Double Dudeney sets for an odd number of vertices

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Abstract

A double Dudeney set in K_n is a multiset of Hamilton cycles in K_n having the property that each 2-path in K_n lies in exactly two of the cycles. In this paper, we construct a double Dudeney set in K_n when $n = p_1p_2\cdots p_s + 2$, where $p_1, p_2, \ldots, p_s$ are different odd prime numbers and s is a natural number.

1 Introduction

A Dudeney set in the complete graph K_n is a set of Hamilton cycles in K_n having the property that each path of length two (2-path) lies on exactly one of the cycles. The length of a path is the number of edges in the path. A Dudeney set in K_n has been constructed when $n \geq 4$ is even [4]. In the case when n is odd, a Dudeney set in K_n has been constructed only when $n = 2^e + 1$ (e is a natural number) [6], $n = p + 2$ (p is an odd prime number and 2 or -2 is a primitive root of $GF(p)$) [1, 3], and in some other cases when $n = p + 2$ [3, 5].

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A double Dudeney set in K_n is a multiset of Hamilton cycles having the property that each 2-path lies on exactly two of the cycles. If there exists a Dudeney set in K_n, there exists a double Dudeney set in K_n. Except for the above n, it is not known whether a double Dudeney set of K_n exists.

In this paper we will prove Theorem 1.1. For part of our proof we will use the same method as used by [4].

Theorem 1.1 *There exists a double Dudeney set in K_n when $n = p_1 p_2 \cdots p_s + 2$, where $p_1, p_2, \ldots, p_s$ are different odd prime numbers and s is a natural number.*

2 Notation and Preliminaries

Let $n \geq 4$ be an even number. Put $m = n-1$ and $r = (m-1)/2$. Let $K_n = (V_n, E_n)$ be the complete graph on n vertices, where V_n is the vertex set and E_n is the edge set. From now on, put $V_n = \{\infty\} \cup Z_m = \{\infty\} \cup \{0, 1, 2, \ldots, m-1\}$, where Z_m is the set of integers modulo m.

For any integer $i, 0 \leq i \leq m-1$, we define the 1-factor F_i:

$$F_i = \{\{\infty, i\}\} \cup \{\{a, b\} \in E_n \mid a, b \neq \infty, a + b \equiv 2i \pmod{m}\}.$$

Let σ be the vertex-permutation $(\infty)(0\ 1\ 2\ \cdots\ m-1)$, and put $\Sigma = \{\sigma^j \mid 0 \leq j \leq m-1\}$. Clearly σ induces a permutation of the edges of K_n; we will also denote this permutation by σ. When $\mathcal{C}$ is a set of cycles or circuits in K_n, define $\Sigma\mathcal{C} = \{C^\tau \mid C \in \mathcal{C}, \tau \in \Sigma\}$.

For any edge $\{a, b\}$ in K_n, we define the length $d(a, b)$:

$$d(a,b) = \begin{cases} (b-a) (\mathrm{mod}\ m) & (a, b \neq \infty) \\ \infty & (\text{otherwise}), \end{cases}$$

and for any two lengths $d_1, d_2 (\neq \infty)$, we define that d_1 and d_2 are equal as lengths when $d_1 = d_2$ or $d_1 = -d_2$ in Z_m.

The following proposition is easy to prove.

Proposition 2.1 *Let H_i $(1 \leq i \leq m-1)$ be a 1-factor in K_n. If $F_0 \cup H_i$ $(1 \leq i \leq m-1)$ is a Hamilton cycle in K_n and $\cup_{i=1}^{m-1} H_i = E_n \setminus F_0$, then $\Sigma\{F_0 \cup H_i \mid 1 \leq i \leq m-1\}$ is a double Dudeney set in K_n.*

Let A be a 1-factor in K_n that satisfies A1 and A2:

A1. $F_0 \cup A$ is a Hamilton cycle in K_n.

A2. If S is the multiset $\{d(a, b) \mid \{a, b\} \in A\}$, then we have $S = \{\infty, 1, 2, \ldots, r\}$, i.e., A has all lengths.

We construct the complete graph $K_{n'}$ by adding a new vertex λ to K_n; that is, put $n' = n + 1$, $K_{n'} = (V_{n'}, E_{n'})$ and $V_{n'} = V_n \cup \{\lambda\}$. Extend σ to be the following permutation of $V_{n'}$, also denoted by σ: $\sigma = (\infty)(\lambda)(0\ 1\ 2\ \cdots\ m-1)$. Again, let $\Sigma = \{\sigma^j | 0 \leq j \leq m-1\}$.

If we insert the vertex λ into all the edges in A, we get a set of 2-paths in $K_{n'}$. Denote this set by A^λ, that is,

$$A^\lambda = \{(a, \lambda, b) \mid \{a, b\} \in A\}.$$

We note that paths are undirected, i.e., $(a, \lambda, b) = (b, \lambda, a)$. $F_0 \cup A^\lambda$ is considered to be a circuit in $K_{n'}$.

Proposition 2.2 (Proposition 2.3 [5]) *Let A be a 1-factor in K_n which satisfies* A1 *and* A2 *above. Assume h_i $(1 \le i \le r)$ is a Hamilton cycle in K_n and $\Sigma\{h_i \mid 1 \le i \le r\}$ is a Dudeney set in K_n. Then*

$$\Sigma(\{F_0 \cup A^\lambda\} \cup \{h_i \mid 1 \le i \le r\})$$

has each 2-path in $K_{n'}$ exactly once.

Proposition 2.3 *Let A_1 and A_2 be 1-factors in K_n which satisfy A1 and A2 above. ($A_1 = A_2$ is allowed.) Assume h_i $(1 \le i \le 2r)$ is a Hamilton cycle in K_n and $\Sigma\{h_i \mid 1 \le i \le 2r\}$ is a double Dudeney set in K_n. Then*

$$\Sigma(\{F_0 \cup A_1^\lambda, F_0 \cup A_2^\lambda\} \cup \{h_i \mid 1 \le i \le 2r\})$$

has each 2-path in $K_{n'}$ exactly twice, where $\{\ \ \}$ means a multiset.

Proof. The proof is similar to the proof of Proposition 2.2. □

Now we refer to the following famous theorem.

Proposition 2.4 *Let m_1, m_2 be natural numbers with $(m_1, m_2) = 1$. Consider an m_2 by m_1 rectangle having $m_2 \times m_1$ cells. If a ball comes in diagonally from the upper left corner and bounces off the edges as in* Figure 2.1*, then the ball passes through each cell exactly once and leaves from the lower right corner when m_1 and m_2 are odd, from the lower left corner when m_1 is odd and m_2 is even, and from the upper right corner when m_1 is even and m_2 is odd.*

Finally, we explain what we mean by exchanging edges between two 1-factors. Let H_1 and H_2 be 1-factors in K_n. Assume that $H_1 \cup H_2$ is not hamiltonian and that we have a cycle C in $H_1 \cup H_2$. Then we exchange edges of H_1 and H_2 via C to obtain two new 1-factors H_1' and H_2':

$$H_1' = (H_1 \setminus C) \cup (H_2 \cap C), \text{ and}$$

$$H_2' = (H_2 \setminus C) \cup (H_1 \cap C).$$

3 Property (B$_n$)

Let $n \ge 4$ be an even number. Put $m = n - 1$ and $r = (m-1)/2$. We denote by (B$_n$) the following property of K_n:
(B$_n$) There exist 1-factors G_i, $1 \le i \le 2r$, in K_n such that
(1) $F_0 \cup G_i$ is a Hamilton cycle in K_n $(1 \le i \le 2r)$,

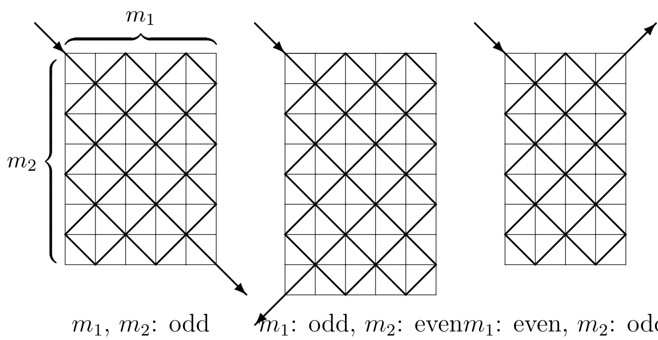

Figure 2.1

(2) $\cup_{i=1}^{2r} G_i = E_n \setminus F_0$,
(3) G_i has an edge of length 1 ($1 \leq i \leq 2r$).

In this terminology, if we put $D = \Sigma\{F_0 \cup G_i \mid 1 \leq i \leq 2r\}$, D is a double Dudeney set in K_n from Proposition 2.1.

Proposition 3.1 *Let $n \geq 4$ be even. If K_n satisfies property* (B_n), *then there exists a double Dudeney set in K_{n+1}.*

Proof. From the assumption, there exist 1-factors G_i, $1 \leq i \leq 2r$, in K_n satisfying (1), (2), (3) of (B_n).

Let θ be the vertex permutation:

$$\theta = \begin{cases} (2\ -2)(4\ -4)(6\ -6)\cdots(r\ \ -r) & (\text{if } m \equiv 1 \pmod 4) \\ (2\ -2)(4\ -4)(6\ -6)\cdots(r-1\ \ -(r-1)) & (\text{if } m \equiv 3 \pmod 4). \end{cases}$$

Then the order of θ is 2 and each edge in F_0 is fixed by θ, i.e., $\theta e = e$ for $e \in F_0$. Put

$$E^{(1)} = \{\{a, b\} \mid d(a, b) = 1\} \setminus \{\{r, -r\}\};$$

then we have $|E^{(1)}| = 2r$.

Claim 3.1 $\theta E^{(1)} = F_r \cup F_{-r} \setminus \{\{\infty, r\}, \{\infty, -r\}\}$.

Since the G_i, $1 \leq i \leq 2r$, satisfy conditions (1) and (2) of (B_n), the 1-factors θG_i, $1 \leq i \leq 2r$, also satisfy conditions (1) and (2) of (B_n), that is, we have,

Claim 3.2
(1) *$F_0 \cup \theta G_i$ is a Hamilton cycle in K_n ($1 \leq i \leq 2r$),*
(2) *$\cup_{i=1}^{2r} \theta G_i = E_n \setminus F_0$.*

Proof. (1) Since $\theta(F_0 \cup G_i) = \theta F_0 \cup (\theta G_i) = F_0 \cup (\theta G_i)$, $F_0 \cup \theta G_i$ is a Hamilton cycle in K_n.
(2) Since $\cup G_i = E_n \setminus F_0$, we have $\theta(\cup G_i) = \theta(E_n \setminus F_0) = \theta E_n \setminus \theta F_0 = E_n \setminus F_0$. □

Therefore we obtain from Proposition 2.1,

Claim 3.3 $\Sigma\{F_0 \cup \theta G_i | 1 \leq i \leq 2r\}$ *is a double Dudeney set in* K_n.

Insert the vertex λ into all edges in F_r and F_{-r} and define F_r^λ and F_{-r}^λ:

$$F_r^\lambda = \{(a, \lambda, b) \mid \{a, b\} \in F_r\} \text{ and } F_{-r}^\lambda = \{(a, \lambda, b) \mid \{a, b\} \in F_{-r}\},$$

where (a, λ, b) is a 2-path. Put $\mathcal{D}^\lambda = \Sigma(\{F_0 \cup F_r^\lambda, F_0 \cup F_{-r}^\lambda\} \cup \{F_0 \cup \theta G_i \mid 1 \leq i \leq 2r\})$.

Claim 3.4 $\mathcal{D}^\lambda$ *has each 2-path in* $K_{n'}$ *exactly twice.*

Proof. From Claim 3.3 and the fact that F_r and F_{-r} satisfy A1 and A2, we obtain Claim 3.4 by Proposition 2.3. □

We would like to leave λ in the 2-path $(\infty, \lambda, r) \in F_r^\lambda$ and λ in the 2-path $(\infty, \lambda, -r) \in F_{-r}^\lambda$, and scatter the remaining $2r$ λs in $F_r^\lambda \cup F_{-r}^\lambda$ over $\{\theta G_i \mid 1 \leq i \leq 2r\}$.

From Claim 3.1, for any i, $1 \leq i \leq 2r$, there is exactly one edge $e_i = \{a_i, b_i\}$ ($a_i, b_i \neq \infty$) that is in both θG_i and $F_r \cup F_{-r}$. Denote by $\theta G_i'$ the set of edges and the 2-path obtained from θG_i by inserting λ into the edge e_i, i.e.,

$$\theta G_i' = \theta G_i \setminus \{\{a_i, b_i\}\} \cup \{(a_i, \lambda, b_i)\}.$$

Define

$$F_r' = F_r \setminus \{\{\infty, r\}\} \cup \{(\infty, \lambda, r)\} \text{ and}$$
$$F_{-r}' = F_{-r} \setminus \{\{\infty, -r\}\} \cup \{(\infty, \lambda, -r)\},$$

where (∞, λ, r) and $(\infty, \lambda, -r)$ are 2-paths. Put

$$\mathcal{D} = \Sigma\left(\{F_0 \cup F_r', F_0 \cup F_{-r}'\} \cup \{F_0 \cup \theta G_i' \mid 1 \leq i \leq 2r\}\right).$$

Then we have

Claim 3.5 $\mathcal{D}$ *is a double Dudeney set in* $K_{n'}$.

Proof. Each element of $\mathcal{D}$ is clearly a Hamilton cycle in $K_{n'}$. The set of all 2-paths in $\mathcal{D}$ and the set of all 2-paths in $\mathcal{D}^\lambda$ are the same. Hence $\mathcal{D}$ has each 2-path in $K_{n'}$ exactly twice by Claim 3.4. Therefore $\mathcal{D}$ is a double Dudeney set in K_n. □

This completes the proof of Proposition 3.1. □

Proposition 3.2 K_{p+1} *satisfies property* (B_{p+1}), *where p is an odd prime number.*

Proof. Put $G_i = F_i$, $1 \leq i \leq p-1$, then the G_i, $1 \leq i \leq p-1$, satisfy (1), (2), (3) of property (B_{p+1}). □

From Propositions 3.1 and 3.2, we obtain,

Proposition 3.3 *There exists a double Dudeney set in* K_{p+2} *where p is an odd prime number.*

4 A proof of Theorem 1.1

To prove Theorem 1.1, we only have to prove Proposition 4.1 from Proposition 3.1.

Proposition 4.1 *K_n satisfies property* (B_n) *when* $n = p_1 p_2 \cdots p_s + 1$, *where* $p_1, p_2, \ldots, p_s$ *are different odd prime numbers and s is a natural number.*

Proof. We will prove the proposition by induction on s. When $s = 1$, the proposition holds from Proposition 3.2. Assume $s \geq 2$. We can assume $p_1 < p_2 < \ldots < p_s$ without loss of generality. Put $m_1 = p_1$, $m_2 = p_2 p_3 \cdots p_s$ and $m = m_1 m_2$. Put $n_l = m_l + 1$ $(l = 1, 2)$ and $n = m + 1$. Note that K_{n_1} satisfies property (B_{n_1}) from Proposition 3.2, and K_{n_2} satisfies property (B_{n_2}) from the hypothesis of the induction. Now we will show that K_n satisfies property (B_n).

For $l = 1, 2$, put $r_l = (m_l - 1)/2$ and consider the complete graph $K_{n_l} = (V_{n_l}, E_{n_l})$, where $V_{n_l} = \{\infty_l\} \cup Z_{m_l} = \{\infty_l\} \cup \{0, 1, 2, \cdots, m_l - 1\}$. Vertices (other than ∞_l) are considered modulo m_l.

Put $r = (m - 1)/2$ and consider the complete graph $K_n = (V_n, E_n)$, where $V_n = \{\infty\} \cup Z_m = \{\infty\} \cup \{0, 1, 2, \cdots, m - 1\}$.

Since $(m_1, m_2) = 1$, Z_m is isomorphic to $Z_{m_1} \times Z_{m_2}$ as additive groups, where $\times$ means a direct product. The isomorphism from Z_m to $Z_{m_1} \times Z_{m_2}$ is given by

$$f : a (\mathrm{mod}\ m) \mapsto (a (\mathrm{mod}\ m_1),\ a (\mathrm{mod}\ m_2)).$$

We identify Z_m and $Z_{m_1} \times Z_{m_2}$ through this mapping. Then we can represent V_n as

$$V_n = \{\infty\} \cup \{(a_1, a_2) \mid a_1 \in Z_{m_1}, a_2 \in Z_{m_2}\}.$$

For any edge $\{\alpha, \beta\}$ in K_n, the length $d(\alpha, \beta)$ is defined as an element of Z_m in Section 2. Since $Z_m \cong Z_{m_1} \times Z_{m_2}$, the length $d(\alpha, \beta)$ is also represented as an element of $Z_{m_1} \times Z_{m_2}$:

$$d(\alpha, \beta) = \begin{cases} ((b_1 - a_1)(\mathrm{mod}\ m_1),\ (b_2 - a_2)(\mathrm{mod}\ m_2)) & (\alpha, \beta \neq \infty) \\ \infty & (\text{otherwise}), \end{cases}$$

where we put $\alpha = (a_1, a_2), \beta = (b_1, b_2)$ when $\alpha, \beta \neq \infty$. And any two lengths $d_1, d_2 (\neq \infty)$ are equal when $d_1 = d_2$ or $d_1 = -d_2$ in $Z_{m_1} \times Z_{m_2}$, for example, lengths $(1, 1)$ and $(-1, -1)$ are equal; $(1, -1)$ and $(-1, 1)$ are equal.

Let $\sigma_l = (\infty_l)(0\ 1\ 2\ \cdots\ m_l - 1)$ be a permutation on V_{n_l}, and put $\Sigma^{(l)} = \langle \sigma_l \rangle$ $(l = 1, 2)$. Put $\sigma = (\infty)(0\ 1\ 2\ \cdots\ m - 1)$ and $\Sigma = \langle \sigma \rangle$. Then σ can be written as $\sigma = (\sigma_1, \sigma_2)$ and it is trivial that $\Sigma \cong \Sigma^{(1)} \times \Sigma^{(2)}$. For $l = 1, 2$, we denote F_0 in K_{n_l} by $F_0^{(l)}$, and we denote F_0 in K_n by $F_{(0,0)}$:

$$\begin{aligned} F_{(0,0)} &= \{\{\infty, 0\}\} \cup \{\{\alpha, \beta\} \in E_n \mid \alpha, \beta \neq \infty, \alpha + \beta \equiv 0 \ (\mathrm{mod}\ m)\} \\ &= \{\{\infty, (0, 0)\}\} \\ &\quad \cup \{\{(a_1, a_2), (b_1, b_2)\} \in E_n \,|\, a_l, b_l \neq \infty_l, a_l + b_l \equiv 0 \ (\mathrm{mod}\ m_l) \ (l = 1, 2)\}. \end{aligned}$$

From our assumption, for $l = 1, 2$, there are 1-factors $G_1^{(l)}, G_2^{(l)}, \ldots, G_{2r_l}^{(l)}$ in K_{n_l} satisfying

(1) $F_0^{(l)} \cup \theta G_i^{(l)}$ is a Hamilton cycle in K_{n_l} $(1 \le i \le 2r_l)$,
(2) $\cup_{i=1}^{2r_l} G_i^{(l)} = E_{n_l} \setminus F_0^{(l)}$,
(3) $G_i^{(l)}$ has an edge of length 1 $(1 \le i \le 2r_l)$.

We denote by v_i and w_j the vertices such that $(\infty_1, v_i) \in G_i^{(1)}$ $(1 \le i \le 2r_1)$, and $(\infty_2, w_j) \in G_j^{(2)}$ $(1 \le j \le 2r_2)$.

Now we define 1-factors in K_n from 1-factors $G_i^{(1)}$, $1 \le i \le 2r_1$, and $G_j^{(2)}$, $1 \le j \le 2r_2$, as follows:

(1) For i $(1 \le i \le 2r_1)$ and j $(1 \le j \le 2r_2)$,

$$\begin{aligned} G_{(i,j)} = \; & \{\{\infty, (v_i, w_j)\}\} \\ & \cup \{\{(v_i, a), (v_i, b)\} \mid \; a, b \ne \infty_2, \{a, b\} \in G_j^{(2)}\} \\ & \cup \{\{(a, w_j), (b, w_j)\} \mid \; a, b \ne \infty_1, \{a, b\} \in G_i^{(1)}\} \\ & \cup \{\{(a_1, a_2), (b_1, b_2)\} \mid a_l, b_l \ne \infty_l \; (l = 1, 2), \\ & \qquad \{a_1, b_1\} \in G_i^{(1)}, \{a_2, b_2\} \in G_j^{(2)}\}. \end{aligned}$$

(2) For i $(1 \le i \le 2r_1)$,

$$\begin{aligned} G_{(i,0)} = \; & \{\{\infty, (v_i, 0)\}\} \\ & \cup \{\{v_i, a), (v_i, b)\} \mid a, b \ne \infty_2, \{a, b\} \in F_0^{(2)}\} \\ & \cup \{\{(a_1, a_2), (b_1, b_2)\} \mid a_l, b_l \ne \infty_l (l = 1, 2), \{a_1, b_1\} \in G_i^{(1)}, \\ & \qquad a_2 + b_2 \equiv 0 \pmod{m_2}\}. \end{aligned}$$

(3) For j $(1 \le j \le 2r_2)$,

$$\begin{aligned} G_{(0,j)} = \; & \{\{\infty, (0, w_j)\}\} \\ & \cup \{\{(a, w_j), (b, w_j)\} \mid a, b \ne \infty_1, \{a, b\} \in F_0^{(1)}\} \\ & \cup \{\{(a_1, a_2), (b_1, b_2)\} \mid a_l, b_l \ne \infty_l (l = 1, 2), a_1 + b_1 \equiv 0 \pmod{m_1}, \\ & \qquad \{a_2, b_2\} \in G_j^{(2)}\}. \end{aligned}$$

It is trivial that these are 1-factors in K_n and any two of these 1-factors have no common edges.

We can represent these 1-factors in K_n geometrically. Since $F_0^{(1)} \cup G_i^{(1)}$ $(1 \le i \le 2r_1)$ is a Hamilton cycle in K_{n_1}, put

$$F_0^{(1)} \cup G_i^{(1)} = (\infty_1, x_{1i}{=}0, x_{2i}, x_{3i}, \ldots, x_{n_1-1,i}{=}v_i),$$

where $x_{si} \in V_{n_1}$ $(1 \le s \le n_1 - 1)$, and

$$\{\infty_1, x_{1i}\} \in F_0^{(1)}, \; \{x_{1i}, x_{2i}\} \in G_i^{(1)}, \; \{x_{2i}, x_{3i}\} \in F_0^{(1)}, \ldots,$$

$$\{x_{n_1-2,i}, x_{n_1-1,i}\} \in F_0^{(1)}, \; \{x_{n_1-1,i}, \infty_1\} \in G_i^{(1)}.$$

Similarly, since $F_0^{(2)} \cup G_j^{(2)}$ $(1 \le j \le 2r_2)$ is a Hamilton cycle in K_{n_2}, put

$$F_0^{(2)} \cup G_j^{(2)} = (\infty_2, y_{1j} = 0, y_{2j}, y_{3j}, \ldots, y_{n_2-1,j} = w_j),$$

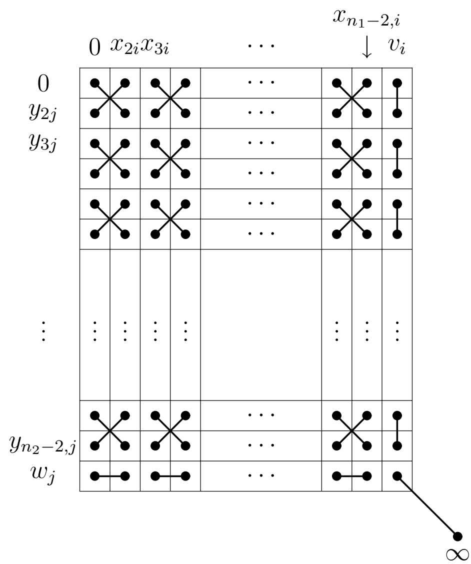

Figure 4.1: $G_{(i,j)}$

where $y_{tj} \in V_{n_2}$ $(1 \le t \le n_2 - 1)$, and

$$
\begin{aligned}
\{\infty_2, y_{1j}\} \in F_0^{(2)},\ \{y_{1j}, y_{2j}\} \in G_j^{(2)},\ \{y_{2j}, y_{3j}\} &\in F_0^{(2)}, \ldots, \\
\{y_{n_2-2,j}, y_{n_2-1,j}\} &\in F_0^{(2)},\ \{y_{n_2-1,j}, \infty_2\} \in G_j^{(2)}.
\end{aligned}
$$

The 1-factor $G_{(i,j)}$ $(1 \le i \le 2r_1, 1 \le j \le 2r_2)$ is represented in Figure 4.1.

In the figures each cell represents a vertex $(\neq \infty)$ in K_n: the cell (x_{si}, y_{tj}) represents the vertex (x_{si}, y_{tj}). The 1-factor $G_{(i,0)}$ $(1 \le i \le 2r_1)$ is represented in Figure 4.2, where we can take any $G_j^{(2)}$ $(1 \le j \le 2r_2)$.

The 1-factor $G_{(0,j)}$ $(1 \le j \le 2r_2)$ is represented in Figure 4.3, where we can take any $G_i^{(1)}$ $(1 \le i \le 2r_1)$.

The 1-factor $F_{(0,0)}$ is represented in Figure 4.4, where we can take any $G_i^{(1)}$ and $G_j^{(2)}$ $(1 \le i \le 2r_1, 1 \le j \le 2r_2)$.

Put

$$
\begin{aligned}
\mathcal{G}_1 &= \{G_{(i,j)} \mid 1 \le i \le 2r_1, 1 \le j \le 2r_2\}, \\
\mathcal{G}_2 &= \{G_{(i,0)} \mid 1 \le i \le 2r_1\}, \\
\mathcal{G}_3 &= \{G_{(0,j)} \mid 1 \le j \le 2r_2\}, \text{ and} \\
\mathcal{G} &= \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3.
\end{aligned}
$$

Claim 4.1

(1) $F_{(0,0)} \cup G_{(i,j)}$ *is a Hamilton cycle* $(1 \le i \le 2r_1, 1 \le j \le 2r_2)$.

(2) $F_{(0,0)} \cup G_{(i,0)}$ *is not a Hamilton cycle* $(1 \le i \le 2r_1)$.

(3) $F_{(0,0)} \cup G_{(0,j)}$ *is not a Hamilton cycle* $(1 \le j \le 2r_2)$.

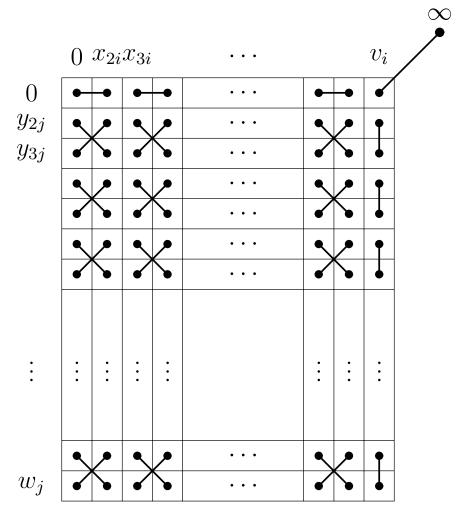

Figure 4.2: $G_{(i,0)}$

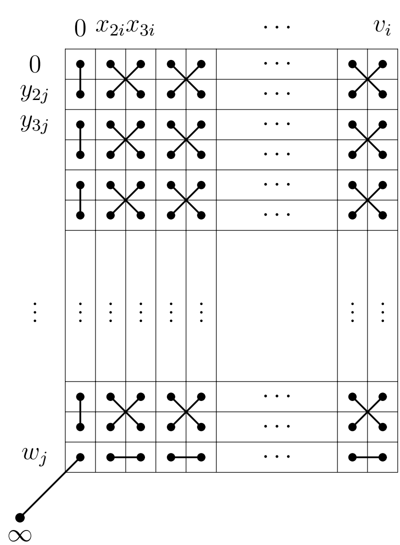

Figure 4.3: $G_{(0,j)}$

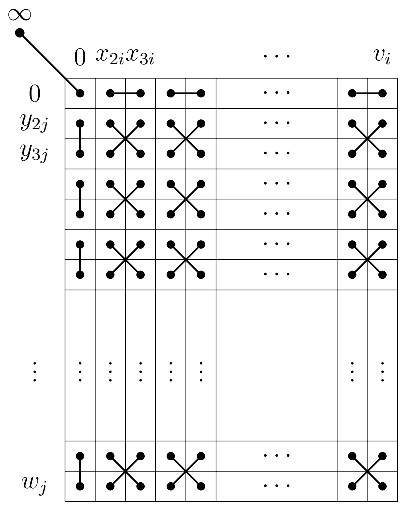

Figure 4.4: $F_{(0,0)}$

Proof.
(1) Combining Figures 4.1 and 4.4, we obtain Figure 4.5.
Then we see $F_{(0,0)} \cup G_{(i,j)}$ is a Hamilton cycle from Proposition 2.4.
(2) Combining Figures 4.2 and 4.4, we obtain Figure 4.6.
Then we see $F_{(0,0)} \cup G_{(i,0)}$ is the union of one cycle of length $m_1 + 1$ and r_2 cycles of length $2m_1$.
(3) Combining Figures 4.3 and 4.4, we obtain Figure 4.7.
Then we see $F_{(0,0)} \cup G_{(0,j)}$ is the union of one cycle of length $m_2 + 1$ and r_1 cycles of length $2m_2$. □

Claim 4.2 *$\mathcal{G}$ satisfies conditions* (2) *and* (3) *of property* (B_n)*, that is,*

$$\cup_{G\in\mathcal{G}} G = E_n \setminus F_{(0,0)},$$

and G $(G \in \mathcal{G})$ has an edge of length $1 = (1,1)$.

Proof. Since $|\cup_{G\in\mathcal{G}} G| = n(n-2)/2$ and $(\cup_{G\in\mathcal{G}} G) \cap F_{(0,0)} = \emptyset$, we have $\cup_{G\in\mathcal{G}} G = E_n \setminus F_{(0,0)}$.

From our assumption, there exists an edge $\{a, b\}$ of length 1 in $G_i^{(1)}$ and there exists an edge $\{c, d\}$ of length 1 in $G_j^{(2)}$. The edges $\{(a, c), (b, d)\}$ and $\{(a, d), (b, c)\}$ are in $G_{(i,j)}$ and their lengths are $1 = (1, 1)$ and $(1, -1)$. So, there exists an edge of length 1 in $G_{(i,j)}$.

As there exists an edge of length 1 in F_0, proofs about $G_{(i,0)}$ and $G_{(0,j)}$ are similar. □

For any $G \in \mathcal{G}_2 \cup \mathcal{G}_3$, $F_{(0,0)} \cup G$ is not a Hamilton cycle from Claim 4.1, so we will

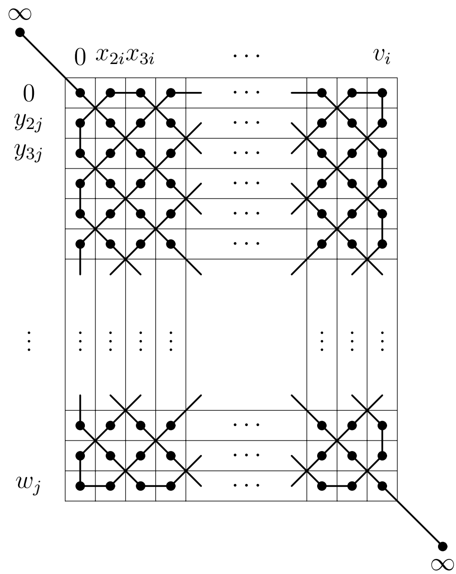

Figure 4.5: $F_{(0,0)} \cup G_{(i,j)}$

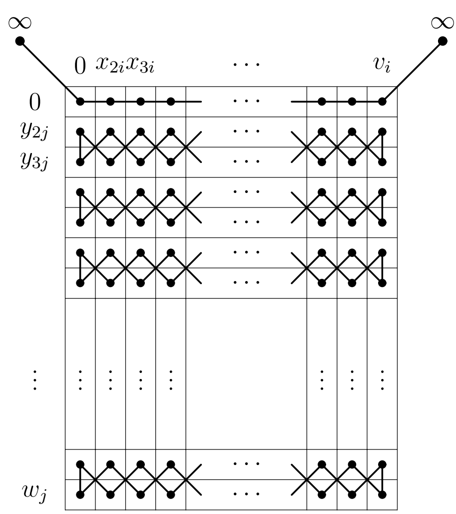

Figure 4.6: $F_{(0,0)} \cup G_{(i,0)}$

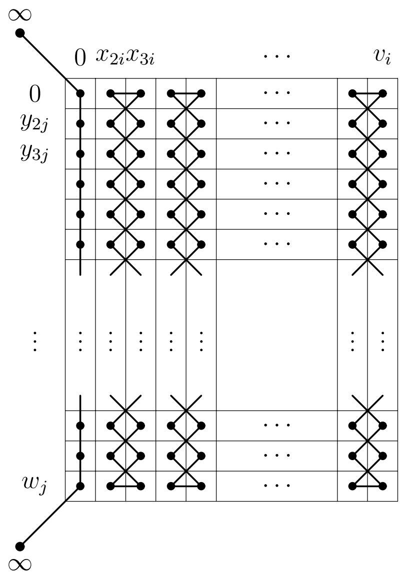

Figure 4.7: $F_{(0,0)} \cup G_{(0,j)}$

exchange edges of 1-factors in $\mathcal{G}_2 \cup \mathcal{G}_3$ and 1-factors in $\mathcal{G}_1$.

Let $G_{(i,0)} \in \mathcal{G}_2$. Consider the union of $G_{(i,0)}$ and $G_{(i,j)}$ $(1 \leq i \leq 2r_1, 1 \leq j \leq 2r_2)$ (Figure 4.8).

It contains r_1 $2m_2$-cycles and one (m_2+1)-cycle. Let C_1 be the (m_2+1)-cycle. Exchange their edges via C_1. Then we obtain (Figures 4.9, 4.10)

$$G_{(i,0)(i,j)} = (G_{(i,0)} \setminus C_1) \cup (G_{(i,j)} \cap C_1); \text{ and}$$

$$G^*_{(i,j)(i,0)} = (G_{(i,j)} \setminus C_1) \cup (G_{(i,0)} \cap C_1).$$

Claim 4.3

(1) $F_{(0,0)} \cup G_{(i,0)(i,j)}$ *is a Hamilton cycle* $(1 \leq i \leq 2r_1, 1 \leq j \leq 2r_2)$.

(2) $F_{(0,0)} \cup G^*_{(i,j)(i,0)}$ *is a Hamilton cycle* $(1 \leq i \leq 2r_1, 1 \leq j \leq 2r_2)$.

(3) *Both* $G_{(i,0)(i,j)}$ *and* $G^*_{(i,j)(i,0)}$ *have an edge of length* 1 $(1 \leq i \leq 2r_1, 1 \leq j \leq 2r_2)$.

Proof.

(1) $F_{(0,0)} \cup G_{(i,0)(i,j)}$ is shown in Figure 4.11, so (1) is trivial.

(2) $F_{(0,0)} \cup G^*_{(i,j)(i,0)}$ is shown in Figure 4.12. We have $(m_1-1, m_2) = 1$ from minimality of p_1. So, (2) holds from Proposition 2.4.

(3) Both $G_{(i,0)}$ and $G_{(i,j)}$ have an edge of length 1 from Claim 4.2. The cycle C_1 doesn't have edges of length 1 because the length of any edge in C_1 is of type $(0, a)$ or ∞. So, after the exchange of edges, both $G_{(i,0)(i,j)}$ and $G^*_{(i,j)(i,0)}$ still have an edge of length 1. □

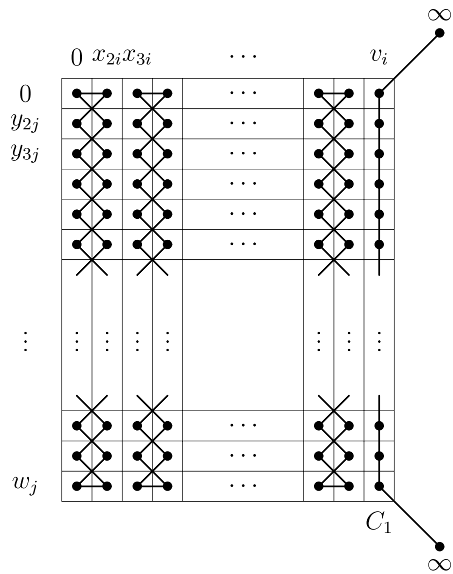

Figure 4.8: $G_{(i,0)} \cup G_{(i,j)}$

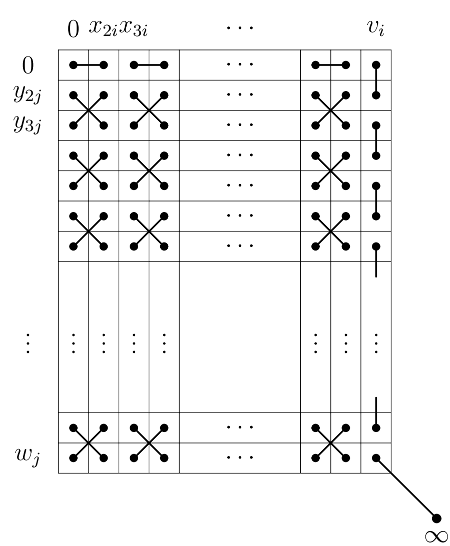

Figure 4.9: $G_{(i,0)(i,j)}$

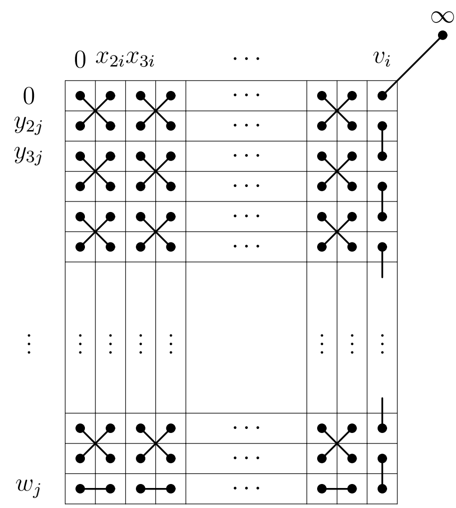

Figure 4.10: $G^*_{(i,j)(i,0)}$

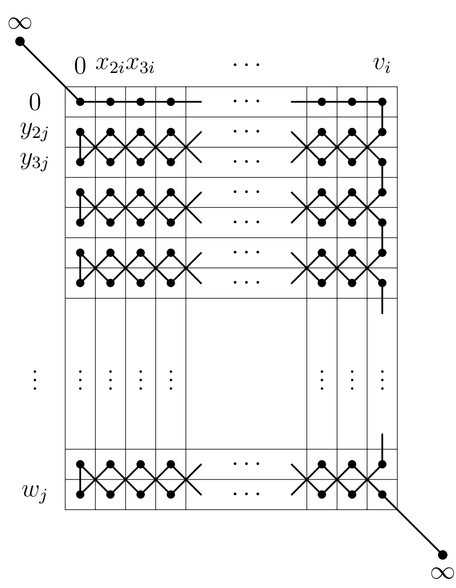

Figure 4.11: $F_{(0,0)} \cup G_{(i,0)(i,j)}$

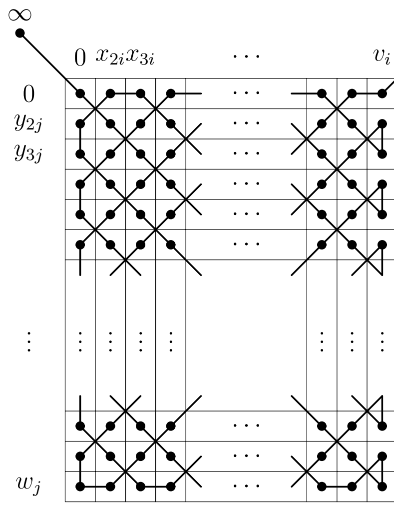

Figure 4.12: $F_{(0,0)} \cup G^*_{(i,j)(i,0)}$

Next, let $G_{(0,j)} \in \mathcal{G}_3$. Consider the union of $G_{(0,j)}$ and $G_{(i,j)}$ $(1 \leq i \leq 2r_1, 1 \leq j \leq 2r_2)$ (Figure 4.13). It contains r_2 $2m_1$-cycles and one (m_1+1)-cycle. Let C_1 be the (m_1+1)-cycle and C_2 the uppermost $2m_1$-cycle.

If $(m_1, m_2-1) = 1$, then exchange edges of $G_{(0,j)}$ and $G_{(i,j)}$ via C_1. If $(m_1, m_2-1) \neq 1$, then exchange edges of $G_{(0,j)}$ and $G_{(i,j)}$ via C_2. Then we obtain (Figures 4.14, 4.15, 4.16, 4.17)

$$G_{(0,j)(i,j)} = (G_{(0,j)} \setminus C) \cup (G_{(i,j)} \cap C); \text{ and}$$

$$G^*_{(i,j)(0,j)} = (G_{(i,j)} \setminus C) \cup (G_{(0,j)} \cap C),$$

where $C = C_1$ if $(m_1, m_2-1) = 1$; $C = C_2$ if $(m_1, m_2-1) \neq 1$.

Claim 4.4
(1) *$F_{(0,0)} \cup G_{(0,j)(i,j)}$ is a Hamilton cycle* $(1 \leq i \leq 2r_1, 1 \leq j \leq 2r_2)$.
(2) *$F_{(0,0)} \cup G^*_{(i,j)(0,j)}$ is a Hamilton cycle* $(1 \leq i \leq 2r_1, 1 \leq j \leq 2r_2)$.
(3) *Both $G_{(0,j)(i,j)}$ and $G^*_{(i,j)(0,j)}$ have an edge of length* 1 $(1 \leq i \leq 2r_1, 1 \leq j \leq 2r_2)$.

Proof. If $(m_1, m_2-1) = 1$, then we exchange edges via C_1. In this case, proofs of (1), (2), (3) are similar to the proof of Claim 4.3. So, we will omit them.

Assume $(m_1, m_2-1) \neq 1$. Then we have $(m_1, m_2-2) = 1$ because m_1 is prime. Since $(m_1, 2) = 1$, (1) holds from Proposition 2.4. Since $(m_1, m_2-2) = 1$, (2) holds from Proposition 2.4.

Next we will prove (3). Both $G_{(0,j)}$ and $G_{(i,j)}$ have an edge of length 1 from Claim 4.2. If C_2 has no edges of length 1, $G_{(0,j)(i,j)}$ and $G^*_{(i,j)(0,j)}$ still have an edge of length 1 trivially.

Assume $G_{(0,j)} \cap C_2$ has an edge of length 1. Let $\{(a,0),(b,c)\}$ be the edge in

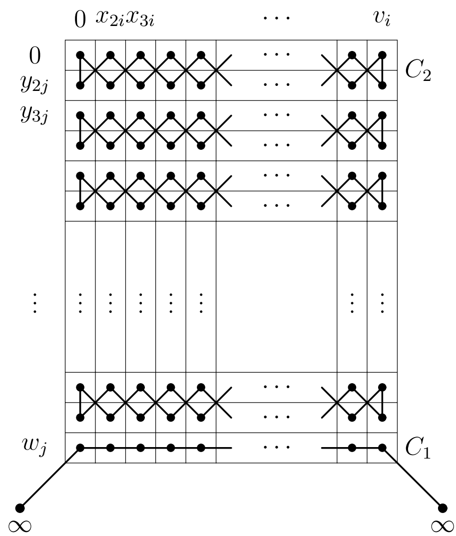

Figure 4.13: $G_{(0,j)} \cup G_{(i,j)}$

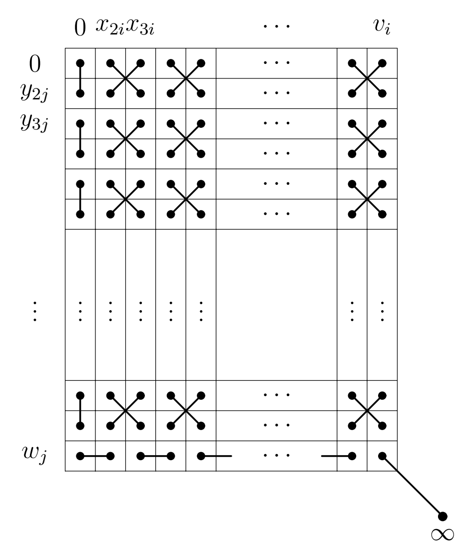

Figure 4.14: $G_{(0,j)(i,j)}$ (the case $(m_1, m_2 - 1) = 1$)

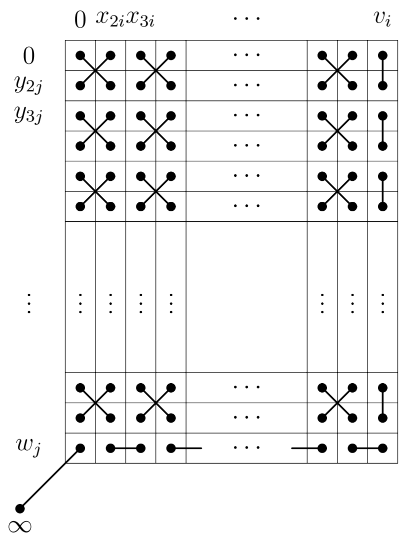

Figure 4.15: $G^*_{(i,j)(0,j)}$ (the case $(m_1, m_2 - 1) = 1$)

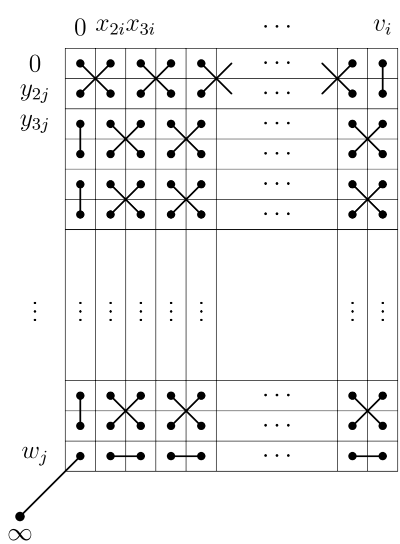

Figure 4.16: $G_{(0,j)(i,j)}$ (the case $(m_1, m_2 - 1) \neq 1$)

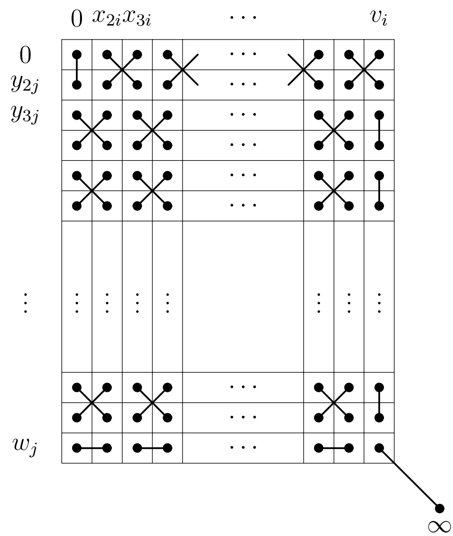

Figure 4.17: $G^*_{(i,j)(0,j)}$ (the case $(m_1, m_2 - 1) \neq 1$)

$G_{(0,j)} \cap C_2$ of length $1 = (1,1)$. Then $(b-a, c)$ or $(a-b, -c)$ is $(1,1)$. There exists an edge $\{e,f\} \in G_i^{(1)}$ of length 1. Then the edges $\{(e,0),(f,c)\}$ and $\{(f,0),(e,c)\}$ belong to $G_{(i,j)} \cap C_2$. One of these edges is of length $1 = (1,1)$. (The other edge is of length $(1,-1)$.) Therefore, after exchanging edges, both $G_{(0,j)(i,j)}$ and $G^*_{(i,j)(0,j)}$ have an edge of length 1.

Assume $G_{(i,j)} \cap C_2$ has an edge of length 1. Then $G_{(0,j)} \cap C_2$ has an edge of length 1.

Therefore we have completed the proof. □

Now we specify 1-factors $G_{(i,j)} \in \mathcal{G}_1$ for exchanging edges of $G_{(i,0)} \in \mathcal{G}_2$ and $G_{(0,j)} \in \mathcal{G}_3$. For $G_{(i,0)} \in \mathcal{G}_2$, we exchange edges of $G_{(i,0)}$ and $G_{(i,-1)}$ when $1 \leq i \leq r_1$; $G_{(i,0)}$ and $G_{(i,1)}$ when $r_1 + 1 \leq i \leq 2r_1$. For $G_{(0,j)} \in \mathcal{G}_3$, we exchange edges of $G_{(0,j)}$ and $G_{(1,j)}$ when $1 \leq j \leq r_2$; $G_{(0,j)}$ and $G_{(-1,j)}$ when $r_2 + 1 \leq j \leq 2r_2$. Put

$$\begin{aligned}
\mathcal{G}'_1 = \ & \{G^*_{(i,-1)(i,0)} \mid 1{\leq}i{\leq}r_1\} \cup \{G^*_{(i,1)(i,0)} \mid r_1{+}1{\leq}i{\leq}2r_1\} \\
& \cup\{G^*_{(1,j)(0,j)} \mid 1{\leq}j{\leq}r_2\} \cup \{G^*_{(-1,j)(0,j)} \mid r_2{+}1{\leq}j{\leq}2r_2\} \\
& \cup\ (\mathcal{G}_1 \setminus \{G_{(i,-1)} \mid 1{\leq}i{\leq}r_1\} \setminus \{G_{(i,1)} \mid r_1{+}1{\leq}i{\leq}2r_1\} \\
& \qquad \setminus\{G_{(1,j)} \mid 1{\leq}j{\leq}r_2\} \setminus \{G_{(-1,j)} \mid r_2{+}1{\leq}j{\leq}2r_2\}), \\
\mathcal{G}'_2 = \ & \{G_{(i,0)(i,-1)} \mid 1{\leq}i{\leq}r_1\} \cup \{G_{(i,0)(i,1)} \mid r_1{+}1{\leq}i{\leq}2r_1\}, \\
\mathcal{G}'_3 = \ & \{G_{(0,j)(1,j)} \mid 1{\leq}j{\leq}r_2\} \cup \{G_{(0,j)(-1,j)} \mid r_2{+}1{\leq}j{\leq}2r_2\}, \text{ and} \\
\mathcal{G}' = \ & \mathcal{G}'_1 \cup \mathcal{G}'_2 \cup \mathcal{G}'_3.
\end{aligned}$$

Claim 4.5 *The 1-factors $G \in \mathcal{G}'$ satisfy (1), (2), (3) of property (B_n), that is,*

(1) *$F_{(0,0)} \cup G$ is a Hamilton cycle in K_n $(G \in \mathcal{G}')$,*

(2) $\cup_{G\in\mathcal{G}'}G = E_n \setminus F_{(0,0)}$,
(3) *G has an edge of length* 1 ($G \in \mathcal{G}'$).

Proof. Condition (1) holds from Claims 4.1, 4.3 and 4.4. Since $\mathcal{G}'$ is obtained by exchanging edges in $\mathcal{G}$, we have $\cup_{G\in\mathcal{G}}G = \cup_{G\in\mathcal{G}'}G$. So (2) holds from Claim 4.2. Condition (3) holds from Claims 4.2, 4.3 and 4.4. □

Hence K_n satisfies property (B_n). This completes the proof of Proposition 4.1. □

Therefore we complete the proof of Theorem 1.1.

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