Progress on the Hall-Number-Two Problem

Ch. Eslahchi

Institute for Studies in Theoretical Physics and Mathematics
and Department of Mathematical Science
Sharif University of Technology
Tehran, Iran

A. J. W. Hilton

Department of Mathematics
The University of Reading
Whiteknights, Reading RG6 2AX England

P. D. Johnson Jr.*

Department of Discrete and Statistical Sciences
120 Math Annex
Auburn University, Alabama 36849 U.S.A.

Abstract

The graphs with Hall number at most 2 form a class of graphs within which the chromatic number equals the choice (list-chromatic) number. This class has a forbidden-induced-subgraph characterization which has not yet been found, although a fairly imposing collection of minimal forbidden induced subgraphs has been assembled. In this paper we add to the collection, most notably adding

(i) K_5 with an ear of length 2 attached;

(ii) K_4 with an ear of any length > 2 attached;

(iii) any cycle together with two triangles based on incident edges on the cycle;

(iv) any odd cycle together with two triangles based on non-incident edges of the cycle; and

(v) any even cycle together with three triangles based on non-incident edges of the cycle.

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1 Introduction

Throughout, G will denote a finite simple graph and L will denote a list assignment to the vertices of G, i.e., a function from $V(G)$ into the collection $\mathcal{F}(C)$ of finite subsets of C, an infinite set (of "colors", or symbols). A proper L-coloring of G is a selection $\varphi(v) \in L(v)$ for all $v \in V(G)$ such that if u and v are adjacent in G, then $\varphi(u) \neq \varphi(v)$. [Alternatively, this last bit can be restated: for each $\sigma \in C$, $\varphi^{-1}(\sigma) = \{v \in V(G) : \varphi(v) = \sigma\}$ is an independent set of vertices in G.]

The study of list colorings, started by Vizing [13] and independently by Erdös, Rubin, and Taylor [2], departs from the question of when (under what conditions on G and L) is there a proper L-coloring of G? The main focus of interest is the choice number, or list chromatic number: $c(G)$ is the smallest positive integer among those m such that there is a proper L-coloring of G whenever $|L(v)| \geq m$ for all $v \in V(G)$. It is clear that $c(G) \geq \chi(G)$, the chromatic number of G, and it is known that $c(G)$ can be quite a bit larger than $\chi(G)$; for instance, $c(K_{m,m}) \sim \log_2 m$ [10]. Curiosity is drawn to the extremes: how much larger than $\chi(G)$ can $c(G)$ be (for instance, can $c(G)/(\chi(G) \log |V(G)|)$ be arbitrarily large?) and, at the other extreme, for which G is $c(G) = \chi(G)$?

Here is a necessary condition for a proper L-coloring which does not directly refer to the size of the lists $L(v)$, $v \in V(G)$. We say that G and L *satisfy Hall's condition* iff for each subgraph H of G,

$$|V(H)| \leq \sum_{\sigma \in C} \alpha(\sigma, L, H) \qquad (*)$$

where $\alpha(\sigma, L, H)$ is the independence number of the subgraph of H induced by $\{u \in V(H); \sigma \in L(u)\}$. To put it another way, if you were trying to properly L-color H, $\alpha(\sigma, L, H)$ would be the largest number of vertices you could color with σ. (This shows why Hall's condition is necessary for the existence of a proper L-coloring of G)

Note that for G and L to satisfy Hall's condition it suffices that $(*)$ holds for induced subgraphs H of G. Note also that if G and L satisfy Hall's condition, then so do G' and L, for any subgraph G' of G. The reason for the name, *Hall's* condition, is that in the case when G is a clique, in which case a proper L-coloring of G is also a *system of distinct representatives* (SDR) of the sets $L(v)$, $v \in V(G)$, Hall's condition (with H confined to induced subgraphs, i.e. subcliques) boils down to the condition, both necessary and sufficient for such an SDR, given in the famous theorem of Phillip Hall [6]. The main result of [7] is that a graph G has the property that Hall's condition is sufficient for the existence of a proper L-coloring if and only if every block of G is a clique.

The *Hall number* of G, denoted $h(G)$, is the smallest positive integer among those m such that there is a proper L-coloring of G whenever Hall's condition is satisfied *and* $|L(v)| \geq m$ for all $v \in V(G)$. The result mentioned in the preceding paragraph can be restated: $h(G) = 1$ if and only if every block of G is a clique.

The Hall number is somewhat contrived and unnatural—when you hear $\chi(G) = 3$ you feel you know something rather straightforward about G, but when you hear that $h(G) = 3$, you look at the definition, and you look at G, and you shake your head. To make matters worse, the Hall number is very badly behaved: removing a single edge can cause the Hall number to go up, or down, by a large amount (see [8]). The Hall number behaves better with respect to vertex removal: if H is an induced subgraph of G, then $h(H) \leq h(G)$ ([9]). Still, removing a single vertex can cause a huge drop in the Hall number, whereas the chromatic and choice numbers can drop by at most one.

But even if one does not regard the Hall number as being of much interest in itself, there is a good reason to work on it, and to endure its caprices: it offers a way in to the study of the extremal equation $c(G) = \chi(G)$. This virtue arises from some fundamental relations that h enjoys with c, χ, and a fourth parameter, the Hall-condition number, that will not play a role here—see [9] for details. One consequence of these relations is that $c(G) = \chi(G)$ if and only if $h(G) \leq \chi(G)$. Thus, in principle, a practical characterization of $\mathcal{H}_k = \{G; h(G) \leq k\}$ for each positive integer k would "solve" the $c = \chi$ problem: given G, determine $k = \chi(G)$ and then check to see if $G \in \mathcal{H}_k$. Of course, determining $\chi(G)$ is "hard", but not as hard as determining $c(G)$, which is on an entirely different level of complexity; see [4], [5], and [12] (Section 4.4).

Furthermore, a "practical characterization" of $\mathcal{H}_k$ exists, although we despair of finding it when $k \geq 3$. Because h is monotone with respect to taking induced subgraphs, $\mathcal{H}_k$ has a forbidden-induced-subgraph characterization: $G \in \mathcal{H}_k$ if and only if G has no induced subgraph which is "critical with Hall number $> k$"; a graph H is critical with Hall number $> k$ if and only if $h(H) > k$ but $h(H - v) \leq k$ for all $v \in V(H)$.

Let us shorten "critical with Hall number $> k$", the terminology used in [9], to "Hall-k^+-critical". Thus, as shown in [9], the Hall-1^+-critical graphs are the cycles C_n, $n \geq 4$ and K_4-minus-an-edge. Note that we have a perfectly good characterization of $\mathcal{H}_1 = \{G; h(G) = 1\}$, namely every-block-is-a-clique, which does not refer to forbidden induced subgraphs. Our aim here is to forge on toward a forbidden-induced-subgraph characterization of $\mathcal{H}_2$, i.e. to add to the list of Hall-2^+-critical graphs begun in [9], but we do not rule out the possibility of an alternative characterization of the "global" variety. We would expect any such characterization to emerge from the forbidden-induced-subgraph characterization.

Why make a fuss about $\mathcal{H}_2$? For one thing, it is next in line after $\mathcal{H}_1$. Also, characterizing the graphs G satisfying $p(G) = 2$, or $p(G) \leq 2$, is a standard and fundamental exercise for positive-integer-valued parameters p. (See [2] for a characterization of the graphs with choice number ≤ 2.) But mainly our interest is piqued by the observation, easily verifiable by previous remarks, that $G \in \mathcal{H}_2$ implies that $c(G) = \chi(G)$. Furthermore, $k = 2$ is the largest value for which $\mathcal{H}_k$ has this property, since for every graph G such that $c(G) > \chi(G)$, we have $c(G) = h(G)$ (see [9]), and there are plenty of such G with $\chi(G) = 2$ and $c(G) = 3$ ($G = K_{3,3}$, for instance).

2 Results and Problems

First we collect the Hall-2^+-critical graphs from [9]. We extend the terminology introduced in [2]: if $m_1, \ldots, m_k$ are positive integers, $\theta(m_1, \ldots, m_k)$ will denote the graph obtained by connecting two vertices by k internally disjoint paths of lengths $m_1, \ldots, m_k$, respectively. (In [2], $k = 3$, only. Clearly we can do without $k = 1$ and $k = 2$. Also, note that $\theta(m_1, \ldots, m_k)$ is simple only if $m_j = 1$ for at most one $j \in \{1, \ldots, k\}$.) If G_1, G_2 are simple graphs, let *cuff*(G_1, G_2, ℓ) denote a graph obtained by connecting copies of G_1 and G_2 by a path of length ℓ; the copies of G_1 and G_2 are to be disjoint except for a single shared vertex when $\ell = 0$, and the connecting path is understood to intersect G_1 and G_2 only at its end-vertices. Of course, attaching the connecting path to different vertices of, say, G_1, may result in different graphs, if G_1 is not vertex-transitive; when the end attachments are not specified, let *cuff*(G_1, G_2, ℓ) stand for the whole class of graphs obtainable by joining G_1 to G_2 as described, with various points of attachment.

Theorem 1 ([9], Theorem 6) *The following are Hall–2^+-critical:*

(a) Cuff(C_m, C_n, ℓ), for any integers, $m \geq n \geq 3$, $\ell \geq 0$, provided $m \geq 4$;

(b) $\theta(m_1, m_2, m_3)$ for any positive integers $m_1 \geq m_2 \geq m_3$ with $m_2 \geq 3$, except possibly if $(m_1, m_2, m_3) = (3, 3, 2)$;

(c) $\theta(m, 2, 2, 1)$ and $\theta(m, 2, 2, 2)$ for any positive integer $m \geq 2$.

The case of $\theta(3, 3, 2)$ was left unsettled in [9], which also leaves the case of $\theta(3, 3, 2, 2)$ unsettled. (This latter has Hall number > 2, as shown in [9], but will only be Hall-2^+-critical if $\theta(3, 3, 2)$ has Hall number 2.) The graph $\theta(3, 3, 2)$ is one of those small-case anomalies that makes finite discrete mathematics so curiously unpredictable and exciting, since $\theta(3, 3, 1)$, $\theta(m, 3, 3)$, $m \geq 3$, and $\theta(m, 3, 2)$, $m \geq 4$, are all Hall-2^+-critical. We will settle the matter of $\theta(3, 3, 2)$ here. The other claims in the following theorem are proven in [9], as part of the proof of Theorem 6 there (Theorem 1, above).

Theorem 2 *The following have Hall number 2:*

(a) C_n, $n \geq 4$;

(b) $\theta(m, 2, 1)$, $m \geq 2$;

(c) $\theta(m, 2, 2)$, $m \geq 2$;

(d) $\theta(3, 3, 2)$.

Corollary 1 *(of part (d)). $\theta(3, 3, 2, 2)$ is Hall-2^+-critical.*

It is worth noting that $\theta(2,2,1)$ is also known as K_4-minus-an-edge.

Now on to new business. An *ear* on a clique K_n is a path from one vertex of K_n to another, with no internal vertex of the path in K_n. A triangle based on an edge of a cycle is just what it sounds like; the triangle together with the cycle make a copy of $\theta(m,2,1)$, where $m+1$ is the length of the cycle. When we refer to more than one triangle based on edges of a cycle, it will be understood that the vertices of the triangles that are not on the cycle are distinct, a different one for each triangle. The following two theorems are closely related—indeed, the claims of Theorem 3 can be inferred from Theorem 4—but it seems more reader-friendly to separate their claims.

Theorem 3 *The following have Hall number 2:*

(a) K_4 with an ear of length 2;

(b) any even cycle with two triangles based on non-incident edges of the cycle;

(c) Cuff(G, K_3, ℓ), *where $\ell \geq 0$, $G = \theta(2,2,1)$, and the point of attachment of the joining path to G is one of the vertices of G of degree 2.*

We suspect that K_3 in Theorem 3(c) can be replaced by K_n for any n, but will leave this conjecture for another time. See Problem 1(b).

Theorem 4 *The following are Hall-2^+-critical:*

(a) K_5 with an ear of length 2;

(b) K_4 with an ear of any length > 2;

(c) K_4 with two disjoint ears of length 2;

(d) two K_4's intersecting in a K_3;

(e) two K_4's intersecting in a K_2;

(f) any cycle with two triangles based on incident edges of the cycle;

(g) any cycle with two triangles based on the same edge of the cycle;

(h) any odd cycle with two triangles based on distinct non-incident edges of the cycle;

(i) any even cycle with three triangles based on distinct non-incident edges of the cycle;

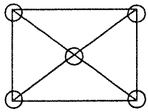
(j) Cuff(G, K_3, ℓ), for any integer $\ell \geq 0$, when G is

(1) K_4 with an ear of length 2, provided the point of attachment to G of the joining path is one of the two vertices of degree 3, or

(2) *$\theta(m,2,1)$ for some integer $m \geq 3$, provided the point of attachment to G of the joining path is the vertex of degree 2 in the only triangle in G [see Fig. 1], or*

(3) *$\theta(2,2,1)$, provided the point of attachment of the joining path to G is one of the vertices of G of degree 3 [see Figure 2].*

(k)



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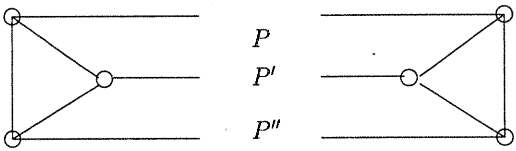


, in which either at least two of P, P', P'' are single edges, or the lengths of all three have the same parity.

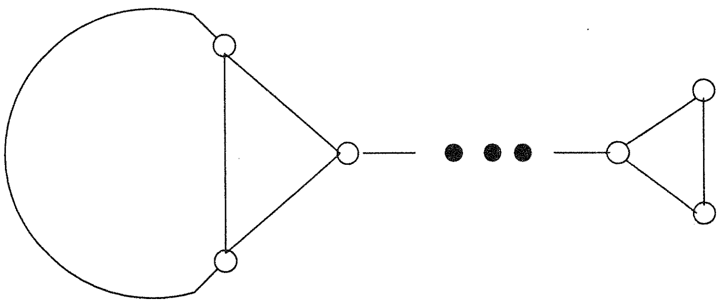


Figure 1: Theorem 4(j)(2).

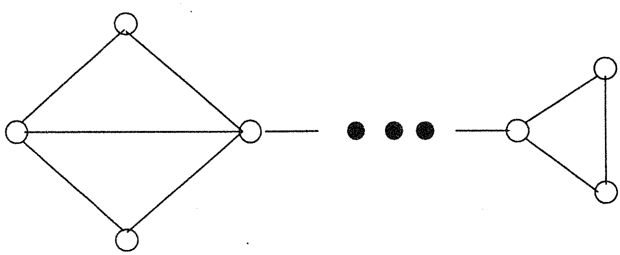


Figure 2: Theorem 4(j) (3)

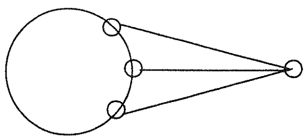
What's next? The graphs whose Hall-2^+-criticality next seems most obviously in question are listed below in Problem 1. In each case, removing any vertex results in a graph with Hall number ≤ 2, so the question of Hall-2^+-criticality rests on whether or not the Hall number is > 2. To show $h(G) > 2$ is simply a matter of finding a list assignment satisfying some requirements. Although this is not necessarily easy (for instance, it took us quite a while to discover an assignment for K_5 with an ear of length 2), it is far less painful than proving $h(G) = 2$, should this be the case.

Problem 1 *Which of the following are Hall–2^+-critical? (See Figure 3).*

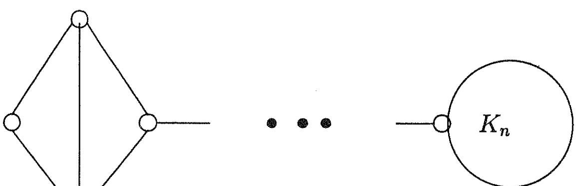
(a) A graph obtained by inserting one or more vertices of degree 2 onto one edge of a K_4.

(b) Cuff($\theta(2,2,1), K_n, \ell$), where $\ell \geq 0$, $n \geq 4$, and the point of attachment of the joining path to $\theta(2,2,1)$ is one of the vertices of degree 2 in $\theta(2,2,1)$.

(c) Cuff(G, K_n, ℓ) where $\ell \geq 0$, $n \geq 3$, G is K_4 with an ear of length 2, and the point of attachment of the joining path to G is the vertex of degree 2, in G.



(a)



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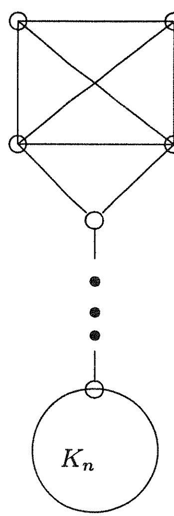


(c)

Figure 3:

When Problem 1 is solved the Hall-number-two problem could be quite close to solution, we estimate, although the final assault will be quite a producton.

The graphs $\theta(2,2,1)$, K_4 or K_5 with an ear of length 2, and the graphs in (d) and (e) of Theorem 4 are special cases of graphs formed by two intersecting cliques.

It would be interesting to know the Hall numbers of such graphs, as a sort of generalization of Hall's theorem. (Note that the main result of [7], referred to in the Introduction, says that any graph formed by sticking cliques together at cut-vertices has Hall number 1.)

Problem 2 *Suppose that a, b, and c are positive integers satisfying $a \geq b > c$. Determine, or estimate, in terms of a, b, and c, the Hall number of the graph consisting of a K_a and a K_b intersecting in a K_c.*

At one extreme, when $c = 1$, the Hall number is 1. At the other, Tuza [11] has shown that h (K_n minus an edge) $= n-2$; thus the Hall number above is $a-1$ when $a = b = c+1$.

3 Proofs and intermediate results

Lemma 1 *If $h(G_0) \leq 2$ and G is obtained by adding a path to G_0, intersecting G_0 only at one end-vertex of the path, then $h(G) \leq 2$.*

Proof: Suppose L is a list assignment to $V(G)$ such that G and L satisfy Hall's condition and $|L(v)| \geq 2$ for all $v \in V(G)$. Since $h(G_0) \leq 2$, G_0 can be properly L-colored. Since the lists on the path each contain at least two colors, it is straightforward to extend the L-coloring of G_0 to a proper L-coloring of G.

Definition A subgraph H of G is *L-tight* if and only if

$$|V(H)| = \sum_{\sigma \in C} \alpha(\sigma, L, H).$$

Clearly, if H is L-tight then in any proper L-coloring of H, each color σ appears on exactly $\alpha(\sigma, L, H)$ vertices of H. The following lemma is borrowed from [1].

Lemma 2 *Suppose that G and L satisfy Hall's condition. Suppose that K is a clique in G. Let L' be obtained from L by removing a symbol τ from every list $L(v)$, $v \in V(K)$, on which it appears. If G and L' do not satisfy Hall's condition then there is an L-tight induced subgraph H of G, intersecting K, such that every maximum independent set of vertices of H, among those bearing τ on their L-lists, contains a vertex of K.*

Proof: Since G and L satisfy Hall's condition, but G and L' do not, for some induced subgraph H of G we have $\sum_{\sigma \in C} \alpha(\sigma, L', H) < |V(H)| \leq \sum_{\sigma \in C} \alpha(\sigma, L, H)$. Going from L' to L by restoring τ to the lists on K from which it was removed does not affect the numbers $\alpha(\sigma, -, H)$, $\sigma \neq \tau$, and can increase $\alpha(\tau, -, H)$ by at most one, and only by that amount if every independent set of $\alpha(\tau, L, H)$ vertices of H bearing τ on their lists includes some vertex of K. The conclusions of the Lemma follow.

With G, L, K, τ, and H as in Lemma 2, for any proper L-coloring φ of H, it must be that τ is the color on a maximum independent set of vertices of H, among those with τ on their lists, because H is L-tight, so $\varphi(v) = \tau$ for some $v \in V(K)$.

Lemma 2 will usually be applied with K being a single vertex.

Definition Suppose that $u, v \in V(G)$, $a \in L(u)$ and $b \in L(v)$. We will say that (the choice of) a at u forces (the choice of) b at v through G if and only if there is a proper L-coloring φ of G with $\varphi(u) = a$, and for every such coloring, $\varphi(v) = b$. If G is a path with end-vertices u and v, the word "along" will be used in place of "through".

The following is extracted from [9], and we omit the proof, which is by induction on the length of the path.

Lemma 3 ([9], Lemma 2) *Suppose that P is a path with vertices $v_0, \dots, v_\ell$, in order, $a \in L(v_0)$, and $|L(v_i)| \geq 2$, $i = 1, \dots, \ell$. Then the choice of a at v_0 forces b at v_ℓ along P if and only if there exist $\sigma_0, \dots, \sigma_\ell$ with $a = \sigma_0$, $b = \sigma_\ell$, such that $L(v_j) = \{\sigma_{j-1}, \sigma_j\}$, $j = 1, \dots, \ell$.*

Corollary 2 *Suppose that P and L are as above, $a_1, a_2 \in L(v_0), b_1, b_2 \in L(v_\ell)$, $a_1 \neq a_2$, and a_i at v_0 forces b_i at v_ℓ along P, $i = 1, 2$. Then $b_1 \neq b_2$ and $L(v_j) = \{a_1, a_2\} = \{b_1, b_2\}$, $j = 1, \dots, \ell$.*

Corollary 3 *Suppose that P and L are as above, $\ell \geq 1$, $a \in L(v_0)$, and $b \in L(v_\ell)$. Suppose that a at v_0 forces b at v_ℓ along P, and b at v_ℓ forces a at v_0 along P. Then $L(v_0) = \cdots = L(v_\ell)$. If ℓ is odd, $a \neq b$ and $L(v_i) = \{a, b\}$, $i = 0, \dots, \ell$. If ℓ is even, $a = b$ and $L(v_i) = \{a, \sigma\}$, $i = 0, \dots, \ell$, for some $\sigma \neq a$.*

Proof: The proof is by induction on ℓ. The result is easy for $\ell = 1$. Suppose that $\ell > 1$. Then $|L(v_i)| = 2$, $i = 0, \dots, \ell$, by Lemma 3. Let $a = \sigma_0, \sigma_1, \dots, \sigma_{\ell-1}, \sigma_\ell = b$ be as in the conclusion of Lemma 3, arising from the supposed forcing of b by a. Since b at v_ℓ forces a at v_0 along P, it must be that $b \in L(v_{\ell-1}) = \{\sigma_{\ell-2}, \sigma_{\ell-1}\}$, by Lemma 3; since $b = \sigma_\ell \neq \sigma_{\ell-1}$, it must be that $b = \sigma_{\ell-2}$. Now, a at v_0 forces $\sigma_{\ell-1}$ at $v_{\ell-1}$ along $P - v_\ell$, and $\sigma_{\ell-1}$ at $v_{\ell-1}$ forces a at v_0 along $P - v_\ell$. By the induction hypothesis, $L(v_0) = \cdots = L(v_{\ell-1})$, and by reversing the roles of a and b, and of v_0 and v_ℓ, we conclude $L(v_1) = \cdots = L(v_\ell)$. Thus all $L(v_j)$ are the same, $j = 0, \dots, \ell$. If ℓ is odd, $\ell - 1$ is even, so $a = \sigma_{\ell-1} \neq b$ and the common list on P is $\{a, \sigma\}$ for some $\sigma \neq a$; since $L(v_{\ell-1}) = \{b, \sigma_{\ell-1}) = \{a, b\}$, it must be that $\sigma = b$. If ℓ is even, $\ell - 1$ is odd, so by the induction hypothesis $a \neq \sigma_{\ell-1}$ and the common list is $\{a, \sigma_{\ell-1}\}$; since $L(v_{\ell-1}) = \{b, \sigma_{\ell-1}\}$, it must be that $a = b$, in this case.

Proof of Theorem 2(d) Let the vertices of $G = \theta(3, 3, 2)$ be labeled as in Figure 4, and suppose that L is a list assignment such that G and L satisfy Hall's condition and $|L(x)| \geq 2$ for all $x \in V(G)$.

By earlier remarks, Theorem 2(a), and Lemma 1, $G - x$ is properly L-colorable for all $x \in V(G)$, so we may as well assume that $|L(x)| = 2$ for $x = x_1, x_2, y_1, y_2, v$. (Otherwise, G would surely be properly L-colorable.) Suppose that $L(v) = \{a, b\}$.

Then we may as well suppose that in each proper L-coloring of the 6-cycle $G - v$, one of u, w is colored a, the other b.

First suppose that some symbol $\tau \notin \{a, b\}$ is in $L(u) \cup L(w)$. Without loss of generality, suppose that $a, \tau \in L(u)$ and $b \in L(w)$. Since all lists are of length ≥ 2, we can properly L-color the path $P: u, y_1, y_2, w, x_2, x_1$ starting with τ at u: since no proper L-coloring of $G - v$ has τ at u, it must be that τ at u forces τ at x_1 along P. By Lemma 3 there exist $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ such that $L(y_1) = \{\tau, \sigma_1\}$, $L(y_2) = \{\sigma_1, \sigma_2\}$, $L(w) = \{\sigma_2, \sigma_3\}$, $L(x_2) = \{\sigma_3, \sigma_4\}$, and $L(x_1) = \{\sigma_4, \tau\}$. Note that $\sigma_1 \neq \tau$, $\sigma_1 \neq \sigma_2$, $\sigma_2 \neq \sigma_3$, $\sigma_3 \neq \sigma_4$, and $\sigma_4 \neq \tau$, although it may be that $\sigma_1 = \sigma_3$ or σ_4 or that $\sigma_2 = \sigma_4$, or that τ is either of σ_2, σ_3. Recall that τ is neither a nor b.

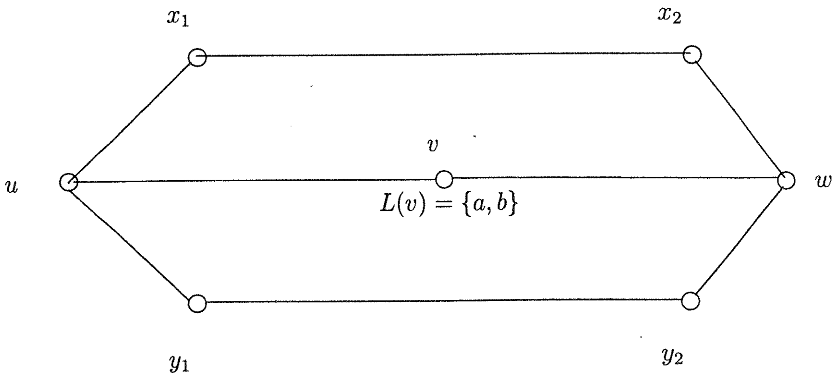


Figure 4:

Since $b \in L(w)$, either $b = \sigma_2$ or $b = \sigma_3$. First, assume that $b = \sigma_3$. Now we can properly L-color G by setting $\varphi(u) = a$, $\varphi(v) = b$, $\varphi(w) = \sigma_2$, $\varphi(y_2) = \sigma_1$, $\varphi(y_1) = \tau$, $\varphi(x_1) = \tau$, and $\varphi(x_2) = \sigma_3 (= b \neq \tau, \sigma_2)$. If $b = \sigma_2$, a similar coloring can be achieved.

So now we may assume that no such τ exists, after all, and $L(u) = L(w) = \{a, b\} = L(v)$.

Since all lists are of length 2, we may properly L-color each path $P_1 : u, x_1, x_2, w$, and $P_2 : u, y_1, y_2, w$, starting with either a or b at u. If there exist proper L-colorings of each path starting and ending with a, or of each starting and ending with b, then we can properly L-color G. So it must be that a at u forces b at w along one of P_1, P_2, and that b at u forces a at w along one of P_1, P_2.

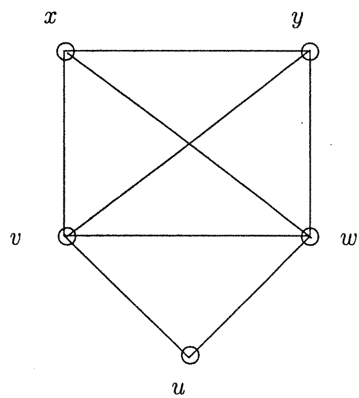
Without loss of generality, suppose that a at u forces b at w along P_1. By Lemma 3 there exists $\sigma_1 \neq a$ such that $L(x_1) = \{a, \sigma_1\} = L(x_2)$. Hall's condition implies that $\sigma_1 \neq b$, for, if $\sigma_1 = b$, L and the 5-cycle u, v, w, x_2, x_1, u do not satisfy the inequality in Hall's condition. [By Theorem 2(a) we know this without checking, because C_5 is not properly colorable with two colors.]

Therefore, it must be along P_2 that b at u forces a at w; therefore, for some $\sigma_2 \neq b$ (and $\sigma_2 \neq a$, for the same reason that $\sigma_1 \neq b$), $L(y_1) = L(y_2) = \{b, \sigma_2\}$. Whether or not $\sigma_1 = \sigma_2$ is not important—let us assume that $\sigma_1 \neq \sigma_2$. We now have that $\alpha(a, L, G) = \alpha(b, L, G) = 2$, $\alpha(\sigma_i, L, G) = 1$, $i = 1, 2$; thus $\sum_{\sigma \in C} \alpha(\sigma, L, G) = 6 < 7 = |V(G)|$, contradicting the assumption that G and L satisfy Hall's condition. □

Proof of Theorem 3(a). Let the vertices of $G = K_4$-with-an-ear-of-length-2 be labeled as in Figure 5, and suppose that L is a list assignment such that G and

L satisfy Hall's condition, and $|L(z)| \geq 2$ for all $z \in V(G)$. Since $G - z$ has Hall number ≤ 2 for each $z \in V(G)$, by previous remarks and Theorem 2(b), it follows that $G - z$ is properly L-colorable for each $z \in V(G)$. Therefore, we may as well suppose that $|L(u)| = 2$; say $L(u) = \{a, b\}$.

We may also suppose that L is *critical* with respect to the requirements it satisfies. This means, in this case, that if $|L(z)| > 2$ and $\sigma \in L(z)$, for some $\sigma \in C$, $z \in V(G)$, then the list assignment L' obtained from L by removing σ from $L(z)$, and changing the L-lists in no other way, will not satisfy Hall's condition with G. [If L is not critical, then remove symbols from lists until none can be removed further without reducing a list to length 1 or violating Hall's condition, and let this "reduced" assignment replace L. Surely a proper coloring with respect to the reduced assignment will be a proper L-coloring.]



$L(u) = \{a, b\}$

Figure 5:

We may as well suppose that in every proper L-coloring of the clique $G - u$, one of v, w is colored a and the other b. Keeping this firmly in mind, we first show that neither $L(v)$ nor $L(w)$ contains a, b, and a third symbol. Suppose, to the contrary, that $a, b \in L(v)$ and $|L(v)| \geq 3$. By the criticality of L, the list assignments obtained by removing a, respectively b, from $L(v)$ must fail to satisfy Hall's condition with G. By Lemma 2 there are L-tight subgraphs H_a, H_b of G, containing v, such that in $H_a(H_b)$, v is in every maximum independent set of vertices among those bearing a (b) on their lists. From the position of v in G, it follows that a (b) occurs on no list of $H_a(H_b)$ other than $L(v)$.

Therefore u is in neither H_a nor H_b; that is, both H_a and H_b are subgraphs of the clique $G - u$. Since $G - u$ is a clique, and Hall's condition is satisfied, $G - u$ is properly L-colorable; let φ be a proper L-coloring of $G - u$. Now, φ restricted to $V(H_a)$ and $V(H_b)$ properly L-colors H_a and H_b; but the tightness of H_a and H_b and the fact that a, resp. b, appears only in $L(v)$ among the lists on H_a, resp. H_b, forces

$\varphi(v) = a$ and $\varphi(v) = b$, an impossibility.

Because Hall's condition is satisfied by G and L, $L(u) \cup L(v) \cup L(w)$ must contain at least one symbol other than a and b. Without loss of generality, suppose that $\tau \in L(v) \backslash \{a, b\}$. By previous remarks and the result of the preceding two paragraphs, $L(v)$ must contain one of a, b, but not both. Without loss of generality, assume $a \in L(v)$ and $b \notin L(v)$, which forces $b \in L(w)$ (because $G - u$ is properly L-colorable and for any proper L-coloring φ of $G - u$, $\{\varphi(v), \varphi(w)\} = \{a, b\}$). Then in every proper L-coloring of $G - u$, v will be colored a and w will be colored b.

Since $G - u$ is a clique, it follows that removing a from $L(v)$ results in a list assignment that does not satisfy Hall's condition with $G - u$, and similarly upon removing b from $L(w)$. Applying Lemma 2, there exist L-tight subgraphs H_1, H_2 of $G - u$, with $v \in V(H_1)$ being the only vertex of H_1 bearing a on its L-list, and $w \in V(H_2)$ being the only vertex of H_2 bearing b on its L-list. From these considerations, the tightness of the H_i, and the fact that all lists are of cardinality at least two, it is easy to see that $|V(H_i)| \geq 3$, $i = 1, 2$.

Case 1: $H_1 = H_2 = G - u$. Then $|L(x) \cup L(y) \cup L(v) \cup L(w)| = 4$ and neither a nor b occurs in $L(x) \cup L(y)$. But then $\alpha(\sigma, L, G) = \alpha(b, L, G) = 1$ and we have $\sum_{\sigma \in C} \alpha(\sigma, L, G) = 4 < |V(G)|$. So this case is impossible.

Notice that the argument in the preceding case shows that if $G - u$ is L-tight, then either a or b must be an element of $L(x) \cup L(y)$.

Case 2: $H_1 = G - u$ and $|V(H_2)| = 3$. By remarks above, it cannot be that both x and y are in H_2. Without loss of generality, assume that $V(H_2) = \{v, w, y\}$.

Let $L(u) \cup L(v) \cup L(x) \cup L(y) = \{a, b, \tau, \sigma\}$ (noting that H_1 is L-tight). Since $L(y)$ contains neither a nor b, we have $L(y) = \{\tau, \sigma\}$. But then H_2 is not L-tight, because $a, b, \tau, \sigma \in L(v) \cup L(w) \cup L(y)$.

The case $|V(H_1)| = 3$ and $H_2 = G - u$ is handled similarly.

Case 3: $|V(H_1)| = |V(H_2)| = 3$.

Subcase 3(i): $H_1 = H_2$. If, say, $V(H_1) = V(H_2) = \{v, w, x\}$, then $L(x)$ contains at least two symbols, neither of them equal to a or b. But then $|L(v) \cup L(w) \cup L(x)| \geq 4$, so $H_1 = H_2$ is not L-tight; thus this subcase is impossible.

Subcase 3(ii): $\{v, w\} \subseteq V(H_1) \cap V(H_2)$ and $H_1 \neq H_2$. Without loss of generality, assume that $V(H_1) = \{v, w, x\}$ and $V(H_2) = \{v, w, y\}$. The tightness of H_1 and H_2 implies that $3 = |L(v) \cup L(w) \cup L(x)| = |L(v) \cup L(w) \cup L(y)|$. Since $a, b, \tau \in L(v) \cup L(w)$ it follows that only a, b, τ lie on the lists of $G - u$, a clique with 4 vertices, contradicting Hall's condition.

Subcase 3(iii): $\{v, w\} \subseteq V(H_1)$ and $v \notin V(H_2)$; then $V(H_2) = \{x, y, w\}$ and $V(H_1)$ is one of $\{v, w, x\}$, $\{v, w, y\}$; without loss of generality, assume that $V(H_1) = \{v, w, x\}$. Then $L(x) = \{b, \tau\} = L(w)$ (because H_1 is L-tight, $a, b, \tau \in L(v) \cup L(w)$, and a appears on no list on H_1 other than $L(v)$). But then b appears on a list, namely $L(x)$, of H_2 other than $L(w)$, an impossibility.

Subcase 3(iv): $\{v, w\} \subseteq V(H_2)$, $w \notin V(H_1)$. This is dismissed by an argument similar to that preceding.

Subcase 3(v): $V(H_1) = \{v, x, y\}$, $V(H_2) = \{w, x, y\}$. Then neither a nor b appears in $L(x) \cup L(y)$. Since $3 = |L(v) \cup L(x) \cup L(y)| = |L(w) \cup L(x) \cup L(y)|$ and

$|L(x)|, |L(y)| \geq 2$, it must be that $L(x) = L(y) = \{\tau, \gamma\}$ for some symbol γ different from a, b, and τ. But then $\alpha(\sigma, L, G) = 1$, $\sigma = a, b, \tau, \gamma$, so $\sum_{\sigma \in C} \alpha(\sigma, L, G) = 4 < 5 = |V(G)|$, contradicting Hall's condition.

The possibilities are exhausted; it must be that G is properly L-colorable, after all. □

Proof of Theorem 3(b) Let G be a graph as described in 3(b), with vertices labeled as in Figure 6.

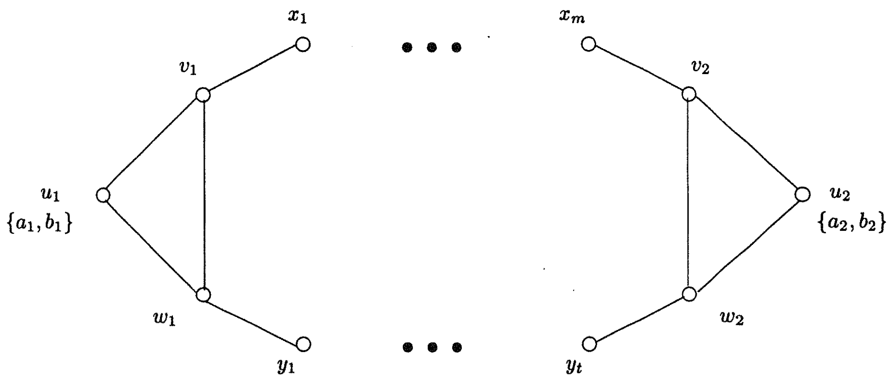


Figure 6:

Note that $m+t$ is even, and either m or t may be zero; $m = 0$, for instance, means that v_1 and v_2 are adjacent. Suppose that L is a list assignment such that G and L satisfy Hall's condition, and $|L(z)| \geq 2$ for all $z \in V(G)$. Suppose that there is no proper L-coloring of G. [For those who abhor proofs by contradiction: every inference below proceeding from the assumption that there is no proper L-coloring of G can be introduced, in the absence of this assumption, by a sentence of the form "We may as well suppose that ..., otherwise there is clearly a proper L-coloring of G." Viewed in this way, the proof constitutes a list of instructions for finding a proper L-coloring of G.] Since, for $z \in V(G)$, $G - z$ is either a graph with every block a clique, or $G - z = \theta(m + t + 3, 2, 1)$, $G - z$ is properly L-colorable. Therefore, $|L(z)| = 2$ for $z \in \{u_1, u_2, x_1, \ldots, x_m, y_1, \ldots, y_t\}$ and $|L(z)| \leq 3$ for $z \in \{v_1, w_1, v_2, w_2\}$.

By previous results and remarks, it is also the case that for every $e \in E(G)$, $h(G-e) \leq 2$, and therefore $G-e$ is properly L-colorable. Therefore, in every proper L-coloring of $G - e$, the ends of e receive the same color. (Otherwise, the proper L-coloring of $G - e$ would also properly L-color G.)

Suppose $L(u_1) = \{a_1, b_1\}$ and $L(u_2) = \{a_2, b_2\}$. It must be that in every proper L-coloring of $G - u_i$, v_i and w_i are colored with a_i and b_i, $i = 1, 2$. This implies that $\{a_i, b_i\} \subseteq L(v_i) \cup L(w_i)$, $i = 1, 2$.

First we show that $L(v_i) \neq \{a_i, b_i\} \neq L(w_i)$, $i = 1, 2$. Suppose, to the contrary, that $L(v_1) = \{a_1, b_1\}$. Let $e = v_1x$ be the edge of G incident to v_1 and to x_1 (if $m \geq 1$) or to v_2 (if $m = 0$). Let φ be a proper L-coloring of $G - e$. By previous remarks, $\varphi(v_1) = \varphi(x)$; without loss of generality, suppose that $a_1 = \varphi(v_1) = \varphi(x)$. Then, ξ, defined by $\xi(u_1) = a_1$, $\xi(v_1) = b_1$, and $\xi = \varphi$ on $V(G)\backslash\{u_1, v_1\}$, properly L-colors G.

Now let ψ be a proper L-coloring of $G - u_2$. By remarks above, $\{a_2, b_2\} = \{\psi(v_2), \psi(w_2)\}$. Without loss of generality, let $\psi(v_2) = a_2$ and $\psi(w_2) = b_2$. Because ψ is a proper coloring, at least one of $\psi(v_1), \psi(w_1)$ must be something other than a_1, b_1; without loss of generality, assume $\psi(v_1) = \tau \notin \{a_1, b_1\}$.

Let P_{high} be the path with vertices $v_1, x_1, \ldots, x_m, v_2$ and let P_{low} be the path with vertices $w_1, y_1, \ldots, y_t, w_2$. First note that in every proper L-coloring of P_{high} with τ at v_1, v_2 must be colored a_2 or b_2; if not, if there were a proper L-coloring of P_{high} with τ at v_1 and some $\sigma \notin \{a_2, b_2\}$ at v_2, then we could put this coloring together with ψ on $\{u_1\} \cup P_{\text{low}}$, and then color u_2 with $a_2 (\neq b_2 = \psi(w_2))$ to obtain a proper L-coloring of G.

So, assuming there is no proper L-coloring of G (as we have been), τ at v_1 forces "a_2 or b_2" at v_2, along P_{high}. Apply Lemma 3 with the b there being either a_2, if $b_2 \notin L(v_2)$, or "a_2 or b_2", i.e., a_2, b_2 combined for the moment into a single color, in $L(v_2)$, in case $\{a_2, b_2\} \subseteq L(v_2)$. Since $L(v_2) \neq \{a_2, b_2\}$, $L(v_2)$ must contain something other than "a_2 or b_2". Then Lemma 3 implies the existence of $\tau = \sigma_0, \sigma_1, \ldots, \sigma_m$ such that $L(x_i) = \{\sigma_{i-1}, \sigma_i\}$, $i = 1, \ldots, m$ and $L(v_2) = \{a_2, \sigma_m\}$ or $\{a_2, b_2, \sigma_m\}$, and $\sigma_m \notin \{a_2, b_2\}$. [This holds as well when $m = 0$; $\sigma_0 = \sigma_m = \tau \notin \{a_2, b_2\}$; recall that $\tau \notin \{a_1, b_1\}$, also.]

Observe that the choice of σ_m at v_2 forces the choice of "not τ", i.e. of any color in $L(v_1)$ besides τ, at v_1, along P_{high}. We know that $L(v_1)$ contains one or both of a_1, b_1 (because one of these must color v_1 in a proper L-coloring of $G - u_1$). It follows that $\psi(w_1) \in \{a_1, b_1\}$, because if, on the contrary, $\psi(w_1) \notin \{a_1, b_1\}$, then we can color P_{high} starting with σ_m at v_2, along to one of a_1, b_1 at v_1, put that together with ψ on P_{low}, and then finish off by coloring u_2 with a_2 and u_1 with whichever of a_1, b_1 is not coloring v_1, to obtain a proper L-coloring of G. Without loss of generality, assume that $\psi(w_1) = b_1$.

By the same sort of reasoning, we may assume that $L(v_1)\backslash\{a_1, b_1, \tau\}$ is empty, so $L(v_1)$ consists of τ and either a_1 alone or both a_1 and b_1.

Since $L(w_2) \neq \{a_2, b_2\}$, $L(w_2)$ contains a symbol γ other than a_2, b_2. Since all lists are of cardinality ≥ 2, we can properly L-color P_{low} starting with γ at w_2. It must be that in every such coloring, w_1 is colored with $\tau = \psi(v_1)$, because otherwise we could put such a coloring together with ψ on P_{high} and finish off a proper coloring of G by coloring u_1, u_2, with no difficulty. That is, γ at w_2 forces τ at w_1 along P_{low}. (Before this, we did not know that $\tau \in L(w_1)$.) By Lemma 3, $|L(w_1)| = 2$, so $L(w_1) = \{b_1, \tau\}$.

Now we show that $L(w_2) = \{b_2, \gamma\}$, and $\gamma = \sigma_m$. We know that b_2, $\gamma \in L(w_2)$. The equality $L(w_2) = \{b_2, \gamma)$, and $\gamma = \sigma_m$, will follow if there is a proper L-coloring of $G - u_1$, with σ_m coloring v_2. [$L(w_1) = \{b_1, \tau\}$ arose from the assumption of a proper L-coloring ψ of $G - u_2$, with $\tau \notin \{a_1, b_1\}$ coloring v_1.] We obtain such a coloring by coloring P_{low} with ψ, and P_{high} with σ_m at v_2, a_1 at v_1—we have already seen that a proper such coloring exists—and, finally, a_2 at u_2.

We hope that you have been keeping accounts! At this point, we have that $L(w_2) = \{b_2, \gamma\}$, $L(w_1) = \{b_1, \tau\}$, $L(v_1) = \{a_1, \tau\}$ or $\{a_1, b_1, \tau\}$, and $L(v_2) = \{a_2, \gamma\}$ or $\{a_2, b_2, \gamma\}$, with a_1, b_1, τ distinct and a_2, b_2, γ distinct. More importantly, γ at w_2 forces τ at w_1 along P_{low}, and τ at w_1 forces γ at w_2 along P_{low} [either by reversing the roles of u_1 and u_2, or—if not, put a proper coloring of P_{low} with τ at w_1 and b_2 at w_2 together with a proper coloring of P_{high} with γ at v_2 and a_1 at v_1 and then finish off with b_1 at u_1 and a_2 at u_2].

By Corollary 3 it follows that $L(w_1) = L(y_1) = \cdots = L(y_t) = L(w_2)$.

Case 1. t is even. Then the length of P_{low} is odd, and by Corollary 3 we have that $\gamma \neq \tau$, so $\gamma = b_1$, $\tau = b_2$, and the common list along P_{low} is $\{b_1, b_2\} = \{\gamma, \tau\}$. In this case we will see that it is possible to properly L-color G. Properly L-color P_{low} with $b_2 = \tau$ at w_1 and $b_1 = \gamma$ at w_2, and color u_i with b_i, $i = 1, 2$. Now we try to properly color P_{high}; just to make things harder, delete b_1 from $L(v_1)$ and b_2 from $L(v_2)$, if either is there; now $L(v_1) = \{a_1, \tau\} = \{a_1, b_2\}$ and $L(v_2) = \{a_2, \gamma\} = \{a_2, b_1\}$, and τ at v_1 forces a_2 at v_2, along P_{high}.

We hope to color P_{high} with a_1 at v_1 and a_2 at v_2, because such a coloring will go well with the coloring already done to make a proper L-coloring of G. We can find such a coloring unless a_1 at v_1 forces γ at v_2 along P_{high}. If that were the case, then by Corollary 2, $L = \{a_1, \tau\} = \{a_2, \gamma\}$ at every vertex of P_{high}. But since $\tau = b_2 \neq a_2$, that would mean that $\tau = \gamma$, contradicting $\tau \neq \gamma$, in this case.

Case 2. t is odd. Then the length of P_{low} is even, so $\gamma = \tau$ and the common list along P_{low} is $\{\tau, \sigma\}$ for some $\sigma \neq \tau$. Since $L(w_1) = \{\tau, b_1\}$ and $L(w_2) = \{\tau, b_2\}$, we have that $\sigma = b_1 = b_2$.

Rashly removing $b_1 = b_2$ from $L(v_1), L(v_2)$, if necessary, we try to properly L-color G by properly coloring P_{low} with τ on w_1 and w_2, $b_1 = b_2$ on u_i, $i = 1, 2$ and starting in with a_1 on v_1, with the hope of coloring P_{high} with a_i on v_i, $i = 1, 2$. Recall that τ on v_1 forces a_2 on v_2, along P_{high}. If a_1 on v_1 forces $\tau = \gamma$ on v_2 along P_{high}, then $L = \{a_1, \tau\} = \{a_2, \tau\}$ at all vertices of P_{high}, by Corollary 2, so $a_1 = a_2$. But the length of P_{high} is even, because the length of P_{low} was, so a_1 at v_1 does *not* force τ at v_2 along P_{high}, after all—it forces $a_2 = a_1$. Thus there is a proper L-coloring of G, after all! □

Lemma 4 *Suppose $v \in V(G)$ and for each non-negative integer ℓ, $G_\ell = Cuff(G, K_3, \ell)$, with the joining path attached to G at v. If $h(G_\ell) = 2$ for some ℓ, then $h(G_\ell) = 2$ for all $\ell = 0, 1, 2, \ldots$.*

Proof: It suffices to show that, for $\ell > 0$, $h(G_\ell) = 2$ if and only if $h(G_0) = 2$. First suppose that $h(G_0) = 2$, $\ell > 0$, and the vertices of G_ℓ are labeled as in Figure 7.

In case $\ell > 1$, let the internal vertices along the joining path P from v to w be $u_1, \ldots, u_{\ell-1}$.

Suppose that L is a list assignment to G_ℓ such that G_ℓ and L satisfy Hall's condition and $|L(z)| \geq 2$ for all $z \in V(G_\ell)$, but there is no proper L-coloring of G_ℓ.

Since G is an induced subgraph of G_0, $h(G) \leq h(G_0) = 2$, so $h(G) = 2$, because if $h(G) = 1$ then every block of G is a clique, so the same would be true of G_ℓ.

By Lemma 1, $G_\ell - x$ and $G_\ell - y$ have Hall number 2. Therefore, it must be that $|L(x)| = |L(y)| = 2$; for if, say, $|L(x)| \geq 3$, then clearly G_ℓ would be properly L-colorable. Also, $|L(w)| \leq 3$.

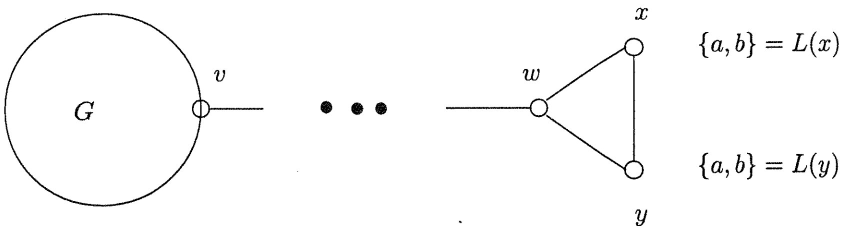


Figure 7:

As in the proof of Theorem 3(a), we may assume criticality: for any list $L(z)$, $z \in V(G_\ell)$, with $|L(z)| \geq 3$, removing any single symbol from $L(z)$ and disturbing no other list results in a list assignment that does not satisfy Hall's condition with G_ℓ. We may also assume that for each $\sigma \in C$, the subgraph $G_\ell(\sigma)$ of G_ℓ induced by $\{z \in V(G_\ell);\ \sigma \in L(z)\}$ is connected; if it is not, replace σ in the lists on the different components of $G_\ell(\sigma)$ by different symbols in C, none previously appearing in any L-lists on G_ℓ. It is straightforward to see that after this replacement, the new list assignment satisfies Hall's condition with G_ℓ, and that there is a proper coloring of G_ℓ from the new assignment if and only if there is one with the old. (The assignment after replacement satisfies Hall's condition with G_ℓ if *and only if* the original assignment does; this is laboriously proven in [1]. The "only if" part of this proposition implies that the replacement also preserves the criticality mentioned above—alternatively, a critical list assignment with every $G_\ell(\sigma)$ connected can be achieved by a sequence of symbol replacements alternating with list pruning.]

By Lemma 1, there is a proper L-coloring φ of $G_\ell - \{x, y\}$. Since $|L(x)| = |L(y)| = 2$ and G_ℓ is not properly L-colorable, it must be that $L(x) \backslash \varphi(w) = L(y) \backslash \varphi(w)$, a singleton. Thus, for some $a \neq b$, $L(x) = L(y) = \{a, b\}$, and at least one of a, b, say b, is in $L(w)$.

First we show that not both of a, b can be in $L(w)$. Suppose, to the contrary, that $\{a, b\} \subseteq L(w)$. Because Hall's condition is satisfied, the triangle with vertices w, x, y is properly L-colorable, so $L(w)$ must contain some symbol $c \notin \{a, b\}$. Thus $L(w) = \{a, b, c\}$.

Removing any of a, b, c from $L(w)$ results in a new list assignment which does not satisfy Hall's condition with G_ℓ. Applying Lemma 2 in the cases of removing a or b we see that there are L-tight induced subgraphs H_a, H_b of G_ℓ, containing w, such that w is in every maximum independent set of vertices of H_τ, among those bearing τ on their L-lists, for $\tau = a, b$. Then neither H_a nor H_b contains either of x, y; i.e., H_a and H_b are subgraphs of G_ℓ-$\{x, y\}$, which is properly L-colorable. A proper L-coloring of G_ℓ-$\{x, y\}$ properly L-colors H_a and H_b, both L-tight—so w would have to be colored a *and* b in such a coloring, an impossibility.

Thus $a \notin L(w)$ and $b,\ c \in L(w)$. Because $G_\ell(a)$ is connected, a appears only in $L(x)$ and $L(y)$, among the lists on G_ℓ. This observation will be useful, very shortly.

G is properly L-colorable (since, as noted above, $h(G) = 2$). Since G_ℓ is not properly L-colorable, it must be the case that for any proper L-coloring of G, whatever v is colored will force b at w along P. From Lemma 3 and Corollary 2 it follows that $|L(w)| = 2$, so $L(w) = \{b, c\}$, and there is only one symbol τ with which v can be colored, in any proper L-coloring of G. If $\ell = 1$, $\tau = c$. Otherwise, if $\ell > 1$, let $L(u_j) = \{\sigma_{j-1}, \sigma_j\}$, $j = 1, \ldots, \ell - 1$ (as in Lemma 3), with $\sigma_0 = \tau$ and $\sigma_{\ell-1} = c$.

In any case, $\tau \neq a$. We define a list assignment $\tilde{L}$ on G_0, with G_0 as in Figure 8, by $\tilde{L} = L$ on $V(G)$ and $\tilde{L}(x) = \tilde{L}(y) = \{a, \tau\}$.

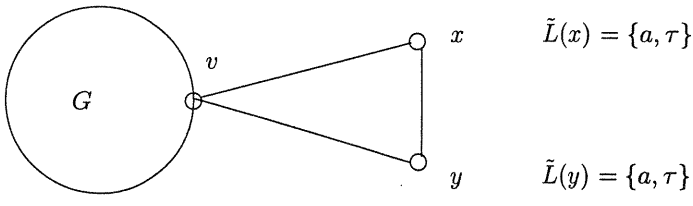


Figure 8:

Clearly G_0 is not properly $\tilde{L}$-colorable, since in every proper L-coloring of G, v must be colored with τ. If we show that G_0 and $\tilde{L}$ satisfy Hall's condition, we will be done with this part of the proof. Suppose that $\tilde{H}$ is an induced subgraph of G_0. If $\tilde{H}$ is a subgraph of G, then $(*)$ holds, with H there replaced by $\tilde{H}$ (and note that $L = \tilde{L}$ on G), so suppose that one or both of x, y are in $\tilde{H}$. If $\tilde{H}$ contains only one, say x, then the fact that $(*)$ holds with H replaced by $\tilde{H} - x$ and L by $\tilde{L}$ implies the same for $\tilde{H}$, since a appears on no L-lists on G. So suppose that $x,\ y \in V(\tilde{H})$, and $(*)$ does not hold, for $\tilde{L}$ and $\tilde{H}$.

Again, the facts that $(*)$ is satisfied by $\tilde{H}$-$\{x, y\}$ and $\tilde{L}$, and that a appears in no L-list on G, implies that we may as well suppose that $\sum_{\sigma \in C} \alpha(\sigma, \tilde{L}, \tilde{H}) = |V(\tilde{H})| - 1$, and that v is not only in $\tilde{H}$, but also is in every maximum independent set of vertices $\tilde{H} - \{x, y\}$, among those with τ on their L-lists. [Otherwise, we would have $\alpha(\tau, \tilde{L}, \tilde{H}) = \alpha(\tau, \tilde{L}, \tilde{H} - \{x, y\}) + 1$, so

$$\begin{aligned}\sum_{\sigma \in C} \alpha(\sigma, \tilde{L}, \tilde{H}) &= \sum_{\sigma \in C} \alpha(\sigma, L, \tilde{H} - \{x, y\}) + 2 \\ &\geq |V(\tilde{H} - \{x, y\})| + 2 = |V(\tilde{H})|.)\end{aligned}$$

Let H be the subgraph of G_ℓ induced by $V(\tilde{H}) \cup \{u_1, \ldots, u_{\ell-1}, w\}$; i.e., H is obtained by joining the triangle with vertices w, x, and y to $\tilde{H} - \{x, y\}$ by the path P, with v being the point of attachment. Clearly $|V(H)| = |V(\tilde{H})| + \ell$. If

$$\sum_{\sigma\in C}\alpha(\sigma,L,H)\le\sum_{\sigma\in C}\alpha(\sigma,\tilde{L},\tilde{H})+\ell \tag{**}$$

then

$$\sum_{\sigma\in C}\alpha(\sigma,L,H)\le|V(\tilde{H})|-1+\ell=|V(H)|-1,$$

contradicting the assumption that G_ℓ and L satisfy Hall's condition.

To see that (**) holds (with equality, in fact), think of $\sum_{\sigma\in C}\alpha(\sigma,\tilde{L},\tilde{H})$ being added to by the lists along P, to get up to $\sum_{\sigma\in C}\alpha(\sigma,L,H)$. The occurrence of a in the lists on x and y contributes 1 to both sums; the occurrence of τ in $\tilde{L}(x)$, $\tilde{L}(y)$ contributes nothing to the sum $\sum_{\sigma}\alpha(\sigma,\tilde{L},\tilde{H})$, because of the earlier conclusion about v being in every maximum independent set of vertices of $\tilde{H}-\{x,y\}$, among those with τ on their lists. For the same reason, the occurrence of $\tau=\sigma_0$ in $L(u_1)$ (or $(L(w)$, if $\ell=1$) contributes nothing to $\sum_\sigma\alpha(\sigma,L,H)$. For $1\le j\le\ell-1$ (supposing $\ell\ge2$), the two appearances of σ_j contribute 1 to this sum, and, finally, b contributes one more, for a total of ℓ, whence (**), with equality.

This account glosses over the possibility that some σ_j might equal σ_{j+2}, including the possibility that b might be the same as $\sigma_{\ell-2}$. Nonetheless, the account is accurate. To see this, recollect that $G_\ell(\sigma)$ is connected, for each $\sigma\in C$; from this and the facts that $L(u_j)=\{\sigma_{j-1},\sigma_j\}$, $j=1,\dots,\ell-1$, and $L(w)=\{b,c\}=\{b,\sigma_{\ell-1}\}$, it is easy to see that τ appears on the lists of an even number of consecutive vertices of P, starting at v, that b appears on an odd number of consecutive vertices of $P-v$, counting back from w, and that each $\sigma\in\{\sigma_1,\dots,\sigma_{\ell-1}\}\backslash\{\tau,b\}$ appears on the lists of a subpath of $P-v$ of even order. It may be that $\sigma_1\in L(v)$, or not. In any of the several cases ($\sigma_1=b$, $\sigma_1\ne b$, $\sigma_1\in L(v)$, or $\sigma_1\notin L(v)$) it is straightforward to see that the claim of the preceding paragraph is true: for $1\le j\le\ell-1$, the appearance of σ_j in $L(u_j)$ and in $L(u_{j+1})$ (where $u_\ell=w$) contributes 1 to the sum $\sum_\sigma\alpha(\sigma,L,H)$, over $\sum_\sigma\alpha(\sigma,\tilde{L},\tilde{H})$, the appearance of τ in $L(u_1)$ contributes nothing, and the appearance of b in $L(x)$, $L(y)$ and $L(w)$ contributes 1. This completes the proof that if $h(G_0)=2$, then $h(G_\ell)=2$ for any $\ell>0$.

Now suppose that $h(G_\ell)=2$ for some $\ell>0$. We want to show that $h(G_0)=2$. As before, we conclude immediately that $h(G)=2$.

Suppose that L is a list assignment to $V(G_0)$ satisfying Hall's condition with G_0, and $|L(z)|\ge2$ for all $z\in V(G_0)$, and suppose that there is no proper L-coloring of G_0. As in the first half of the proof, we aim for a contradiction by producing a list assignment $\tilde{L}$, this time to $V(G_\ell)$, from which there is no proper coloring of G_ℓ, although $|\tilde{L}(z)|\ge2$ for all $z\in V(G_\ell)$ and G_ℓ and $\tilde{L}$ satisfy Hall's condition.

Also as before, we may assume that L is critical, i.e., if $|L(z)|\ge3$ then removing any single symbol from $L(z)$ results in a list assignment that does not satisfy Hall's condition with G_0.

Let the vertices of G_0 and G_ℓ be labeled as in Figures 8 and 7 (and ignore the lists in those figures). Since $h(G)=2$, G_0-x and G_0-y are properly L-colorable, by Lemma 1, so $|L(x)| = |L(y)| = 2$. The proper L-colorability of G then implies, as in the first part of the proof, that $L(x) = L(y) = \{a, b\}$, say, and in every proper L-coloring of G, v is colored with a or with b.

Because Hall's condition is satisfied, the K_3 induced by v, x, and y is properly L-colorable, so $L(v)$ contains a symbol not in $\{a, b\}$. We show that $L(v)$ does not contain both a and b. If $a, b \in L(v)$ then $|L(v)| \geq 3$; by criticality and Lemma 2 there exist L-tight subgraphs H_a, H_b of G_0, each containing v, with v in every maximum independent set of vertices of H_a, resp. H_b, among those with a, resp. b, on their lists. Then neither x nor y is a vertex in either H_a or H_b; i.e., H_a, H_b are subgraphs of G. There is a proper L-coloring of G, and the properties of H_a, H_b imply that in any such, v must be colored with both a and b, an impossibility.

Thus exactly one of a, b, say b, is in $L(v)$, so in every proper L-coloring of G, v is colored b.

Make a list assignment $\tilde{L}$ to G_ℓ by taking $\tilde{L} = L$ on $V(G)$, $\tilde{L}(x) = \tilde{L}(y) = \{\sigma, \tau\}$ and $\tilde{L}(w) = \{\tau, \gamma\}$, $\gamma \neq \sigma$, where τ, σ are new symbols that appear nowhere in the lists on the vertices of G, and so is γ, if $\ell > 1$; if $\ell = 1$, $\gamma = b$. If $\ell > 1$, equip the internal vertices of P with lists of 2 symbols each, so that b at v forces τ at w, along P. Clearly G_ℓ is not properly $\tilde{L}$-colorable. It remains to show that G_ℓ and $\tilde{L}$ satisfy Hall's condition. Suppose H is an induced subgraph of G_ℓ. We want to show that

$$\sum_{\sigma \in C} \alpha(\sigma, \tilde{L}, H) \geq |V(H)|. \qquad (*)'$$

Let $H_1 = H \cap G$, i.e., the subgraph induced (in G) by $V(G) \cap V(H)$. We have that $\sum_{\sigma \in C} \alpha(\sigma, L, H_1) \geq |V(H_1)|$; from the relation of H to H_1, and the nature of the new lists on $G_\ell - V(G)$, it is straightforward to see that there is only one set of circumstances in which $(*)'$ could fail: H contains $v, u_1, \ldots, u_{\ell-1}, w, x$, and y, v is in every maximum independent set of vertices of H_1, among those with b on their L-lists (so the occurrence of b in $\tilde{L}(u_1)$, or $\tilde{L}(w)$ if $\ell = 1$, contributes nothing to $\sum_{\sigma \in C} \alpha(\sigma, \tilde{L}, H)$), and H_1 is L-tight.

But these circumstances regarding H_1, L, b, and v cannot hold, for consider H_2, the subgraph of G_0 induced by $V(H_1) \cup \{x, y\}$. If v is in every maximum independent set of vertices in H_1 with b on their lists, then $\alpha(b, L, H_1) = \alpha(b, L, H_2)$. Meanwhile, clearly $\alpha(a, L, H_2) \leq \alpha(a, L, H_1) + 1$. So if H_1 is L tight, we would have

$$\begin{aligned} |V(H_2)| = |V(H_1) + 2 &= \sum_{\sigma \in C} \alpha(\sigma, L, H_1) + 2 \\ &\geq \sum_{\sigma \in C} \alpha(\sigma, L, H_2) - 1 + 2, \end{aligned}$$

contradicting that G_0 and L satisfy Hall's condition. □

Proof of Theorem 3(c) By Lemma 4, it suffices to prove the result for $\ell = 0$. Let $A =$*Cuff*$(\theta(2, 2, 1), K_3, 0)$, as described in the theorem, be labeled as shown in Figure 9. Suppose that L is a list assignment to $V(A)$, satisfying Hall's condition with A, with $|L(z)| \geq 2$ for all $z \in V(A)$, and suppose that there is no proper L-coloring of A; we aim to prove a contradiction. As in the proofs of Lemma 4 and Theorem 3(a), we can assume that L is critical.

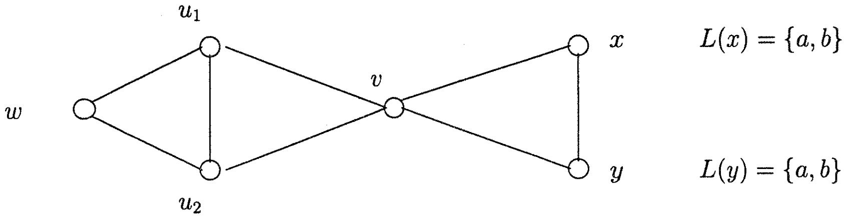


Figure 9:

Since $L - x$ and $L - y$ have Hall number 2, and $L - w$ has Hall number 1, it must be that $|L(x)| = |L(y)| = |L(w)| = 2$. Since $A - \{x, y\}$ is properly L-colorable, and A isn't, it must be that $L(x) = L(y) = \{a, b\}$, say, and in every proper coloring of $G = A - \{x, y\}$, v is colored with one of a, b.

Since the triangle $T(v, x, y)$ with vertices v, x, y is properly L-colorable, there is a symbol $c \notin \{a, b\}$ in $L(v)$. As in earlier proofs, we use criticality to show that $L(v)$ cannot contain both a and b. If, on the contrary, $\{a, b\} \subseteq L(v)$, then $|L(v)| \geq 3$; thinking of removing either of a, b from v, by criticality and Lemma 2 there exist L-tight subgraphs H_a, H_b of A such that v is in every maximum independent set of vertices of H_τ among those bearing τ on their lists, $\tau = a, b$. Thus neither H_a nor H_b contains either of x, y; i.e., both H_a and H_b are subgraphs of $G = A - \{x, y\}$. But G is properly L-colorable, and any proper L-coloring of G colors H_a and H_b, as well, which leads to the absurd conclusion that v has to be colored both a and b, in such a coloring.

So $L(v)$ contains only one of a, b—say b, and in every proper coloring of G, v is colored with b. We have that b, $c \in L(v)$; next we show that $L(v) = \{b, c\}$. If not, then $|L(v)| \geq 3$, and, thinking of removing b from $L(v)$, we still have the L-tight subgraph H_b of G, referred to above, with v in every maximum independent set of vertices of H_b, among those with b on their lists. Let H be the subgraph of A induced by $V(H_b) \cup \{x, y\}$. Then $\alpha(b, L, H) = \alpha(b, L, H_b)$, $\alpha(a, L, H) = \alpha(a, L, H_b) + 1$, and clearly $\alpha(\sigma, L, H) = \alpha(\sigma, L, H_b)$ for all $\sigma \in C \backslash \{a, b\}$, so, because H_b is tight,

$$\begin{aligned}\sum_{\sigma \in C} \alpha(\sigma, L, H) &= \sum_{\sigma \in C} \alpha(\sigma, L, H_b) + 1 \\ &= |V(H_b)| + 1 < |V(H_b)| + 2 = |V(H)|,\end{aligned}$$

contradicting the assumption that A and L satisfy Hall's condition.

So $L(v) = \{b, c\}$. Now, observe that $A - u_1v$ is a graph with every block a clique, and so has Hall number 1. Therefore, there is a proper L-coloring of $A - u_1v$, and in every such, u_1 and v must receive the same color; that color must be c, since in every proper L-coloring of $T(v,x,y)$, v is colored c. Similarly, in every proper L-coloring of $A - u_2v$, u_2 and v are colored c. It follows not only that $c \in L(u_1) \cap L(u_2)$, but also that there are at least two different proper L-colorings of the triangle $T(u_1, u_2, w)$ with vertices u_1, u_2, w, in one of which u_1 is colored c, and in the other u_2 is colored c. Furthermore, in any proper L-coloring of that triangle, one or the other of u_1, u_2 must be colored c—otherwise, a coloring of w, u_1, and u_2 could be extended to a proper L-coloring of A.

Next we observe that $c \notin L(w)$; if, on the contrary, $c \in L(w)$, then, since w is not colored c in any proper coloring of $T(u_1, u_2, w)$, and $c \in L(u_1) \cap L(u_2)$, it must be that $L(u_1) = L(u_2) = \{c, d\}$ for some d (distinct from c, but not necessarily from b or a). However, it then follows that $(*)$ fails for $H = A - w$ (by direct calculation; it also follows that Hall's condition is violated somehow because H is not properly L-colorable), contradicting the assumption that A and L satisfy Hall's condition.

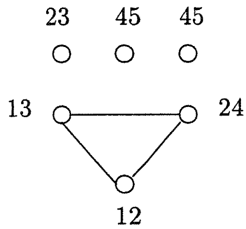
So, recalling that $|L(w)| = 2$, we have that $c \notin L(w) = \{d, e\}$, for some d, $e \in C$. Also, by the observation above about $H = A - w$, it is not possible that $L(u_1) = L(u_2) = \{f, c\}$ for any symbol $f \in C$.

From this and previous conclusions about $T(u_1, u_2, w)$, it must be that d, $e \in L(u_1) \cup L(u_2) \subseteq \{c, d, e\}$. But then, by direct computation, $(*)$ fails with H replaced by A, contradicting the assumption that Hall's condition is satisfied by A and L. □

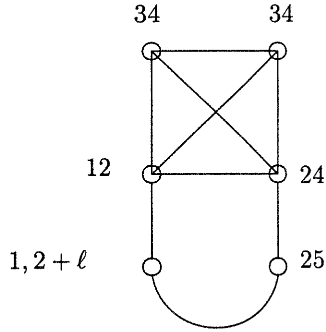
Proof of Theorem 4. By previous results, either proven here or in [7] or in [9], for each graph A claimed to be Hall-2^+-critical in Theorem 4, and each $z \in V(A)$, $h(A - z) \leq 2$, so all that remains is to produce, for each A, a list assignment L to $V(A)$, satisfying Hall's condition with A, such that $|L(z)| \geq 2$ for all $z \in V(A)$, and such that there is no proper L-coloring of A. These list assignments are given pictorially, using positive integers for colors (without brackets and commas, sometimes, so 12 stands for $\{1, 2\}$, for example). In the cases listed under (j) assignments are given for the case $\ell = 0$ only, and this suffices to show that $h(A) > 2$ for all ℓ, by Lemma 4. Regarding part (d), the theorem of Tuza [11] mentioned earlier implies that $h(A) = 3$; we give an assignment showing $h(A) > 2$, anyway.

In every case, it is straightforward to see that no proper coloring is possible. Verifying that Hall's condition is satisfied is a little harder; in each case, check that $(*)$ holds with $H = A$, and then verify that $A - z$ is properly colorable for each $z \in V(A)$, from the given list assignment.

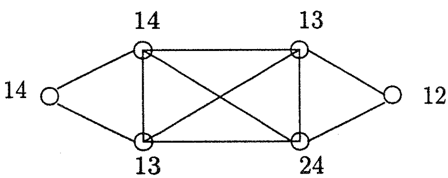
The graphs in all but (d), (e), (g), and (j)(3) are line graphs, and the list assignments in most of these cases are due entirely to the first author; they will also appear, in edge assignment form, in his paper [3], which completely characterizes the line graphs with Hall number ≤ 2.



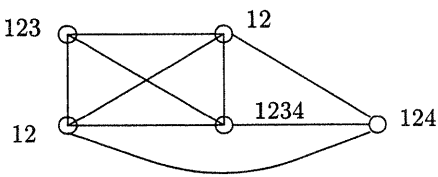
(a), nine edges omitted



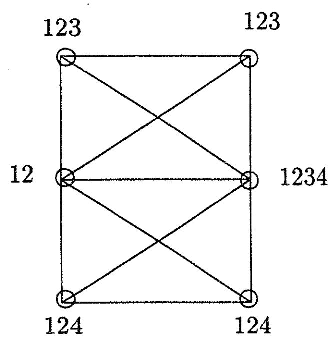
(b), with $\ell =$ ear length; in case $\ell > 3$, the lists on the vertices between 25 and 1, $2 + \ell$ are $\{k-1, k\}$, $k = 6, \ldots, 2 + \ell$



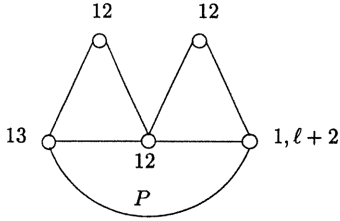
(c)



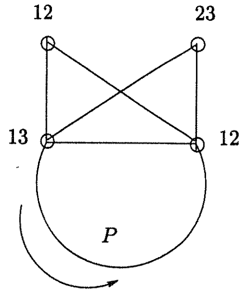
(d)



(e)

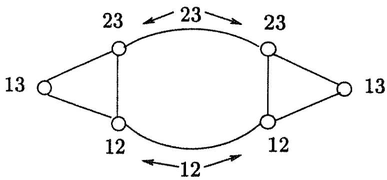


3 forces 1 along P
The cycle is of length $\ell + 2 \geq 3$;
P is of length ℓ
(f)

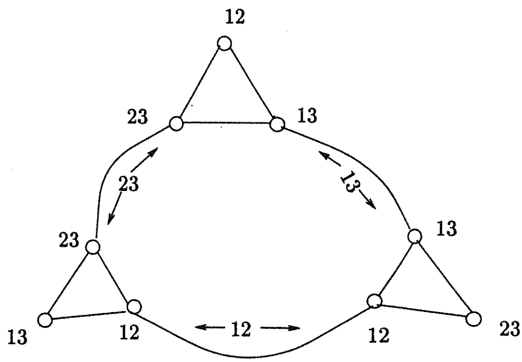


3 forces 2 along P

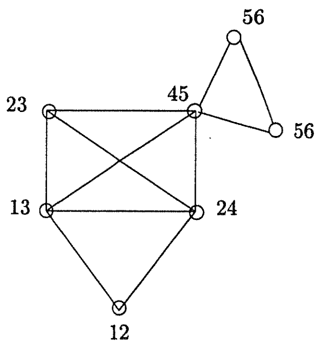
(g)



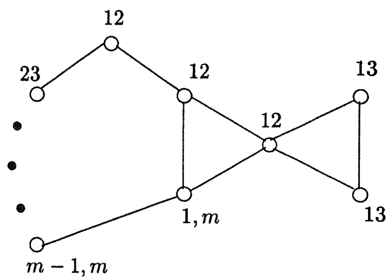
(h)



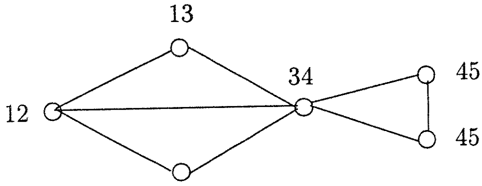
(i)



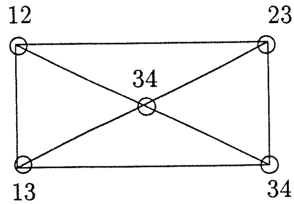
(j)(1)



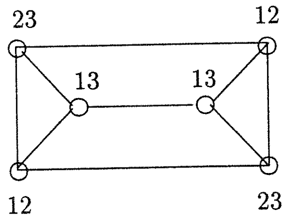
(j)(2)



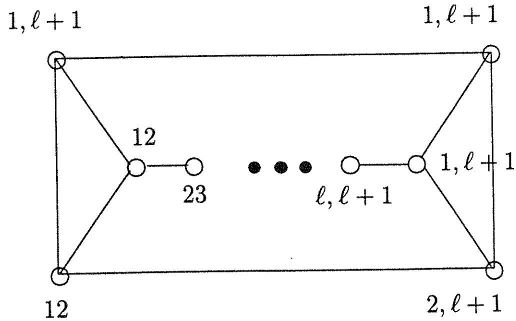
(j)(3)



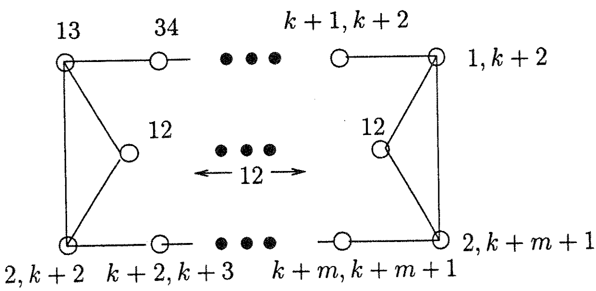
(k)



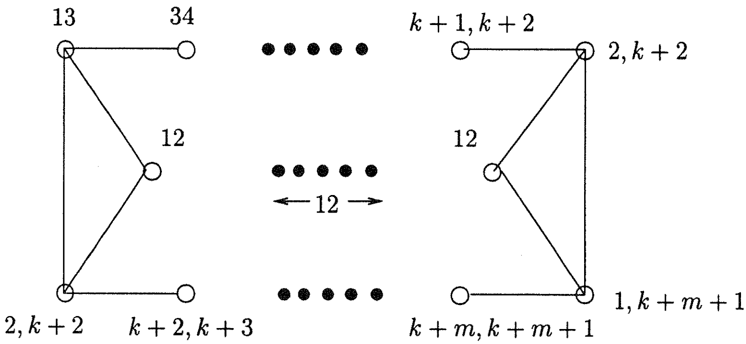
(ℓ); P, P', P''
all of length one



(ℓ); P, P'' of length one,
P' of length $\ell > 1$



(ℓ); P, P', P'' of lengths $k, \ell, m \geq 2$,
all even



(ℓ); P, P', P'' of lengths $k, \ell, m \geq 3$, all odd

This completes the proof of Theorem 4. □

The last two list assignments show that the graph in Theorem 4(ℓ) has Hall number > 2 whatever the lengths of P, P', P''; but if, say, P and P' have lengths of different parity, and the length of P'' is > 1, then the graph has a proper induced subgraph of the type of Theorem 4(h), and so is not Hall-2^+-critical.

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References

[1] M. M. Cropper, J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman, and P. D. Johnson Jr., Extending the disjoint-representatives theorems of Hall, Halmos, and Vaughan to list-multicolorings of graphs. Manuscript submitted, 1998.

[2] P. Erdös, A. Rubin, and H. Taylor, Choosability in graphs, Congressus Numerantium 26 (1979), 125-157.

[3] Ch. Eslahchi, Characterization of graphs with Hall index 2, Australas. J. Combin. 20 (2000), 13-21.

[4] S. Gutner, Choice Numbers of Graphs, M. Sc. thesis, Tel Aviv Univ., 1992.

[5] S. Gutner and M. Tarsi, Some results on $(a:b)$ - choosability, to appear.

[6] P. Hall, On representatives of subsets, Jour. London Math. Soc. 10 (1935), 26-30.

[7] A. J. W. Hilton and P. D. Johnson Jr., Extending Hall's theorem, *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, Physica-Verlag, Heidelberg, 1990, 360-371.

[8] A. J. W. Hilton and P. D. Johnson Jr., The Hall number, the Hall index, and the total Hall number of a graph, Discrete Applied Math., 94 (1999), 227-245.

[9] A. J. W. Hilton, P. D. Johnson Jr., and E. B. Wantland, The Hall number of a simple graph, Congressus Numerantium 121 (1996), 161-182.

[10] D. G. Hoffman and P. D. Johnson Jr., On the choice number of $K_{m,n}$, Congressus Numerantium 98 (1993), 105-111.

[11] Zs. Tuza, Choice perfect graphs and Hall numbers. Manuscript, 1997.

[12] Zs. Tuza, Graph colorings with local constraints - a survey, Discuss. Math. Graph Theory 17 (1997), 161-228.

[13] V. G. Vizing, Colouring the vertices of a graph in prescribed colours (Russian), Diskret. Analiz. 29, Metody Diskret. Anal. V. Teorii Kodov i Shem, 101 (1976), 3-10 (MR 58 16371).

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