On an extremal subfamily of an extremal family of nearly strongly regular graphs

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Abstract

We continue the classification of the regular simple graphs in which, for some t, any two adjacent vertices have exactly t common neighbors, and the union of their neighbor sets misses exactly two vertices. Previously it was shown that for any such graph with n vertices, if $t > 0$ then $t + 8 \leq n \leq 3t + 6$. Here we show that there is exactly one such graph on $n = 3t + 6$ vertices, for each $t = 1, 2, \ldots$, namely $K_{t+2,t+2,t+2}$ minus a two-factor consisting of triangles.

Let G be a simple graph. For an edge $e \in E(G)$ with end-vertices u, v let $t(e) = |N(u) \cap N(v)|$ and let $J(e) = |N(u) \cup N(v)|$ (with neighborhoods taken in G, of course). Let $t(G) = |E(G)|^{-1} \sum_{e \in E(G)} t(e)$, and let $J(G) = \max_{e \in E(G)} J(e)$. It is shown in [3] that if G has m edges and n vertices, then $4m \leq n(J(G) + t(G))$, with equality if and only if G is regular and t is a constant function (equivalently, G is regular and J is a constant function; observe that $J(e) + t(e) = d(u) + d(v)$). This conclusion also holds if $J(G)$ is defined to be an arithmetic mean, and $t(G)$ is a max. Clearly this result generalizes Mantel's famous theorem, i.e. Turan's theorem with $r = 2$ (see [5]).

To agree with the notation in [3] and [4], let us denote by $ET(n, J, t)$ the set of extremal graphs for the inequality above, on n vertices with $J = J(G)$ and $t = t(G)$.

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That is, $ET(n,J,t)$ consists of the regular graphs on n vertices, of degree $2^{-1}(J+t)$, with each pair of adjacent vertices having exactly t common neighbors. These graphs are "nearly strongly regular"; they are regular, and, in the lingo of strongly regular graphs (see [5]), there is a common value λ of $|N(u)\cap N(v)|$ for adjacent vertices u and v (namely, $\lambda = t$), but there might not be a μ (a common value of $|N(u)\cap N(v)|$ for non-adjacent distinct u and v).

Since the totality of such graphs include the strongly regular graphs, we despair of ever achieving a complete catalog, indexed by n, J, and t, of such graphs. But some interesting results have been produced by fixing certain values of $J = J(n)$. In [2] it is shown that $\bigcup_{n,t} ET(n,n,t)$ consists of the regular Turàn graphs, i.e., the complete r–partite graphs (for various r) with parts of equal size. In [3] it is shown that $\bigcup_{n,t} ET(n,n-1,t)$ consists of the complements of the strongly regular graphs with $\mu = 1$. (In general, if $G \in str(n,k,\lambda,\mu)$ then $\bar{G} \in ET(n, n-\mu, n+\mu-2-2k)$.)

In [4] it is shown that $G \in ET(n,n-2,0)$ if and only if n is even and either G is bipartite and regular (so $G = K_{\frac{n}{2},\frac{n}{2}}$ minus a one-factor) or G is one of the two non-bipartite graphs given in [4]. (It has since been pointed out that this result for $n \geq 10$ is an easy consequence of a famous theorem of Andraśfai, Erdös and Sós [1].) It is also proven in [4] that for $t > 0$, if $ET(n,n-2,t)$ is non-empty then $t+8 \leq n \leq 3t+6$, and that in the case $t=1$, the unique graph in $ET(9,7,1)$ is the line graph of $K_{3,3}$.

Our aim here is to show that the extreme $n = 3t+6$ in the result just mentioned is achievable for every $t \geq 1$, and that the graph achieving it is unique.

Theorem 1 *Suppose that t is a positive integer. Then $G \in ET(3t+6, 3t+4, t)$ if and only if $G = K_{t+2,t+2,t+2} - F$, where F is the set of edges of a 2-factor of $K_{t+2,t+2,t+2}$ consisting of triangles.*

Remark. The graph G described above is the complement of the line graph of $K_{3,t+2}$.

Proof. It is straightforward to verify that $K_{t+2,t+2,t+2} - F \in ET(3t+6, 3t+4, t)$.

Suppose that $G \in ET(3t+6,3t+4,t)$. G is regular with degree $2^{-1}(3t+4+t) = 2t+2$.

For adjacent vertices $u,v \in V(G)$, let $T = T(u,v) = N(u)\cap N(v)$, $A = A(u,v) = N(u) \setminus (T \cup \{v\})$, $B = B(u,v) = N(v) \setminus (T\cup\{u\})$, and $X = X(u,v) = \{x,y\} = V(G) \setminus (N(u)\cup N(v))$. Observe that $|A| = |B| = t+1$.

Claim 1. There are no edges among the vertices of T, and every vertex of T is adjacent to each of x and y.

Proof. Suppose that $w \in T$; w has $t-1$ neighbors in common with u, other than v, and these must be in $T \cup A$. Suppose that w is adjacent to s vertices of T. Then w is adjacent to $t-1-s$ vertices of A, and, similarly, of B. Since w might be adjacent to one or both of x,y, and is adjacent to both u and v, we have $2t+2 = d(w) \leq s + 2(t-1-s)+2+2 = 2t+2-s$. It follows that $s = 0$ and that

w is adjacent to both x and y, which establishes the claim.

The first assertion of Claim 1, that there are no edges among the vertices of T, is equivalent to: G contains no K_4's.

It is a consequence of the proof of Claim 1 that each $w \in T$ has $t-1$ neighbors in A and in B and each of these sets of $t-1$ neighbors are independent (i.e., there are no edges among them), because they are in $T(u,w)$, or in $T(v,w)$.

For a subset S of $V(G)$, let $\langle S \rangle$ denote the subgraph of G induced by S.

Claim 2. Each of $\langle A \rangle$, $\langle B \rangle$ has exactly t edges.
Proof. It suffices to prove the claim for A. Let d_A denote degree within $\langle A \rangle$. Each $a \in A$ has t common neighbors with u, and these are in $A \cup T$. By remarks above, counting the number of edges between A and T we have $t(t-1) = \sum_{a \in A}(t - d_A(a)) = t|A| - \sum_{a \in A} d_A(a) = t(t+1) - 2|E(\langle A \rangle)|$, which clearly implies the claim.

For $a \in A$, it is clear from the proof of Claim 2 that $d_A(a) = t - |N(a) \cap T|$.

Claim 3. If $t > 1$ then there is at most one vertex of A which is adjacent to no vertices of T, and if there is one, then $\langle A \rangle \cong K_{1,t}$. (Of course, the same holds for B.)
Proof. If there were two vertices of A each adjacent to no vertices of T, then each would have degree t in $\langle A \rangle$, and so would jointly be incident to $2t-1$ edges in $\langle A \rangle$. But $2t - 1 \leq t$ only if $t \leq 1$, so if $t > 1$ it is impossible that there could be two such vertices, by Claim 2.

If there is one such vertex, it is of degree t in $\langle A \rangle$, which is of order $t+1$ with only t edges, by Claim 2. Thus $\langle A \rangle \cong K_{1,t}$.

Claim 4. If $t = 2$ or $t \geq 5$ then A contains a vertex which is adjacent to no vertex of T (so $\langle A \rangle \cong K_{1,t}$, by the preceding claim). If $t = 3$ the only possibility for $\langle A \rangle$ besides $K_{1,3}$ is P_4. If $t = 4$ the only possibility for $\langle A \rangle$ besides $K_{1,4}$ is $K_1 + C_4$.
Proof. If $t = 2$ then $\langle A \rangle$ is a simple graph with 3 vertices and 2 edges, so $\langle A \rangle \cong K_{1,2}$ and the vertex of degree 2 in $\langle A \rangle$ is adjacent to no vertex of T.

Suppose that each vertex of A is adjacent to something in T. Therefore, by previous remarks, each vertex of A belongs to an independent set of $t-1$ vertices in A. Thus $d_A(a) \leq 2$ for each $a \in A$. Suppose $t = 3$. The only graphs on 4 vertices with 3 edges and maximum degree 2 are $K_1 + K_3$ and P_4; $K_1 + K_3$ is not possible because the K_3 together with u would make a K_4, contradicting Claim 1.

If a, $a' \in A$ are adjacent, then a, a' can have no common neighbors in T. (For if $a, a' \in N(w), w \in T$, then a, a', w, and u induce a K_4 in G, contradicting Claim 1.) On the other hand, each is adjacent to at least $t-2$ vertices of T, since each is of degree ≤ 2 in $\langle A \rangle$. Therefore, $2(t-2) \leq |T| = t$, so $t \leq 4$.

If $t = 4$ the preceding shows that every vertex of A must have degree 2 or 0 in $\langle A \rangle$ (otherwise, we would have $t - 2 + t - 1 \leq t$). The only graph on 5 vertices, with degrees 2 or 0, with exactly 4 edges, is $K_1 + C_4$. The claim is proven.

Of course, the conclusions of Claim 4 hold with A replaced by B.

Now suppose that $\langle A\rangle \cong K_{1,t} \cong \langle B\rangle$, whatever the value of $t > 1$. Let a_0, b_0 be the central vertices of degree t in $\langle A\rangle, \langle B\rangle$, respectively, and let $A' = A \setminus \{a_0\}$, $B' = B \setminus \{b_0\}$. From remarks preceding, $\langle A' \cup T\rangle$ and $\langle B' \cup T\rangle$ are regular bipartite graphs of degree $t-1$ with bipartitions A', T and B', T, respectively; thus they are isomorphic to $K_{t,t}$ minus a one-factor. Let $w_1, \ldots, w_t$, $a_1, \ldots, a_t$, and $b_1, \ldots, b_t$ be orderings of T, A', and B', respectively, such that for each $j \in \{1, \ldots, t\}$, w_j is adjacent to each vertex in A' except a_j, and to each vertex in B' except b_j.

Claim 5. Suppose that $t > 1$ and $\langle A\rangle \cong \langle B\rangle \cong K_{1,t}$, with $w_1, \ldots, w_t$, $a_1, \ldots, a_t$, $b_1, \ldots, b_t$, a_0, b_0, A' and B' as above. Suppose that x is adjacent to no vertex of B'. Then $G \cong K_{t+2,t+2,t+2} - F$ as claimed in the Theorem.

The tripartition of $V(G)$ is $T \cup \{a_0, b_0\}$, $A' \cup \{v, y\}$, and $B' \cup \{u, x\}$. The two-factor of which F is the set of edges is composed of the triangles $a_j b_j w_j$, $j = 1, \ldots, t$, $a_0 v x$, and $b_0 u y$.

Proof. By Claim 1 x is adjacent to every vertex of T, so x must have t common neighbors with each of these. Since T is an independent set and x is adjacent to neither of u, v, these common neighbors all lie in $A' \cup B' \cup \{y\}$. Since, by assumption, x has no neighbors in B', and since each vertex of T is adjacent to only $t-1$ vertices of A', it follows that x is adjacent to every vertex of A', and to y. Since $|A' \cup T \cup \{y\}| = 2t+1$, x must be adjacent to exactly one of a_0, b_0.

Now, x and y already have t common neighbors in T, so y's $t+1$ neighbors outside of $\{x\} \cup T$ must be vertices not adjacent to x. Also, y needs $t-1$ common neighbors (other than x) with each vertex of T. It follows that y is adjacent to all the vertices of B', and to the vertex in $\{a_0, b_0\}$ that x is not adjacent to.

Each $a_j \in A'$ and $w_i \in T$ to which a_j is adjacent (i.e., $i \neq j$) have common neighbors u and x, and they must have $t-2$ others; these can only be in B'. Thus a_j is adjacent to at least $t-2$ and at most $t-1$ vertices of B'. Since the degree of a_j is $2t+2$, and a_j is not adjacent to any of the vertices of A', nor to y, v, or w_j, it must be that a_j is adjacent to $t-1$ vertices of B', and to b_0. It is easy to see that the vertex of B' that a_j is not adjacent to must be b_j, if a_j is to fulfill its common neighbor obligations with the w_i, $i \neq j$.

By the symmetry of the situation at this point, we see that because b_0 is adjacent to all of A', a_0 must be adjacent to all of B'. One is adjacent to x, the other to y, which fills their degree count to $2t+2$. So a_0, b_0 are not adjacent.

Finally, note that b_0 must have t common neighbors with each $a \in A'$, and b_0 is not adjacent to a_0, u, or any vertex in T. Since a and b_0 have only $t-1$ common neighbors in B', it must be that b_0 and x are adjacent. Symmetrically, a_0 and y are adjacent. The conclusion of the Claim is now easy to verify.

Claim 6. Suppose $t \geq 2$ and $\langle A\rangle \cong \langle B\rangle \cong K_{1,t}$. Then the conclusion of Claim 5 holds (and so the Theorem is proven, in these cases).

Proof. Let $a_0, \ldots, a_t$, $b_0, \ldots, b_t$, $w_1, \ldots, w_t$, A' and B' be as in Claim 5. Suppose $i, j \in \{1, \ldots, t\}$, $i \neq j$. Then a_i and w_j are adjacent, and neither is adjacent to a_j, nor to w_i, so $X(a_i, w_j) = \{a_j, w_i\}$. It follows from Claim 1 that $N(a_i) \cap N(w_j) \subseteq N(a_j) \cap N(w_i)$; therefore $N(a_i) \cap N(w_j) = N(a_j) \cap N(w_i)$.

Since x is adjacent to each of $w_1, \ldots, w_t$, it follows that if one vertex of A' is adjacent to x, then they all are. The same holds for B' (and for y). So if x has neighbors both in A' and in B', then $A' \cup B' \cup T \subseteq N(x)$; therefore $d(x) = 2t+2 \geq 3t$, which is impossible if $t > 2$.

By Claim 5, we are done unless $t = 2$ and both x and y are adjacent to all vertices of $A' \cup B'$. In this case, we see that a_1 and w_2 have three common neighbors, namely, u, x, and y. Since $3 \neq 2$, the case $t = 2$ is finished, and the claim is proven.

As mentioned above, it is shown in [4] that in the case $t = 1$, the only possibility for G is the line graph of $K_{3,3}$, which is self-complementary, so the Theorem holds in this case. The cases $t = 3$, $\langle A \rangle = P_4$ and $t = 4$, $\langle A \rangle = K_1 + C_4$ remain. We will, in each case, show that the proposed form of $\langle A \rangle$ is impossible.

$\mathbf{t = 3}, \langle \mathbf{A} \rangle = \mathbf{P_4}$. Let the vertices of $\langle A \rangle$ along the path one way or the other be a_1, a_2, a_3, a_4. As noted previously, adjacent vertices in A have no common neighbors in T. The adjacencies between T and A are determined by this and the fact that a_1 and a_4 each have two neighbors in T, a_2 and a_3 one each. Let the vertices of T be w_1, w_2, w_3, such that a_1 is adjacent to w_2, w_3, a_2 is adjacent to w_1, a_3 is adjacent to w_2, and a_4 is adjacent to w_1, w_3.

Then, clearly, $X(a_2, w_1) = \{w_2, w_3\}$, so $N(a_2) \cup N(w_1)$ covers B and $N(a_2) \cap N(w_1) \subseteq N(w_2) \cap N(w_3)$, by Claim 1. Therefore, a_2 and w_1 have no common neighbors in B, because such a neighbor would have to be adjacent to w_1, w_2, and w_3. (This is impossible because $\langle B \rangle = P_4$ or $K_{1,3}$, and so has no isolated vertices.) So a_2 must be adjacent to both x and y, in order that a_2 have 3 common neighbors (namely, x, y, and u) with w_1.

By a similar argument, a_3 is adjacent to both x and y. But, because neither a_2 nor a_3 is adjacent to v, their common neighbors must be in $N(v)$, by Claim 1. This contradiction dismisses this case.

$\mathbf{t = 4}, \langle \mathbf{A} \rangle = \mathbf{K_1 + C_4}$. Let the isolated vertex in $\langle A \rangle$ be a_0, and let a_1, a_2, a_3, and a_4 be the vertices around the cycle. Since adjacent vertices in $\langle A \rangle$ have no common neighbor in T, and since each vertex $a \in A$ has $4 - d_A(a)$ neighbors in T, we may as well suppose that $T = \{w_1, w_2, w_3, w_4\}$ with a_1 and a_3 each adjacent to each of w_1, w_2, and a_2 and a_4 each adjacent to each of w_3, w_4.

Then $X(a_1, w_1) = X(a_1, w_2) = \{w_3, w_4\}$. Thus $B \subseteq N(a_1) \cup N(w_i)$, $i = 1, 2$, and $N(a_1) \cap N(w_i) \subseteq N(w_3) \cap N(w_4)$, $i = 1, 2$, by Claim 1. Now, a_1 and w_1 must have at least one common neighbor in B, since they have four common neighbors, and outside of B the only candidates besides u are x and y. By the comments preceding, any common neighbor of a_1 and w_1 in B must be adjacent to at least three vertices of T, namely w_1, w_3, and w_4.

Suppose $\langle B \rangle = K_1 + C_4$ and let b_0 be the isolated vertex in $\langle B \rangle$. By the paragraph above, b_0 is the only possible common neighbor of a_1 and w_1 in B, so a_1 is adjacent to b_0, x and y. Similarly, so is a_2. But a_1 and a_2 are adjacent, and neither is adjacent to v, so by Claim 1 all their common neighbors must be in $N(v)$, which x and y are not.

Therefore, $\langle B \rangle = K_{1,4}$. Let b_0 be the vertex of B adjacent to no vertices of T, and let b_1, b_2, b_3, and b_4 be the other vertices of B, with b_i adjacent to w_j if and only if $i \neq j$. Since common neighbors of a_1 and w_1 are in $N(w_3) \cap N(w_4)$, a_1 is adjacent to neither of b_3, b_4.

Now, a_1 must have 5 neighbors total in $\{x, y\} \cup B$, so a_1 must be adjacent to b_1, b_2, b_0, x, and y. Similarly, a_2 must be adjacent to x and y. This leads to the same contradiction as above: a_1 and a_2 have common neighbors outside of $N(v)$. This completes the proof of the Theorem. □

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