A characterization of domination 4-relative-critical graphs of diameter 5

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Abstract

Let G be a spanning subgraph of $K(s,s)$ and let H be the complement of G relative to $K(s,s)$; that is, $K(s,s) = G \oplus H$ is a factorization of $K(s,s)$. The graph G is γ-relative-critical if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for all $e \in E(H)$, where $\gamma(G)$ denotes the domination number of G. The 2-relative-critical graphs and 3-relative-critical graphs are characterized in [7]. In [7], it is shown that the diameter of a connected 4-relative-critical graph is at most 5. In this paper, we construct five families of connected 4-relative-critical graphs of diameter 5 and show that a graph G is a connected 4-relative-critical graph of diameter 5 if and only if G belongs to one of these five families.

1 Introduction

A set D of vertices of a graph $G = (V, E)$ is a *dominating set* if every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is the domination number of G and is denoted by $\gamma(G)$. A graph G is said to be *γ-domination critical*, or just *γ-critical*, if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for every edge e in the complement $\overline{G}$ of G. This concept of γ-critical graphs has been studied by, among others, Blitch [1], Sumner [9], Sumner

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and Blitch [8], and Wojcicka [10]. For a more thorough survey of these concepts, see Chapter 5 of [5] and Chapter 16 of [6]. Terminology and notation not defined here may be found in [2].

If G is a spanning subgraph of a graph F, then the graph $F - E(G)$ is the *complement of G relative to F* with respect to a fixed embedding of G into F. The idea of a relative complement of a graph was suggested by Cockayne [3] and is studied in [4]. In [7], domination critical graphs with respect to relative complements are investigated.

Let $G \oplus H = K(s,s)$ be a factorization of the complete bipartite graph $K(s,s)$. (If G and H are graphs on the same vertex set but with disjoint edge sets, then $G \oplus H$ denotes the graph whose edge set is the union of their edge sets.) Notice that if there is a unique (proper) 2-coloring of the vertices of G with each color coloring s vertices, then the graph H is unique. That is, if G is uniquely embeddable in $K(s,s)$, then H is unique. In particular, if G is a connected spanning subgraph of $K(s,s)$, then G is uniquely embeddable in $K(s,s)$.

We say that G is a *γ-critical s-relative graph*, or simply a *γ-relative-critical graph*, if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for all $e \in E(H)$. We denote the relative complement H of G by $\overline{G}$. The rest of this paper deals only with relative complements with the exception of $\overline{K}_2$, so confusion with complements in the ordinary sense is unlikely. Hence, for notational convenience, we shall also denote a γ-relative-critical graph simply as a γ-critical graph. The 2-critical graphs and 3-critical graphs are characterized in [7], as are disconnected 4-critical graphs. Furthermore it is shown in [7] that the diameter of a connected 4-critical graph is at most 5. That this bound is sharp may be seen by considering, for example, the connected 4-critical 5-relative graph with diameter 5 shown in Figure 1.

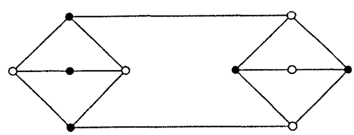

Figure 1: A connected 4-critical 5-relative graph with diameter 5.

Our aim in this paper is to characterize the (connected) 4-critical graphs of diameter 5. We construct five families of 4-critical graphs of diameter 5 and show that a graph G is a connected 4-critical s-relative graph of diameter 5 if and only if G belongs to one of these five families.

For this purpose, we introduce the following notation. Let G be a connected 4-critical s-relative graph. If u and v are non-adjacent vertices in different partite sets of G, then $\gamma(G+uv) = 3$ and so there exists a set W of cardinality 3 that dominates $G+uv$. Since W does not dominate G, it must be that exactly one of u and v, say v, belongs to W and that W dominates all of G except u. Thus, $S = W - \{v\}$ is a set of cardinality 2 such that $S \cup \{v\}$ dominates $G-u$ and we write $[v, S] \to u$. In particular, when we write $[v, S] \to u$ it is understood that u is not dominated by S.

2 Five Families of 4-Critical Graphs of Diameter 5

In this section, we construct five families $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_5$ of connected 4-critical s-relative graphs of diameter 5, and we let $\mathcal{G} = \cup_{i=1}^{5} \mathcal{G}_i$. For each $G \in \mathcal{G}$, the vertex set of G is partitioned into five sets $V_0, V_1, \dots, V_5$ where $|V_0| = 1$. Let E be the set of all edges between V_i and V_{i+1} for $i = 0, 1, \dots, 4$. In what follows, we describe each of the five families in turn.

Let $G \in \mathcal{G}_1$. Then $|V_0| = |V_1| = |V_2| = 1$, $|V_4| = 2k$, and $|V_5| = 2$, while V_3 is partitioned into two sets $V_{3,1}$ and $V_{3,2}$ with $|V_{3,1}| = k$ and $|V_{3,2}| = k-1$ where $k \geq 2$. Let E_1 be a set of edges between $V_{3,1}$ and V_4 that induces a collection of k (vertex-disjoint) paths on three vertices with each center vertex in $V_{3,1}$. Then $E(G) = E - E_1$.

Let $G \in \mathcal{G}_2$. Then $|V_0| = |V_1| = |V_5| = 1$, $|V_2| \geq 2$, and $|V_4| = 2k$, while V_3 is partitioned into three sets $V_{3,1}$, $V_{3,2}$ and $V_{3,3}$ with $|V_{3,1}| = |V_2|$, $|V_{3,2}| = k$, and $|V_{3,3}| = k-1$ where $k \geq 2$. Let $E_{2,1}$ be a set of edges between V_2 and $V_{3,1}$ that induces a perfect matching between these two sets. Let $E_{2,2}$ be a set of edges between $V_{3,2}$ and V_4 that induces a collection of k (vertex-disjoint) paths on three vertices with each center vertex in $V_{3,2}$. Then $E(G) = E - (E_{2,1} \cup E_{2,2})$.

Let $G \in \mathcal{G}_3$. Then $|V_0| = |V_1| = |V_5| = 1$ and $|V_2| = \ell$, where $\ell \geq 1$, while V_3 is partitioned into two sets $V_{3,1}$ and $V_{3,2}$ with $|V_{3,1}| = k$ and $|V_{3,2}| = k + \ell + 1$, where $k \geq 2$. Furthermore, V_4 is partitioned into two sets $V_{4,1}$ and $V_{4,2}$ with $|V_{4,1}| = 2k$ and $|V_{4,2}| = 2$. Let $E_{3,1}$ be a set of edges between $V_{3,1}$ and $V_{4,1}$ that induces a collection of k (vertex-disjoint) paths on three vertices with each center vertex in $V_{3,1}$. Let $E_{3,2}$ be the set of edges between $V_{4,2}$ and V_5. Then $E(G) = E - (E_{3,1} \cup E_{3,2})$.

Let $G \in \mathcal{G}_4$. Then $|V_0| = |V_5| = 1$, $|V_1| = |V_4| = 2$, and $|V_2| = |V_3| \geq 2$. Let $E_{5,1}$ be the set of edges between V_2 and V_3 that induces a perfect matching between these two sets. Then $E(G) = E - E_{5,1}$.

Let $G \in \mathcal{G}_5$. Then $|V_0| = |V_5| = 1$ and $|V_1| = |V_4| = 2$, while V_2 is partitioned into two sets $V_{2,1}$ and $V_{2,2}$ with $|V_{2,1}| = k$ and $|V_{2,2}| = \ell$, where $k \geq 2$ and $\ell \geq 1$. Furthermore, V_3 is partitioned into two sets $V_{3,1}$ and $V_{3,2}$ with $|V_{3,1}| = k$ and $|V_{3,2}| = \ell$. Let $E_{5,1}$ be the set of edges between a vertex of V_1 and the vertices of $V_{2,1}$, and let $E_{5,2}$ be the set of edges between a vertex of V_4 and the vertices of $V_{3,1}$. Let $E_{5,3}$ be a set of edges between $V_{2,2}$ and $V_{3,2}$ that induce a perfect matching between these two sets. Then $E(G) = E - (E_{5,1} \cup E_{5,2} \cup E_{5,3})$.

It is straightforward to check that each $\mathcal{G}_i$, $1 \leq i \leq 5$, is a family of 4-critical graphs of diameter 5. Note that the graph G in Figure 1 is in $\mathcal{G}_4$.

3 Main Results

We shall prove:

Theorem 1 *Let G be a connected 4-critical s-relative graph having diameter 5. If there exists a vertex in G with at least two vertices at distance 5 from it, then $G \in \mathcal{G}_1$.*

Theorem 2 *Let G be a connected 4-critical s-relative graph having diameter 5. If each vertex of eccentricity 5 in G has a unique vertex at distance 5 from it, then $G \in \mathcal{G} - \{\mathcal{G}_1\}$.*

As an immediate consequence of Theorems 1 and 2 and the construction of the graphs in $\mathcal{G}$, we have the following characterization of connected 4-critical graphs of diameter 5.

Theorem 3 *A graph G is a connected 4-critical s-relative graph of diameter 5 if and only if $G \in \mathcal{G}$.*

4 Proof of Main Results

Let G be a connected 4-critical s-relative graph having diameter 5. Let u and v be vertices of G with $diam\, G = d(u, v) = 5$. Let $u = v_0, v_1, \ldots, v_5 = v$ be a shortest u-v path. For $i = 0, 1, \ldots, 5$, let $V_i = \{x \mid d(u, x) = i\}$. Necessarily, $V_0 = \{u\}$ and $v_i \in V_i$ for $i = 1, 2, \ldots, 5$. The partite sets of G are $V_0 \cup V_2 \cup V_4$ and $V_1 \cup V_3 \cup V_5$. Hence

$$|V_0| + |V_2| + |V_4| = s = |V_1| + |V_3| + |V_5|.$$

If all edges between V_i and V_{i+1} are present, then we shall say that $[V_i, V_{i+1}]$ is *full*. In particular, $[V_0, V_1]$ is full.

Before proceeding further, we introduce some notation. If s and t are non-adjacent vertices in different partite sets of G, then as pointed out in the introduction, there is a set T of cardinality 2 such that $[s, T] \to t$ or $[t, T] \to s$. For the discussion, it is convenient to consider T to be an ordered set, the first element of which belongs to a set V_i of smallest index. That is, if $T = \{s, t\}$ where $s \in V_i$ and $t \in V_j$, then $i \leq j$. Furthermore, if $T = S$, then we let $S = \{x, y\}$.

4.1 Proof of Theorem 1

We may assume that v_0 is the vertex with at least two vertices at distance 5 from it, that is, $|V_5| \geq 2$.

Claim 4 $|V_i| = 1$ *for some* i, $1 \leq i \leq 4$.

Proof. Suppose that $|V_i| \geq 2$ for all $i \neq 0$. Consider $G + v_2v_5$. Then, $[v_2, S] \to v_5$ or $[v_5, S] \to v_2$. Suppose $[v_5, S] \to v_2$. Since v_0 must be dominated, it follows that $x \in V_0 \cup V_1$ and since $|V_5| \geq 2$, it must be the case that $y \in V_4 \cup V_5$. But to dominate $V_2 - \{v_2\}$, x must be in V_1, and hence, $V_1 - \{x\}$ is not dominated. Thus, $[v_2, S] \to v_5$. Since v_0 and $V_5 - \{v_5\}$ must be dominated, it follows that $x \in V_0 \cup V_1$ and $y \in V_4 \cup V_5$.

If $x = v_0$, then $V_2 - \{v_2\}$ is not dominated. Thus, $x \in V_1$, x dominates $V_2 - \{v_2\}$, and v_2 dominates $V_1 - \{x\}$. If $y \in V_4$, then at least one vertex of V_4 is not dominated. Hence, $y \in V_5$ implying that $|V_5| = 2$. Furthermore, y dominates V_4 and v_2 dominates V_3.

Now consider $G + v_0y$. Then $[v_0, T] \rightarrow y$ or $[y, T] \rightarrow v_0$. If $[v_0, T] \rightarrow y$, then since y dominates V_4, no vertex of V_4 is in T. Furthermore, v_5 must be dominated by T, and so $v_5 \in T$. But then the other vertex in T must dominate $V_2 \cup V_3$, which is impossible since $|V_2| \geq 2$ and $|V_3| \geq 2$. On the other hand, if $[y, T] \rightarrow v_0$, then T must dominate v_5, and so one vertex of T is in $V_4 \cup V_5$. The remaining vertex of T must dominate $V_1 \cup V_2$, which is impossible since $|V_1| \geq 2$ and $|V_2| \geq 2$. Hence for at least one $i \in \{1, 2, 3, 4\}$, $|V_i| = 1$. □

Claim 5 $|V_4| \geq 2$.

Proof. Suppose $|V_4| = 1$ and consider $G + v_2v_5$. Then $[v_2, S] \rightarrow v_5$ or $[v_5, S] \rightarrow v_2$. Suppose $[v_5, S] \rightarrow v_2$. Then $x \in V_0 \cup V_1$ in order to dominate v_0, while $y \in V_4 \cup V_5$ in order to dominate $V_5 - \{v_5\}$ since $|V_5| \geq 2$. Since v_4 dominates $V_4 \cup V_5$, we may assume that $y = v_4$. But then x must dominate $V_0 \cup V_1 \cup (V_2 - \{v_2\})$ and v_4 must dominate V_3. Hence, $\{x, v_2, v_4\}$ dominates G, contradicting the fact that $\gamma(G) = 4$. Thus, $[v_2, S] \rightarrow v_5$. Now $x \in V_0 \cup V_1$ to dominate v_0. Since $|V_5| \geq 2$ and $y \notin N[v_5]$, it follows that $y \in V_5$ and $|V_5| = 2$. But then v_4 can replace y in S, contradicting the fact that no vertex of S is in $N[v_5]$. Thus, $|V_4| \geq 2$. □

Claim 6 $|V_3| \geq 2$.

Proof. Suppose $|V_3| = 1$. Consider $G + v_0v_3$. Then $[v_0, S] \rightarrow v_3$ or $[v_3, S] \rightarrow v_0$. Suppose $[v_0, S] \rightarrow v_3$. Then $x \in V_1 \cup V_2$ to dominate V_2. Hence, y must dominate $V_4 \cup V_5$, a contradiction since, by assumption, $|V_5| \geq 2$ and, by Claim 5, $|V_4| \geq 2$. Thus, $[v_3, S] \rightarrow v_0$. Now no vertex of V_1 is in S since $V_1 = N(v_0)$. To dominate V_1, $x \in V_2$. Furthermore, y must be in $V_4 \cup V_5$ to dominate V_5. Since $|V_5| \geq 2$, $y \in V_4$. If $x \in N(v_3)$, $\{v_0, v_3, y\}$ dominates G, contradicting the fact that $\gamma(G) = 4$. Hence, $x \notin N(v_3)$ and v_3 dominates $V_2 - \{x\}$. Consider now $G + xv_5$. Then $[x, T] \rightarrow v_5$ or $[v_5, T] \rightarrow x$. Let $T = \{w, z\}$. If $[x, T] \rightarrow v_5$, then $w \in V_0 \cup V_1$ to dominate v_0 and $z \in V_4 \cup V_5$ to dominate $V_4 \cup (V_5 - \{v_5\})$. Since $|V_4| \geq 2$, it follows that $z \in V_5$. But v_3 is not dominated by $T \cup \{x\}$, a contradiction. If $[v_5, T] \rightarrow x$, then since x dominates V_1, $w = v_0$ to dominate v_0. Now z must dominate $(V_2 - \{x\}) \cup \{v_3\} \cup (V_5 - \{v_5\})$, which is impossible. Therefore, $|V_3| \geq 2$. □

Claim 7 $|V_1| = 1$.

Proof. By Claims 5 and 6, we know that $|V_3| \geq 2$ and $|V_4| \geq 2$. By Claim 4 at least one of V_1 and V_2 have cardinality one. Assume $|V_1| \geq 2$ and hence $|V_2| = 1$. Furthermore, v_2 dominates V_1; otherwise, $diam(G) > 5$, a contradiction. Thus, $[V_1, V_2]$ is full, as is $[V_2, V_3]$.

Suppose $|V_4| \geq 3$. Let $u_4 \in V_4$ and consider $G + v_1u_4$. Then $[v_1, S] \to u_4$ or $[u_4, S] \to v_1$. Suppose $[v_1, S] \to u_4$. Since $|V_1| \geq 2$, it follows that $x \in V_0 \cup V_1 \cup V_2$ to dominate $V_1 - \{v_1\}$. Hence, y must dominate $(V_4 - \{u_4\}) \cup V_5$. But this is impossible since $|V_4 - \{u_4\}| \geq 2$ and $|V_5| \geq 2$. Hence, $[u_4, S] \to v_1$.

In order to dominate v_0, $x \in V_1$. In order to dominate $V_4 - \{u_4\}$, $y \in V_3 \cup V_5$ since $|V_4| \geq 3$. Furthermore, $\{u_4, y\}$ must dominate $V_3 \cup V_4 \cup V_5$. If $y \in V_3$, then $\{v_0, u_4, y\}$ dominates G, a contradiction. Therefore, $y \in V_5$ implying that u_4 dominates V_3. Since u_4 is an arbitrary vertex of V_4, $[V_3, V_4]$ is full. Furthermore, $|V_1| = 2$. Now if any vertex $w_4 \in V_4$ dominates V_5, then $\{v_0, v_3, w_4\}$ dominates G, a contradiction. Thus no vertex of V_4 dominates V_5. In particular, u_4 is not adjacent to y. Also, y dominates $V_4 - \{u_4\}$ and u_4 dominates $V_5 - \{y\}$.

Consider $G + v_0v_3$. Then $[v_0, T] \to v_3$ or $[v_3, T] \to v_0$. Suppose $[v_0, T] \to v_3$. Let $T = \{w, z\}$. In order to dominate v_2, $w \in V_1 \cup V_3$. If $w \in V_1$, then we can replace w with any vertex in $V_3 - \{v_3\}$. Hence we may assume that $w \in V_3 - \{v_3\}$. But now z must dominate V_5, which is impossible since $|V_5| \geq 2$ and no vertex of V_4 dominates V_5. Hence, $[v_3, T] \to v_0$. Then $w = v_2$ to dominate V_1. Once again, z must dominate V_5, a contradiction. Therefore, we must have $|V_4| = 2$. However, $2 + |V_4| = |V_0| + |V_2| + |V_4| = |V_1| + |V_3| + |V_5| \geq 6$, and so $|V_4| \geq 4$. Thus we have a contradiction, implying that $|V_1| = 1$. □

Claim 8 $|V_2| = 1$.

Proof. Suppose $|V_2| \geq 2$. Let $u_2 \in V_2$ and $u_5 \in V_5$, and consider $G + u_2u_5$. Then $[u_5, S] \to u_2$ or $[u_2, S] \to u_5$. If $[u_5, S] \to u_2$, then we can choose v_1 to be in S (to dominate v_0), contradicting the fact that no vertex of S is in $N[u_2]$. Hence, $[u_2, S] \to u_5$. Now v_0 must be dominated, so we may assume that $x = v_1$. Then y must dominate $(V_5 - \{u_5\}) \cup V_4$. Since $|V_4| \geq 2$, it follows that $y \in V_5$ and $|V_5| = 2$. Furthermore, u_2 dominates V_3. Since u_2 is an arbitrary vertex of V_2, it follows that $[V_2, V_3]$ is full. Since u_5 is an arbitrary vertex of V_5, it follows that $[V_4, V_5]$ is full.

Consider $G + v_0v_5$. Then $[v_0, T] \to v_5$ or $[v_5, T] \to v_0$. Let $T = \{w, z\}$. Suppose $[v_5, T] \to v_0$. Then $V_1 \cup V_2$ must be dominated and $v_1 \notin T$; hence, it must be the case that $w \in V_2$ to dominate v_1. Since $|V_2| \geq 2$, z is in $V_2 \cup V_3$ to dominate $V_2 - \{x\}$. But then $V_5 - \{v_5\}$ is not dominated, a contradiction. Hence, $[v_0, T] \to v_5$. Then $w \in V_1 \cup V_2 \cup V_3$ to dominate V_2 and $z \in V_4 \cup V_5$ to dominate $V_5 - \{v_5\}$. If $w = v_1$, then z must dominate $V_3 \cup V_4 \cup (V_5 - \{v_5\})$, which is not possible. Hence, $w \in V_2 \cup V_3$. If $w \in V_2$, then since $|V_2| \geq 2$, $z \in V_2 \cup V_3$ to dominate $V_2 - \{w\}$, contradicting the fact that $z \in V_4 \cup V_5$. Therefore, $w \in V_3$. Since $|V_3| \geq 2$, $z \in V_4$, contradicting the fact that no vertex in $N[v_5]$ is in T. Thus, we conclude that $|V_2| = 1$. □

Claim 9 $|V_5| = 2$.

Proof. Suppose $|V_5| \geq 3$ and consider $G + v_2v_5$. Then $[v_2, S] \to v_5$ or $[v_5, S] \to v_2$. If $[v_5, S] \to v_2$, we can choose $v_1 \in S$, a contradiction. Thus, $[v_2, S] \to v_5$. Now S must dominate v_0, and so either v_0 or v_1 is in S. The other vertex in S must dominate $V_4 \cup (V_5 - \{v_5\})$. But since $|V_4| \geq 2$ and $|V_5 - \{v_5\}| \geq 2$, no single vertex can dominate $V_4 \cup (V_5 - \{v_5\})$. Hence, $|V_5| = 2$. □

Claim 10 $[V_4, V_5]$ *is full.*

Proof. Let $V_5 = \{u_5, w_5\}$ and consider $G + v_2u_5$. As before, $[u_5, S] \to v_2$ cannot occur. Thus, $[v_2, S] \to u_5$. In order to dominate v_0, we may assume $x = v_1$. The remaining vertex y of S must dominate $V_4 \cup \{w_5\}$. Since $|V_4| \geq 2$, $y = w_5$ and w_5 dominates V_4. Similarly, by considering $G + v_2w_5$, u_5 dominates V_4. Hence, $[V_4, V_5]$ is full. □

By the above claims, $|V_0| = |V_1| = |V_2| = 1$, $|V_5| = 2$, $|V_3| \geq 2$ and $|V_4| \geq 2$. Furthermore, $[V_2, V_3]$ and $[V_4, V_5]$ are full. Since $|V_0|+|V_2|+|V_4| = s = |V_1|+|V_3|+|V_5|$, we note that $|V_4| = |V_3| + 1$.

Suppose $|V_4| = 3$ (and so $|V_3| = 2$). Then a vertex of V_3 must be adjacent to at least two vertices of V_4. Let $V_3 = \{u_3, v_3\}$. If u_3 is adjacent to v_2 only, then v_3 dominates V_4 and the graph G would not be 4-critical since $\gamma(G + v_1v_4) = 4$, a contradiction. Hence, u_3 is adjacent to at least one vertex of V_4. However, we can now find a vertex of V_3 and a vertex of V_4 that together dominate $V_3 \cup V_4$. But then we can dominate G with three vertices, a contradiction. Hence, $|V_4| \geq 4$.

Let $u_4 \in V_4$ and consider $G + v_1u_4$. Since v_0 must be dominated, the case $[u_4, S] \to v_1$ cannot occur. Hence, $[v_1, S] \to u_4$. Let $S = \{x, y\}$. Now y must dominate V_5 and $y \notin V_5$, and so $y \in V_4 - \{u_4\}$. Since x dominates $V_4 - \{u_4, y\}$, and $|V_4| \geq 4$, $x \in V_3$. Hence, y dominates $V_3 - \{x\}$ and x dominates $V_4 - \{u_4, y\}$. However, if we now consider $G + v_1y$, then we must have $[v_1, T] \to y$ and $T = \{x, u_4\}$. In particular, x is adjacent to every vertex of V_4 except for u_4 and y, and each of u_4 and y dominates $V_3 - \{x\}$. Since u_4 is an arbitrary vertex in V_4, it follows that the edges of G that are missing between V_3 and V_4 induce a collection of $|V_4|/2 \geq 2$ (vertex-disjoint) paths on three vertices with each center vertex in V_3. The $|V_4|/2 - 1$ vertices in V_3 that are not center vertices each dominate V_4 in G. Hence, $G \in \mathcal{G}_1$. This completes the proof of Theorem 1.

4.2 Proof of Theorem 2

In this case, V_5 consists only of the vertex v_5. Furthermore, each vertex of V_1 is adjacent to some vertex of V_2 and each vertex of V_2 is adjacent to some vertex of V_3 (for otherwise v_5 would have at least two vertices at distance 5 from it).

Claim 11 $|V_3| \geq 2$.

Proof. If $|V_3| = 1$, then v_3 dominates $V_2 \cup V_4$, whence $\{v_0, v_3, v_5\}$ would be a dominating set of G, a contradiction. Hence, $|V_3| \geq 2$. □

Claim 12 *If* $|V_1| = 1$, *then* $G \in \mathcal{G}_2 \cup \mathcal{G}_3$.

Proof. Let $A = N(v_5)$. Then $A \subseteq V_4$. If A consists only of the vertex v_4, then G is not 4-critical as may be seen by adding the edge v_1v_4. Hence, $|A| \geq 2$. Let $B = V_4 - A$ (possibly, $B = \emptyset$). Let C be the set of vertices in V_3 each of which dominates V_4 (possibly, $C = \emptyset$), and let $D = V_3 - C$. Then $[V_4, C]$ is full.

Let $u_2 \in V_2$ and consider $G + u_2v_5$. Then $[u_2, S] \to v_5$ or $[v_5, S] \to u_2$. If $[v_5, S] \to u_2$, then we can choose $v_1 \in S$, a contradiction. Hence, $[u_2, S] \to v_5$. Then $x = v_1$ to dominate v_0 and $y \in V_3$ to dominate V_4 (since $|A| \geq 2$). Note that, $y \in C$. Furthermore, u_2 dominates $V_3 - \{y\}$. Since u_2 is an arbitrary vertex of V_2, every vertex of V_2 dominates at least $|V_3| - 1$ vertices of V_3. Moreover, $[V_2, D]$ is full.

Case 1: $[V_2, V_3]$ is not full.

We show then that $G \in \mathcal{G}_2$. Now, $|V_2| \geq 2$ and there is a vertex u_2 in V_2 that is not adjacent to a vertex $u_3 \in V_3$. Since $[V_2, D]$ is full, $u_3 \in C$. Consider $G + u_2u_3$. Then $[u_2, T] \to u_3$ or $[u_3, T] \to u_2$. If $[u_3, T] \to u_2$, then we can choose $v_1 \in T$, contradicting the fact that there is no vertex of T in $N(u_2)$. Hence, $[u_2, T] \to u_3$. Then $v_1 \in T$ to dominate v_0 and $v_5 \in T$ to dominate v_5 (since no vertex of V_4 can be in T). Hence, $V_4 = A$ (and so, $B = \emptyset$). Thus, if V_2 contains a vertex w_2 that dominates V_3, then $\{v_1, w_2, v_5\}$ dominates G, a contradiction. It follows that each vertex of V_2 is not adjacent to exactly one vertex of V_3 and this vertex belongs to C.

If there is a vertex of A, say a, that dominates V_3, then $\{v_1, a, c\}$ would dominate G, where $c \in C$, a contradiction. Hence no vertex of A dominates V_3. In particular, $|D| \geq 1$.

Now, $|V_3| = |V_2| + |V_4| - 1 \geq 3$. Let u_3 be a vertex in C that does not dominate V_2. Consider $G + v_0u_3$. If $[v_0, R] \to u_3$, then $v_5 \in R$ to dominate v_5 (since no vertex of A can be in R). The remaining vertex of R must dominate $(V_3 - \{u_3\}) \cup V_2$, which is impossible since $|V_2| \geq 2$ and $|V_3| \geq 3$. Hence, $[u_3, R] \to v_0$. Thus, R contains a vertex x in V_2 to dominate v_1. The remaining vertex of R is in $A \cup \{v_5\}$ to dominate v_5. Hence, $\{x, u_3\}$ dominates V_2, and so x must be the only vertex of V_2 that is not adjacent to u_3. Since u_3 is an arbitrary vertex of C that does not dominate V_2, each vertex of C is not adjacent to at most one vertex of V_2. Thus, the edges of G that are missing between V_2 and C induce a matching from V_2 to a subset of C. In particular, $|C| \geq |V_2|$.

We show next that each vertex of D is adjacent to some vertex of A. Suppose $d \in D$ is adjacent to no vertex of A. Then the neighborhood of d is V_2. Consider $G + v_1v_4$. Then $[v_1, K] \to v_4$. Thus, K contains a vertex u_2 of V_2 to dominate d. The remaining vertex of K must dominate v_5 and the vertex of C that is not adjacent to u_2. Hence, $|A| = 2$. However, $|V_2| + 2 = |V_2| + |A| = |C| + |D| + 1 \geq |V_2| + |D| + 1 \geq |V_2| + 2$. Thus we must have equality throughout, and so $|V_2| = |C|$ and $|D| = 1$. This implies that each vertex of C is not adjacent to a (unique) vertex of V_2. We now consider $G + v_0d$. Then $[v_0, W] \to d$ or $[d, W] \to v_0$. Let $W = \{w, z\}$. If $[d, W] \to v_0$, then w belongs to V_2 to dominate v_1. Hence, $v_5 \in W$ to dominate $A \cup \{v_5\}$. But then the vertex of C that is not adjacent to w is not dominated, a contradiction. Hence, $[v_0, W] \to d$. Now z belongs to $A \cup \{v_5\}$ to dominate v_5. If $z = v_5$, then w must dominate $V_2 \cup C$, which is impossible. On the other hand, if $z \in A$, then w must dominate $V_2 \cup (A - \{z\})$, which is impossible. Therefore, each vertex of D must have been adjacent to some vertex of A. In particular, $\{v_1\} \cup A$ dominates G, and so $|A| \geq 3$.

Let A_1 be the set of vertices in A that are not adjacent to exactly one vertex of D, and let $A_2 = A - A_1$. If $A_2 \neq \emptyset$, then each vertex of A_2 is not adjacent to at least two vertices of D.

Let $a_1 \in A_1$, and let d_1 be the vertex of D not adjacent to a_1. If d_1 dominates $A-\{a_1\}$, then $\{a_1, d_1, v_1\}$ dominates G, a contradiction. Thus there is a vertex a_2 in A, different from a_1, that is not adjacent to d_1. Consider $G+v_1a_1$. Then $[v_1, L] \to a_1$. Thus, L contains a vertex ℓ_1 of A to dominate v_5. The remaining vertex of L, say ℓ_2, must dominate $(A \cup D) - (N[\ell_1] \cup \{a_1\})$. If $\ell_2 \in A$, then $|A| = 3$ and $|D| \leq 2$. Since each vertex of A is not adjacent to at least one vertex of D, $|D| = 2$. Let d_2 be the vertex of D different from d_1, and let a_3 be the vertex of A different from a_1 and a_2. Then, a_1d_2, a_2d_2, and a_3d_1 are edges in G. But then $\{v_1, d_2, a_3\}$ dominates G, a contradiction. Hence, $\ell_2 \notin A$. Thus, we must have $\ell_2 = d_1$, implying that $\ell_1 = a_2$. Now, a_2 dominates $V_3 - \{d_1\}$, and d_1 dominates $A - \{a_1, a_2\}$. It follows that $D - N[a_1] = D - N[a_2] = \{d_1\}$ and $A - N[d_1] = \{a_1, a_2\}$.

We show next that $A_2 = \emptyset$. Suppose $u_4 \in A_2$. Then there are at least two vertices, say d_1 and d_2, in D that are not adjacent to u_4. Thus, $|D| \geq 2$, and so $|A| \geq 3$. We consider $G + v_1u_4$. Then $[v_1, M] \to u_4$. Thus, M contains a vertex w_4 of A to dominate v_5. Suppose $w_4 \in A_1$. Then the vertex, d_3 say, of D that is not adjacent to w_4 must be adjacent to u_4, since each vertex of A_1 is an endvertex of a path component on three vertices in the relative complement of G. Thus, $|D| \geq 3$, and so $|A| \geq 4$. Hence the vertex m of M, different from w_4, must belong to V_3 to dominate $(A - \{u_4, w_4\})$. Since $d_3 \notin M$, d_3 is then not dominated by M, a contradiction. Hence, $w \in A_2$. If $|A| \geq 4$, then the vertex m of M, different from w_4, must belong to V_3 to dominate $(A - \{u_4, w_4\})$. But at least one vertex of $D - N(w_4)$ is not dominated. Hence, $|A| = 3$, and so $|D| = 2$. Since u_4 and w_4 both belong to A_2, the two vertices of D are not adjacent to u_4 and w_4 and must therefore be adjacent to the vertex of A different from u_4 and w_4. But then this vertex of A dominates V_3, a contradiction. We deduce, therefore, that $A_2 = \emptyset$, that is, $A = A_1$. Thus the edges of G missing between A and D induce a collection of $|D| \geq 2$ (vertex-disjoint) paths on three vertices with each center vertex in D, and hence, $G \in \mathcal{G}_2$.

Case 2: $[V_2, V_3]$ is full.

Then $B \neq \emptyset$, for otherwise, $\{v_1, v_2, v_5\}$ dominates G, a contradiction. Each vertex of A is not adjacent to at least one vertex of D, for otherwise if $a \in A$ dominates D, then $\{a, c, v_1\}$ dominates G for any vertex c of C. Let $u_4 \in A$ and consider $G + v_1u_4$. If $[u_4, W] \to v_1$, then we can choose $v_1 \in W$, a contradiction. Hence, $[v_1, W] \to u_4$. In order to dominate v_5, W contains a vertex, w_4 say, in $A - \{u_4\}$. The remaining vertex, w say, of W must therefore dominate B and the vertices in D that are not adjacent to w_4. In particular, this implies that each vertex of D is adjacent to some vertex of V_4, for otherwise such a vertex w would not exist.

Suppose $w \in B$. Then $|B| = 1$ and $|A| = 2$. If w dominates D, then $\{v_1, w, v_5\}$ is a dominating set of G, a contradiction. Hence, there is a vertex $d \in D$ not adjacent to w. Since $\{w, w_4\}$ dominates D, every vertex in D that is not adjacent to w_4 (respectively, w) is adjacent to w (respectively, w_4). In particular, d is adjacent to w_4. We now consider $G + dw$. If $[d, Y] \to w$, then v_1 is in Y. The remaining vertex of Y dominates $(V_3 - \{d\}) \cup \{v_5\}$. Hence, $u_4 \in Y$ and u_4 dominates $V_3 - \{d\}$. Since u_4 is not adjacent to at least one vertex of D, d is therefore the only vertex of D

not adjacent to u_4. Consider now $G + v_1w_4$. Then $[v_1, X] \to w_4$ and u_4 must belong to X to dominate v_5. But then the remaining vertex of X must dominate both d and w, which is impossible. Hence, $[w, Y] \to d$. Then $v_1 \in Y$ to dominate v_0 and therefore $v_5 \in Y$ to dominate $A \cup \{v_5\}$. Thus, d is the only vertex of D that is not adjacent to w. Consider now $G + v_1w_4$. Then $[v_1, X] \to w_4$ and u_4 must belong to X to dominate v_5. The remaining vertex of X must dominate w. Consequently, u_4 must be adjacent to d. Thus, d dominates A. Consider now $G + v_0d$. If $[d, R] \to v_0$, then R contains a vertex of V_2 to dominate v_1. The remaining vertex of R must dominate $B \cup V_5$, which is impossible. Hence, $[v_0, R] \to d$. Thus, R must contain v_5. The remaining vertex of R must therefore dominate $V_2 \cup (V_3 - \{d\}) \cup \{w\}$ which is impossible unless $|C| = 1$ and $D = \{d\}$. But then each of u_4 and w_4 dominates D, a contradiction. Hence, $w \notin B$.

Since $w \notin B$, $w \in D$ and w dominates $B \cup (A - \{u_4, w_4\})$. Furthermore, w is the only vertex of D not adjacent to w_4. We now consider $G + v_1w_4$. Then $[v_1, X] \to w_4$. Let $X = \{x_1, x_2\}$. In order to dominate v_5, $x_2 \in A - \{w_4\}$. Thus, x_1 dominates B and the vertices in D that are not adjacent to x_2. As shown above, x_1 must belong to D. But x_1 is therefore not adjacent to w_4, and so $x_1 = w$. Hence, w is the only vertex of D not adjacent to x_2. Consequently, $x_2 = u_4$. Thus, w is the only vertex of D not adjacent to each of u_4 and w_4, while u_4 and w_4 are the only vertices of A that are not adjacent to w. Hence we have established that the edges of G that are missing between A and D induce a collection of $|A|/2$ (vertex-disjoint) paths on three vertices with each center vertex in D. Moreover, each center vertex dominates B.

Suppose $|B| = 1$. Let $B = \{b\}$. If b dominates D, then $\{v_1, b, v_5\}$ dominates G, a contradiction. Hence, there is a vertex $d \in D$ that is not adjacent to b. As shown above, d dominates A. Consider $G + bd$. If $[d, Q] \to b$, then $v_1 \in Q$. The remaining vertex of Q belongs to $A \cup \{v_5\}$ to dominate v_5. But then a vertex of D that is not adjacent to two vertices of A will not be dominated. Hence, $[b, Q] \to d$. Since d dominates A, we must have $v_5 \in Q$. Once again, $v_1 \in Q$. Thus, b dominates $V_3 - \{d\}$. Consider now $G + v_0d$. If $[d, R] \to v_0$, then R contains a vertex of V_2 to dominate v_1. The remaining vertex of R must dominate $\{b, v_5\}$ which is impossible. Hence, $[v_0, R] \to d$. Thus, R must contain v_5 since d dominates A. The remaining vertex of R must therefore dominate $V_2 \cup (V_3 - \{d\}) \cup \{b\}$ which is impossible. Hence, $|B| \geq 2$.

Consider $G + v_1b$, where $b \in B$. Then $[v_1, P] \to b$. If P contains a vertex of A, then the remaining vertex of P must dominate a vertex of D and a vertex of A that is not adjacent with this vertex of D, which is impossible. Hence, $v_5 \in P$. The remaining vertex must dominate $V_3 \cup (B - \{b\})$. This is possible only if $|B| = 2$ and the vertex of B, different from b, dominates V_3. Similarly, b dominate V_3. Hence, $G \in \mathcal{G}_3$. □

In what follows, we may assume that each vertex of eccentricity 5 has degree at least 2, for otherwise, by Claim 12, $G \in \mathcal{G}_2 \cup \mathcal{G}_3$. In particular,

$$|V_1| \geq 2 \qquad \text{and} \qquad |V_4| \geq 2.$$

Thus, $s = |V_1| + |V_3| + |V_5| \geq 5$.

Claim 13 $|V_2| \geq 2$.

Proof. Suppose $|V_2| = 1$. Then $[V_1, V_2]$ and $[V_2, V_3]$ are full. Let A denote the set of vertices in V_4 that are adjacent to v_5, and let $B = V_4 - A$. If $B = \emptyset$, then $\{v_0, v_2, v_5\}$ dominates G, a contradiction. Hence, $|B| \geq 1$ and $|V_4| \geq 3$. Let $a \in A$ and let $b \in B$. We consider $G + v_1 a$. Now $[v_1, W] \to a$ or $[a, W] \to v_1$. Suppose $[v_1, W] \to a$. We may assume $v_2 \in W$ (to dominate $V_1 - \{v_1\}$). But then the remaining vertex of W must dominate both b and v_5, which is impossible. Hence, $[a, W] \to v_1$. Then W must contain a vertex of V_1 to dominate v_0. Thus, the remaining vertex, w say, of W belongs to V_3 to dominate $V_4 - \{a\}$. But then $\{a, v_0, w\}$ dominates G, a contradiction. Hence, $|V_2| \geq 2$. □

Claim 14 $|V_1| = 2$.

Proof. Suppose $|V_1| \geq 3$. Let $u_1 \in V_1$ and let $u_4 \in V_4$. We now consider adding the edge $u_1 u_4$. Suppose $[u_4, S] \to u_1$. Then, S must contain a vertex x (say) of $V_1 - \{u_1\}$ (to dominate v_0). But then the remaining vertex of S must dominate $(V_1 - \{x, u_1\}) \cup (V_4 - \{u_4\})$, which is impossible. Hence, $[u_1, S] \to u_4$. Then x must dominate $V_1 - \{u_1\}$ while the remaining vertex y of S must dominate $(V_4 - \{u_4\}) \cup \{v_5\}$. Hence, $|V_4| = 2$ and $y \in V_4 - \{u_4\}$. Since u_4 is an arbitrary vertex in V_4, it follows that $[V_4, V_5]$ is full. Let $V_4 = \{u_4, w_4\}$. We show next that $[V_1, V_2]$ is full.

Claim 14.1 $[V_1, V_2]$ *is full.*

Proof. Suppose $[V_1, V_2]$ is not full. We may assume u_1 is not adjacent to some vertex u_2 in V_2. Then x must belong to V_2 and $\{u_1, x\}$ dominate $V_1 \cup V_2$. Thus, $x = u_2$. This shows that each vertex of V_1 (V_2) is adjacent to all except possibly one vertex of V_2 (V_1). If u_2 dominates V_3, then $\{u_1, u_2, v_5\}$ dominates G. Hence, u_2 is not adjacent to some vertex in V_3 which we call u_3. We now consider adding the edge $u_1 u_2$. If $[u_2, T] \to u_1$, then T contains a vertex in V_1 (to dominate v_0). The remaining vertex of T must dominate $\{u_3, v_5\} \cup V_4$, which is impossible. Hence, $[u_1, T] \to u_2$. Thus, T contains a vertex $t \in V_0 \cup (V_2 - \{u_2\})$ (to dominate $V_1 - \{u_1\}$). In order to dominate $V_4 \cup V_5$, $v_5 \in T$. Thus, t dominates $V_1 \cup V_3$ and $t \in V_2 - \{u_2\}$. If some vertex z of V_1 dominates V_2, then $\{v_5, t, z\}$ dominates G, a contradiction. Hence each vertex of V_1 must be not adjacent to exactly one vertex of V_2. Thus we can partition V_2 into two sets C and D such that $[V_1, C]$ is full and $[V_1, D]$ is full except for the edges of a perfect matching between V_1 and D. In particular, $|V_2| = |C| + |D| = |C| + |V_1| \geq 4$. This in turn implies $|V_3| \geq 3$.

We now consider adding the edge $v_0 u_3$. If $[u_3, W] \to v_0$, then W contains a vertex of $V_4 \cup V_5$ to dominate v_5. The remaining vertex of W must then dominate both $V_1 \cup \{u_2\}$, which is impossible. Hence, $[v_0, W] \to u_3$. Now W contains a vertex, w say, in $V_4 \cup V_5$ to dominate v_5 and a vertex in $V_3 - \{u_3\}$ to dominate V_2. Since $|V_3| \geq 3$, w must dominate vertices in V_3 as well as v_5. Hence, $w \in V_4$, say $w = u_4$. Since W does not dominate u_3, the vertices u_3 and u_4 are not adjacent. Hence, u_1 is not adjacent to u_2, u_2 is not adjacent to u_3, and u_3 is not adjacent to u_4.

We now consider adding the edge u_1w_4. If $[u_1, Z] \to w_4$, then $u_4 \in Z$. But then the remaining vertex of Z must dominate both u_2 and u_3, which is impossible. Hence, $[w_4, Z] \to u_1$. Thus, Z contains a vertex of $V_1 - \{u_1\}$ to dominate v_0. Since $|V_1| \geq 3$, the remaining vertex of Z must dominate a vertex of V_1 as well as u_4, which is impossible. Hence, $[V_1, V_2]$ must be full. □

We now return to the proof of Claim 14. By Claim 14.1, $[V_1, V_2]$ is full. Hence no vertex of V_2 dominates V_3, for otherwise a vertex from V_1, a vertex from V_2, and the vertex v_5 form a dominating set of G, a contradiction. Since $|V_1| > |V_4| = 2$, we must have $|V_2| > |V_3|$ (and so $|V_2| \geq 3$). Hence there exists a vertex u_3 in V_3 that is not adjacent to at least two vertices, say u_2 and w_2, in V_2.

We show firstly that u_3 dominates V_4. If $[v_5, Z] \to u_2$, then, since $[V_1, V_2]$ is full and Z does not dominate u_2, we must have $v_0 \in Z$ (to dominate v_0). But then the remaining vertex of Z must dominate $(V_2 - \{u_2\}) \cup V_3$, which is impossible since $|V_2| \geq 3$ and $|V_3| \geq 2$. Hence, $[u_2, Z] \to v_5$. Then Z contains a vertex of $V_0 \cup V_1$ (to dominate v_0). The remaining vertex of Z must dominate $V_4 \cup \{u_3\}$. Hence, $u_3 \in Z$ and u_3 dominates V_4.

We now consider adding the edge v_0u_3. If $[u_3, W] \to v_0$, then W contains a vertex of $V_4 \cup V_5$ (to dominate v_5). The remaining vertex of W must then dominate $V_1 \cup \{u_2, w_2\}$, which is impossible. Hence, $[v_0, W] \to u_3$. Since W contains no vertex of V_4, $v_5 \in W$ (to dominate v_5). The remaining vertex of W must dominate $V_2 \cup (V_3 - \{u_3\})$. Hence, $|V_3| = 2$ and the vertex of V_3 different from u_3 dominates V_2. But since each vertex of V_2 must be not adjacent with some vertex of V_3, each vertex of V_2 is not adjacent to u_3. But this is a contradiction since at least one vertex of V_2 must be adjacent with u_3. Hence our assumption that $|V_1| \geq 3$ in incorrect. Thus, $|V_1| = 2$, completing the proof of Claim 14. □

Claim 14 shows that each vertex of eccentricity 5 has degree exactly 2. In particular, v_5 is adjacent to exactly two vertices of V_4. Let u_4 and w_4 be the two neighbours of v_5. Thus, $N(v_5) \cap V_4 = \{u_4, w_4\}$. Further, let $V_1 = \{u_1, w_1\}$.

Claim 15 $|V_4| = 2$.

Proof. Suppose $|V_4| \geq 3$. Let $A = \{u_4, w_4\}$ and $B = V_4 - A$. Then $|B| \geq 1$. Let $u_2 \in V_2$ and consider $G + u_2v_5$. Then $[v_5, S] \to u_2$ or $[u_2, S] \to v_5$. Suppose $[v_5, S] \to u_2$. Then S must contain a vertex in $V_3 \cup B$ (to dominate B). The remaining vertex of S must dominate $V_0 \cup V_1$. Hence, $v_0 \in S$. But then the vertex of S different from v_0 must dominate $(V_2 - \{u_2\}) \cup V_3 \cup B$, which is impossible. Hence, $[u_2, S] \to v_5$. Now S must contain a vertex of $V_0 \cup V_1$ (to dominate v_0). The remaining vertex of S must dominate V_4 and therefore belongs to V_3. In particular, u_2 is adjacent to every vertex of V_3 except possibly for one vertex. Since u_2 was an arbitrary vertex of V_2, each vertex of V_2 is adjacent to every vertex of V_3 except possibly for one vertex.

We now consider adding the edge u_1u_4. If $[u_1, T] \to u_4$, then $w_4 \in T$ (to dominate v_5) and the remaining vertex of T belongs to $V_1 \cup V_2$ (to dominate w_1). But then B is not dominated by $T \cup \{u_1\}$. Hence, $[u_4, T] \to u_1$. Then $w_1 \in T$ (to dominate v_0).

The remaining vertex u_3 (say) of T belongs to V_3 and dominates $V_4 - \{u_4\}$. Thus, u_4 dominates all of V_3, except for possibly one vertex of V_3. If u_3 dominates V_2, then $\{v_0, u_3, u_4\}$ dominates G, a contradiction. Hence we may assume that u_3 is not adjacent to u_2. However, $[u_2, S] \to v_5$. Now S must contain a vertex of $V_0 \cup V_1$ (to dominate v_0). The remaining vertex of S must dominate $\{u_3\} \cup V_4$. Thus, $u_3 \in S$. In particular, u_2 is adjacent to every vertex in V_3 different from u_3 and u_3 is adjacent to u_4. This shows that u_4 dominates V_3 and u_3 dominates V_4. Similarly, w_4 dominates V_3. Hence, $[V_3, A]$ is full.

Let $b \in B$ and consider $G + bu_1$. Suppose $[b, W] \to u_1$. Then $w_1 \in W$ (to dominate v_0) and $v_5 \in W$ (to dominate $V_4 \cup A$). Hence, $|B| = 1$ and b dominates V_3. Hence, $[V_3, V_4]$ is full. If a vertex, say u_3, of V_3 dominates V_2, then $\{v_0, u_3, u_4\}$ dominates G, a contradiction. Thus, every vertex of V_3 is not adjacent to some vertex of V_2. However, $|V_0| + |V_2| + |V_4| = |V_1| + |V_3| + |V_5|$, and so we must have $|V_3| = |V_2| + 1$. By the Pigeonhole Principle, at least one vertex of V_2 is not adjacent to at least two vertices of V_3. However, this contradicts our earlier observation that each vertex of V_2 is adjacent to every vertex of V_3 except possibly for one vertex. Hence, $[u_1, W] \to b$.

Now W contains a vertex of $V_1 \cup V_2$ (to dominate w_1) and a vertex of $A \cup V_5$ (to dominate v_5). Hence, $v_5 \in W$ and W contains a vertex u_2 (say) of V_2 that dominates $\{w_1\} \cup V_3$. This implies that $\{u_1, u_2\}$ dominates $V_0 \cup V_1 \cup V_2 \cup V_3$ and $|B| = 1$. However, since $[u_4, T] \to u_1$, we have shown that T contains a vertex u_3 in V_3 that dominates V_4. Hence, $\{v_0, u_3, u_4\}$ dominates G, a contradiction. Consequently, $|V_4| = 2$. □

By Claim 15, $V_4 = \{u_4, w_4\}$.

Claim 16 *If each vertex in V_1 is adjacent to all except possibly one vertex of V_2, then $[V_1, V_2]$ is full.*

Proof. Suppose u_1 is not adjacent to a vertex u_2 in V_2. Then $w_1 u_2$ must be an edge and, by assumption, u_1 dominates $V_2 - \{u_2\}$. If u_2 dominates V_3, then $\{u_1, u_2, v_5\}$ dominates G, a contradiction. Hence there must be a vertex u_3 (say) in V_3 that is not adjacent to u_2.

We now consider adding the edge $v_0 u_3$. If $[u_3, W] \to v_0$, then W contains a vertex of $V_2 - \{u_2\}$ (to dominate V_1). The remaining vertex of W must then dominate both u_2 and v_5, which is impossible. Hence, $[v_0, W] \to u_3$. Now W contains a vertex in $V_4 \cup V_5$ to dominate v_5. The remaining vertex, w say, of W must dominate V_2. If $w = w_1$, then z must also dominate $V_3 - \{u_3\}$, which is impossible. Hence, $w \in V_3 - \{u_3\}$ and w dominates V_2. If $|V_3| = 2$, then, letting w_2 denote the vertex of V_2 different from u_2, $\{w_1, w_2, v_5\}$ dominates G, a contradiction. Hence, $|V_3| \geq 3$. Since the vertex of W different from w must dominate vertices in V_3 as well as v_5, we may assume $u_4 \in W$. Since W does not dominate u_3, u_3 and u_4 are not adjacent vertices. Hence, u_1 is not adjacent to u_2, u_2 is not adjacent to u_3, and u_3 is not adjacent to u_4.

We now consider adding the edge $u_1 u_2$. If $[u_2, T] \to u_1$, then $w_1 \in T$ (to dominate v_0). The remaining vertex of T must then dominate $\{u_3, v_5\} \cup V_4$, which is impossible. Hence, $[u_1, T] \to u_2$. Then T contains a vertex to dominate w_1 and a vertex of $V_4 \cup V_5$ to dominate v_5. Hence, $v_5 \in T$ and the remaining vertex t (say) of T must belong to V_2 and dominates $\{w_1\} \cup V_3$. If w_1 dominates V_2, then $\{w_1, t, v_5\}$ dominates G, a contradiction. Therefore, w_1 is not adjacent to a vertex w_2 in V_2 (distinct from u_2). Thus there must be a vertex w_3 (say) in V_3 that is not adjacent to w_2.

Suppose that $u_3 \neq w_3$. We now consider adding the edge u_1w_4. If $[u_1, Z] \to w_4$, then $u_4 \in Z$. But then the remaining vertex of Z must dominate both u_2 and u_3, which is impossible. Hence, $[w_4, Z] \to u_1$. Thus, $w_1 \in Z$ (to dominate v_0). If w_3w_4 is not an edge of G, then the remaining vertex of Z must dominate w_2 and w_3, which is impossible. Hence, w_3 and w_4 are adjacent. We now consider adding the edge v_0w_3. If $[w_3, K] \to v_0$, then K contains a vertex of $V_2 - \{w_2\}$ (to dominate V_1). The remaining vertex of K must then dominate both w_2 and v_5, which is impossible. Hence, $[v_0, K] \to w_3$. Now K contains either u_4 or v_5 (to dominate v_5). In any event, the remaining vertex of K must dominate both u_2 and u_3, which is impossible. Hence, $u_3 = w_3$.

We now consider adding the edge u_2v_5. If $[u_2, F] \to v_5$, then F contains a vertex of $V_3 - \{u_3\}$ (to dominate V_4). The remaining vertex of F must then dominate both v_0 and u_3, which is impossible. Hence, $[v_5, F] \to u_2$. Now F contains a vertex, f say, to dominate V_3 and a vertex in $V_0 \cup V_1$ to dominate v_0. Since f dominates V_3, $f \notin \{u_2, w_2\}$. Since $|V_3| \geq 3$, $f \notin V_3$. Hence the vertex of F different from f must dominate all of v_0, u_2, and w_2, which is impossible. Therefore, $[V_1, V_2]$ must be full. □

A symmetrical argument yields the following result.

Claim 17 *If each vertex in V_4 is adjacent to all except possibly one vertex of V_3, then $[V_3, V_4]$ is full.*

Claim 18 *If both $[V_1, V_2]$ and $[V_3, V_4]$ are full, then $G \in \mathcal{G}_4$.*

Proof. If any vertex u_2 in V_2 dominates V_3, then $\{u_1, u_2, v_5\}$ dominates G, a contradiction. Similarly, if any vertex u_3 in V_3 dominates V_2, then $\{v_0, u_3, u_4\}$ dominates G, a contradiction. Hence each vertex in V_2 is not adjacent to some vertex of V_3 and each vertex of V_3 is not adjacent to some vertex of V_2. Suppose a vertex $u_2 \in V_2$ is not adjacent to two vertices, say a and b, in V_3. Then $|V_2| = |V_3| \geq 3$, since each vertex in V_2 is adjacent to at least one vertex in V_3. We now consider adding the edge u_2v_5. If $[u_2, W] \to v_5$, then we may assume u_1 is in W. The remaining vertex of W must dominate $\{a, b\} \cup V_4$, which is impossible. Hence, $[v_5, W] \to u_2$. Thus, $v_0 \in W$. The remaining vertex of W must dominate $(V_2 - \{u_2\}) \cup V_3$, which is impossible (since $|V_2| = |V_3| \geq 3$). Thus each vertex of V_2 is not adjacent to exactly one vertex of V_3. Similarly, each vertex of V_3 is not adjacent to exactly one vertex of V_2. Hence, $[V_2, V_3]$ is full except for the edges of a perfect matching between V_2 and V_3. Thus, $G \in \mathcal{G}_4$. □

By Claims 16, 17, and 18, we may assume that a vertex in V_1 is not adjacent to at least two vertices in V_2 or a vertex in V_4 is not adjacent to at least two vertices in V_3, for otherwise $G \in \mathcal{G}_4$. Without loss of generality, we may assume that u_1 has the smallest degree of the four vertices u_1, w_1, u_4 and w_4. Then, u_1 is not adjacent to at least two vertices of V_2. Let A be the set of vertices in V_2 that are not adjacent to u_1. Further, let $B = V_2 - A$. Since each vertex of V_1 is adjacent to some vertex of V_2, $|B| \geq 1$. By assumption, $|A| \geq 2$. Hence, $|V_2| = |V_3| \geq 3$. We now consider $G + u_1a$ where $a \in A$.

Claim 19 *If there is a vertex a_1 in A such that $[u_1, W] \to a_1$, then $G \in \mathcal{G}_5$.*

Proof. The set W contains a vertex of $V_4 \cup V_5$ (to dominate v_5) and a vertex of V_2 (to dominate $(A - \{a_1\}) \cup \{w_1\}$). Hence, $v_5 \in W$. Thus, W contains a vertex a_2 in $A - \{a_1\}$ and this vertex dominates $(A - \{a_1\}) \cup V_3$. Thus, $A = \{a_1, a_2\}$ and a_2 dominates V_3. Before proceeding further, we prove two claims.

Claim 19.1 $[A, V_3]$ *is full.*

Proof. Suppose a_1 is not adjacent to a vertex, u_3 say, in V_3. Then we must have $[a_2, M] \to u_1$. Thus, $w_1 \in M$ (to dominate v_0). The remaining vertex of W must dominate $V_4 \cup V_5$, and so $v_5 \in M$. In particular, we note that w_1 dominates V_2.

We now consider $G + v_0u_3$. Then $[u_3, L] \to v_0$ or $[v_0, L] \to u_3$. Let $L = \{\ell_1, \ell_2\}$. If $[u_3, L] \to v_0$, then $\ell_2 \in V_4 \cup V_5$ to dominate v_5. But then ℓ_1 must dominate both u_1 and a_1, which is impossible. Hence, $[v_0, L] \to u_3$. If $\ell_2 = v_5$, then ℓ_1 must dominate $V_2 \cup (V_3 - \{u_3\})$, which is impossible since $|V_3| \geq 3$. Hence, we may assume that $\ell_2 = u_4$, and so the vertices u_3 and u_4 are not adjacent.

We now consider $G + a_1u_3$. If $[a_1, K] \to u_3$, then K contains a vertex from $V_4 \cup V_5$ to dominate v_5. The remaining vertex of K must dominate both u_1 and a_2, which is impossible. Hence, $[u_3, K] \to a_1$. Now, K contains a vertex from $V_0 \cup V_1$ to dominate v_0 and a vertex from $V_4 \cup V_5$ to dominate v_5. Thus, $v_0 \in K$. The remaining vertex of K must dominate $(V_3 - \{u_3\}) \cup \{u_4, v_5\}$. Hence, $u_4 \in K$. In particular, u_4 dominates $V_3 - \{u_3\}$ and u_3 dominates $V_2 \cup \{w_4\}$.

Since $[V_3, V_4]$ is not full, and since u_4 is adjacent to every vertex of V_3 except for u_3, Claim 17 implies that w_4 is not adjacent to two vertices of V_3, say d_1 and d_2. If $[d_1, T] \to w_4$ where $T = \{t_1, t_2\}$, then $t_1 \in V_0 \cup V_1$ and $t_2 \in V_4 \cup V_5$. But then at least one of d_2 and u_3 will not be dominated. Hence, $[w_4, T] \to d_1$. Now, $t_1 \in V_0 \cup V_1$ and $t_2 \in V_3 \cup V_4$ to dominate d_2 and u_4. Thus, $t_1 = v_0$ and $t_2 = d_2$. Moreover, d_2 dominates V_2 and $V_3 - N(w_4) = \{d_1, d_2\}$. Similarly, d_1 dominates V_2.

If some vertex $b \in B$ dominates V_3, then $\{w_1, b, v_5\}$ dominates G. Hence, each vertex in B is not adjacent to some vertex of V_3. However, $[\{u_3, d_1, d_2\}, B]$ is full. Since $|V_2| = |V_3|$, some vertex, w_3 say, of V_3 is not adjacent to at least two vertices of B, say b_1 and b_2. Consider $G+v_0w_3$. If $[v_0, Z] \to w_3$, then $v_5 \in Z$ (since w_3 is adjacent to both u_4 and w_4) and the remaining vertex of Z must dominate $V_2 \cup (V_3 - \{w_3\})$, which is impossible since $|V_2| = |V_3| \geq 4$. On the other hand, if $[w_3, Z] \to v_0$, then Z contains a vertex in $V_4 \cup V_5$. The remaining vertex of Z must dominate $V_1 \cup \{b_1, b_2\}$, which is impossible. Therefore, a_1 dominates V_3, and so $[A, V_3]$ is full. □

Claim 19.2 $[V_3, V_4]$ *is not full.*

Proof. Suppose $[V_3, V_4]$ is full. Then each vertex u_3 in V_3 is not adjacent to a vertex of B, for otherwise $\{v_0, u_3, u_4\}$ would dominate G. However, $|V_3| = |B| + 2$. Hence, by the Pigeonhole Principle, there is a verex b in B that is not adjacent to two vertices of V_3, say u_3 and w_3.

We show that the vertices w_1 and b are not adjacent. Consider $G + bv_5$. If $[b, T] \to v_5$, then T contains a vertex in $V_1 \cup V_2$. The remaining vertex of T must dominate $\{u_3, w_3\} \cup V_4$, which is impossible. Thus, $[v_5, T] \to b$. If $v_0 \in T$, then the remaining vertex of T must dominate $V_2 \cup \{u_3, w_3\}$, which is impossible. Hence, $v_0 \notin T$. Thus, since no neighbour of b is in T, $w_1 \in T$ and w_1 is not adjacent to b.

We now consider $G + v_0u_3$. If $[v_0, W] \to u_3$, then $v_5 \in W$. The remaining vertex of W must then dominate $V_2 \cup (V_3 - \{u_3\})$, which is impossible. On the other hand, if $[u_3, W] \to v_0$, then W contains a vertex of $V_4 \cup V_5$. The remaining vertex of W must then dominate both w_1 and b, which is impossible. Therefore, $[V_3, V_4]$ cannot be full. □

We now return to the proof of Claim 19. By Claim 19.2, $[V_3, V_4]$ is not full. Hence, by Claim 17, there is a vertex of V_4, say u_4, that is not adjacent to two vertices of V_3, say c_1 and c_2. Let $C = \{c_1, c_2\}$ and let $D = V_3 - C$. Then u_4 dominates D, for otherwise u_4 would have smaller degree than u_1, contradicting our choice of u_1. We show next that $[C, V_2]$ is full. If $[c_1, Z] \to u_4$, then $w_4 \in Z$ (to dominate v_5) and $v_0 \in Z$ (to dominate $V_0 \cup V_1$). In particular, c_1 dominates V_2. Similarly, if $[c_2, Z] \to u_4$, then c_2 dominates V_2. Thus, if $[c_1, Z] \to u_4$ and $[c_2, Z] \to u_4$, then $[C, V_2]$ is full. On the other hand, if $[u_4, Z] \to c_1$ or $[u_4, Z] \to c_2$, then similar arguments to those used to establish Claim 19.1 show that $[C, V_2]$ is full.

Claim 19.3 *If w_1 dominates B, then $G \in \mathcal{G}_5$.*

Proof. Suppose w_1 dominates B. Let $b \in B$. Then b is not adjacent to at least one vertex of D, for otherwise $\{w_1, b, v_5\}$ would dominate G. Suppose b is not adjacent to two vertices, say d_1 and d_2, of D. Consider $G + bv_5$. Then $[b, T] \to v_5$ or $[v_5, T] \to b$. Let $T = \{t_1, t_2\}$. If $[b, T] \to v_5$, then $t_1 \in V_0 \cup V_1$ and so t_2 must dominate $\{d_1, d_2\} \cup V_4$, which is impossible. On the other hand, if $[v_5, T] \to b$, then $t_1 = v_0$ and so t_2 must dominate $(V_2 - \{b\}) \cup V_3$, which is impossible. Hence, b is not adjacent to exactly one vertex of D. Since b is an arbitrary vertex of B, each vertex of B is not adjacent to exactly one vertex of D.

We show next that each vertex of D is not adjacent to exactly one vertex of B. Let $d \in D$ and suppose that d is not adjacent to two vertices of B. If d is adjacent to w_4, then by considering $G + v_0d$ we arrive at a contradiction. Hence, d and w_4 are not adjacent. Let b be a vertex of B not adjacent to d and consider $G + bv_5$. Then $[b, T] \to v_5$ or $[v_5, T] \to b$. Let $T = \{t_1, t_2\}$. If $[b, T] \to v_5$, then $t_1 \in V_0 \cup V_1$ and so t_2 must dominate $\{d, u_4, w_4\}$, which is impossible. On the other hand, if $[v_5, T] \to b$, then $t_1 = v_0$ and so t_2 must dominate $(V_2 - \{b\}) \cup V_3$, which is impossible. Hence, each vertex of D is not adjacent to exactly one vertex of B. Since $|B| = |D|$, it follows that $[V_2, V_3]$ is full except for the edges of a perfect matching between B and D.

Suppose now that w_4 is not adjacent to some vertex, say d, of D. Let b be the vertex of B that is not adjacent to d, and consider $G + bv_5$. If $[b, T] \to v_5$, then one vertex of T belongs to $V_0 \cup V_1$. The remaining vertex of T must then dominate both w_4 and d, which is impossible. Hence, $[v_5, T] \to b$. Thus, $v_0 \in T$. The remaining vertex of T must dominate $(V_2 - \{b\}) \cup V_3$, which is impossible. Hence, w_4 must dominate D. Thus, $G \in \mathcal{G}_5$. $\square$

A symmetrical argument yields the following result.

Claim 19.4 *If w_4 dominates D, then $G \in \mathcal{G}_5$.*

By Claims 19.3 and 19.4, we may assume that w_1 is not adjacent to some vertex of B and that w_4 is not adjacent to some vertex of D, for otherwise $G \in \mathcal{G}_5$.

Suppose w_1 is not adjacent to exactly one vertex, say b, of B. Then b is not adjacent to a vertex, say d, of D, for otherwise $\{w_1, b, v_5\}$ would dominate G. We now consider $G + v_0d$. If $[d, W] \to v_0$, then one vertex of W belongs to $V_4 \cup V_5$. The remaining vertex of W must then dominate both w_1 and b, which is impossible. Hence, $[v_0, W] \to d$. If $v_5 \in W$, then the remaining vertex of W must dominate $V_2 \cup (V_3 - \{d\})$, which is impossible. Hence, $w_4 \in W$. This implies that the vertices d and w_4 are not adjacent. We now consider $G + u_1u_4$. By symmetry, we may assume that $[u_1, Z] \to u_4$. Then $w_4 \in Z$. The remaining vertex of Z must dominate $\{w_1, d\} \cup A$, which is impossible. Hence, w_1 is not adjacent to at least two vertices of B. A symmetrical argument shows that w_4 is not adjacent to at least two vertices of D.

We again consider $G + u_1u_4$. By symmetry, we may assume that $[u_1, Z] \to u_4$. Then $w_4 \in Z$. The remaining vertex of Z must then dominate $A \cup (V_3 - N(w_4))$, which is impossible since $|A| \geq 2$ and $|V_3 - N(w_4)| \geq 2$. This completes the proof of Claim 19. $\square$

Claim 20 *If $[a, W] \to u_1$ for every $a \in A$, then $G \in \mathcal{G}_5$.*

Proof. Let $a \in A$ and suppose $[a, W] \to u_1$. Then, $w_1 \in W$ (to dominate v_0). The remaining vertex of W must dominate $V_4 \cup V_5$, and so $v_5 \in W$. In particular, w_1 dominates V_2 and the vertex a dominates V_3. Since $[a, W] \to u_1$ for every $a \in A$, this shows that $[A, V_3]$ is full.

If some vertex $b \in B$ dominates V_3, then $\{w_1, b, v_5\}$ dominates G. Hence, each vertex in B is not adjacent to some vertex of V_3. Suppose some vertex $b \in B$ is not adjacent to at least two vertices in V_3. Consider $G + bv_5$. If $[v_5, Z] \to b$, then $v_0 \in Z$ and the remaining vertex of Z must dominate A as well as $V_3 - N(b)$, which is impossible since $|V_3 - N(b)| \geq 2$ and $|A| \geq 2$. Hence, $[b, Z] \to v_5$. Thus, w_1 or v_0 is in Z (to dominate v_0). The remaining vertex of Z must dominate $V_4 \cup (V_3 - N(b))$, which is impossible. Hence each vertex of B is not adjacent to exactly one vertex of V_3.

Let D be the set of vertices in V_3 that are not adjacent to some vertex of B, and let $C = V_3 - D$. Then $|D| \leq |B|$ and therefore $|C| \geq |A| \geq 2$. Thus, since $[C, V_2]$ is full, $[V_3, V_4]$ cannot be full, for otherwise $\{v_0, c, u_4\}$ would dominate G for any vertex

c in C. This in turn implies, by Claim 17, that at least one vertex in V_4, say u_4, is not adjacent to at least two vertices in V_3. But then proceeding as above, we can show that w_4 dominates V_3, C is the set of vertices in V_3 that are not adjacent to u_4, and each vertex in D is not adjacent to exactly one vertex of B. Hence, $G \in \mathcal{G}_5$. $\square$

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