-4-2i; \frac{n-7}{2}) \mid i = 0\ldots, \frac{n-7}{4}\}$

$K_7 = \{(\frac{n-3}{2}, \frac{n-5}{2}; n-4), (\frac{n-3}{2}, n-3; \frac{n-9}{2}), (\frac{n-3}{2}, n-4; \frac{n-11}{2}), (n-1, n-3; n-4), (\frac{n-1}{2}, n-5; \frac{n-11}{2}), (\frac{n-1}{2}, n-4; \frac{n-9}{2}), (\frac{n-1}{2}, n-3; \frac{n-7}{2}), (\frac{n-1}{2}+2i, n-5-2i; \frac{n-11}{2}), (\frac{n-1}{2}+2i, n-3-2i; \frac{n-7}{2}), (n-1, \frac{n-5}{2}; \frac{n-7}{2}), \mid i = 0\ldots, \frac{n-7}{4}\}$

$K_7 = \{(\frac{n-3}{2}, \frac{n-5}{2}; n-4), (\frac{n-3}{2}, n-1; \frac{n-7}{2}), (n-3, \frac{n-5}{2}+2y; \frac{n-11}{2}+2y;) \mid y = 1\ldots, \frac{n+3}{4}\} \cup (n-1, \frac{n-5}{2}+2y; \frac{n-7}{2}+2y;) \mid y = 0\ldots, \frac{n-3}{4}\}$

A CRITICAL SET OF SIZE $|S|+3$ when $a = \frac{n+1}{2}$. Begin by taking C_2, for $a = \frac{n-3}{2}$ and take the partial latin square $C_2 \setminus \{(\frac{n+3}{2}, n-2; \frac{n-1}{2}), (\frac{n+5}{2}, n-2; \frac{n+1}{2})\} \cup \{(0, n-2; n-2), (1, \frac{n-3}{2}; 0), (2, \frac{n-3}{2}; 1), (\frac{n+3}{2}, \frac{n-3}{2}; \frac{n-1}{2}), (\frac{n+5}{2}, \frac{n-3}{2}; \frac{n+1}{2}), (\frac{n+3}{2}, n-2; 0), (\frac{n+5}{2}, n-2; 1)\}$.

A CRITICAL SET OF SIZE $|S|+1$ when $a = \frac{n+3}{2}$. Begin by taking C_2, for $a = \frac{n+1}{2}$ and take the partial latin square $C_2 \setminus \{(1, \frac{n-3}{2}; \frac{n-1}{2}), (2, \frac{n-3}{2}; \frac{n+1}{2})\} \cup \{(1, \frac{n-3}{2}; 0), (2, \frac{n-3}{2}; 1), (\frac{n+3}{2}, \frac{n-3}{2}; \frac{n-1}{2}), (\frac{n+5}{2}, \frac{n-3}{2}; \frac{n+1}{2}), (\frac{n+3}{2}, n-2; 0), (\frac{n+5}{2}, n-2; 1)\}$.

(Received 31/3/99; revised 14/1/2000)