Regular vertex diameter critical graphs

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Abstract

A graph is called *vertex diameter critical* if its diameter increases when any vertex is removed. Regular vertex diameter critical graphs of every valency $k \geq 2$ and diameter $d \geq 2$ exist, raising the question of identifying the smallest such graphs. We describe an infinite family of k-regular vertex diameter critical graphs of diameter d with at most $kd + (2k - 3)$ vertices. This improves the previously known upper bound for all odd valencies k.

1 Introduction

Let X be a graph with no loops or multiple edges (for general graph-theoretic definitions and background see Godsil and Royle [3] or Diestel [2]). If u and v are vertices of X, then the *distance* $d(u, v)$ from u to v is the length of the shortest path from u to v. The *diameter* $\mathrm{diam}(X)$ of a graph X is the maximum distance between any pair of vertices of X.

If a vertex v is deleted from a graph X, then the diameter of the resulting graph $X - v$ can be less than, equal to, or greater than the diameter of X. We will call a graph *vertex diameter critical* if for all vertices $v \in V(X)$, we have

$$\mathrm{diam}(X - v) > \mathrm{diam}(X).$$

Caccetta and El-Batanouny [1] recently proved that *regular* vertex diameter critical graphs exist for all valencies $k \geq 2$ and diameters $d \geq 2$, thereby proving an old conjecture of Plesnik [4]. They also addressed the question of determining the function $f(k, d)$ — the least number of vertices for which there is a k-regular vertex diameter critical graph of diameter d. It is straightforward to see that $f(2, 2) = 5$, and $f(2, d) = 2d$ for $d \geq 3$, and Caccetta and El-Batanouny [1] proved that

$$f(k, 2) = \begin{cases} 2k + 1, & \text{if } k \text{ is even;} \\ 2k + 2, & \text{if } k \text{ is odd.} \end{cases}$$

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For $k \geq 3$ and $d \geq 3$, they conjectured that

$$f(k,d) = \begin{cases} 4d-2, & \text{if } k=3; \\ kd+1, & \text{if } k \text{ is even;} \\ (k+1)d, & \text{otherwise.} \end{cases}$$

Except when the valency k satisfies $k \equiv 1 \bmod 4$, they presented graphs of these sizes, providing constructive upper bounds on $f(k,d)$.

In this paper, for all $k \geq 3$ and $d \geq 4$, we construct at least one k-regular vertex diameter critical graph of diameter d with $kd + (k-1)$ vertices when d is even, and $kd + (2k-3)$ vertices when d is odd. This fills the gap for odd valencies congruent to 1 mod 4, and for all sufficiently large d represents an improved bound for the remaining odd valencies.

This paper concentrates first on the cubic (that is, 3-regular) case, as this presents all the essential features of the construction in an accessible manner. Later sections present the straightforward generalization from the cubic case to higher valencies, and the final section summarizes the consequences of these constructions for the bounds on $f(k,d)$.

2 A family of cubic graphs

For $n \geq 3$, let $L(n)$ denote a graph on $2n-2$ vertices that is constructed in the following manner: Take two vertex-disjoint paths

$$x_1 \sim x_2 \sim \cdots \sim x_{n-2} \qquad y_1 \sim y_2 \sim \cdots \sim y_{n-2},$$

and join each vertex x_i to y_i. Then add a vertex t which is adjacent to x_1 and y_1, and a vertex h which is adjacent to x_{n-2} and y_{n-2}. (Figures 1 and 2 will immediately make this definition clear.) All vertices have degree three except for h and t, which we will call the *head* and *tail* of $L(n)$, respectively.

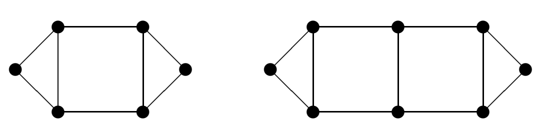

Figure 1: The graphs $L(4)$ and $L(5)$

We can form a cubic graph from these configurations by joining an arbitrary sequence of them in a chain, using an edge from the head of one configuration to the tail of the next in the sequence, and joining the head of the final configuration to the tail of the first one. Figure 3 shows the cubic graph that results when we use $L(4)$, $L(4)$ and $L(5)$ in this construction. We denote this graph by $\mathcal{L}(4,4,5)$, and in general use $\mathcal{L}(n_1, n_2, \ldots, n_s)$ for the graph obtained by joining $L(n_1)$, $L(n_2)$, $\ldots$, $L(n_s)$ in a cycle in an analogous fashion. We will call the induced subgraphs

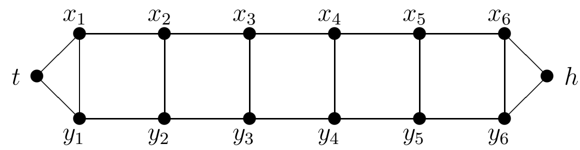

Figure 2: The graph $L(8)$

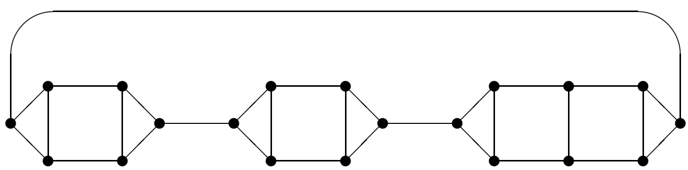

Figure 3: The cubic graph $\mathcal{L}(4,4,5)$

$L(n_1)$, $L(n_2)$, $\cdots$, $L(n_s)$ the *constituents* of $\mathcal{L}(n_1, n_2, \ldots, n_s)$. We need to distinguish the edges connecting the vertices x_i to y_i in the constituents, and so we call these *vertical*. All the remaining edges are called *horizontal*, even the ones incident with the head or tail vertices (which are not actually *drawn* horizontally in the figures).

First we will prove that $\mathcal{L}(4,4,5)$ is a vertex diameter critical graph. Although this is a consequence of our main theorem, it is useful to have a concrete demonstration of the principles underlying the general proof.

2.1 THEOREM. *The graph $X = \mathcal{L}(4,4,5)$ is a vertex diameter critical cubic graph with diameter six and 20 vertices.*

PROOF. First we define a labelling

$$\varphi : V(X) \mapsto \{0, 1, \ldots, 12\}$$

as follows: label the left-most vertex (in Figure 3) with 0, and label every other vertex with i if the shortest left-to-right path from 0 to that vertex has length i. (Glancing at Figure 4 is probably easier than unpacking this description.)

If two vertices, say u and v, are in the same constituent, then their distance is at most three. However if they are in different constituents, then the distance between them is given by

$$d(u,v) = \min\{\varphi(v) - \varphi(u), \varphi(u) - \varphi(v)\},$$

where all arithmetic on labels is performed modulo 13. Moreover, a shortest path between vertices in different constituents uses only horizontal edges. Therefore there are pairs of vertices at distance six (for example, 0 and either of the two vertices

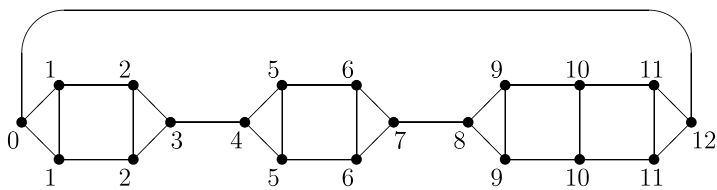

Figure 4: The special labelling of $\mathcal{L}(4,4,5)$

labelled 6), but none at distance seven. We conclude that the diameter of X is exactly six.

Now we must show that X is vertex diameter critical. It is easy to see that if we remove a "head" or "tail" vertex u, then the vertices with labels $\varphi(u) \pm 1$ are now at distance 11. It remains to show that removing any other vertex causes the diameter to increase. So suppose that we remove a vertex u that is neither a head or a tail vertex. Because each constituent $L(n)$ has $n \geq 4$, there is a neighbour v of u that is neither a head or a tail vertex, and such that

$$\varphi(v) = \varphi(u) \pm 1.$$

Without loss of generality, we will assume that $\varphi(v) = \varphi(u) + 1$ (otherwise we simply relabel from right to left). Now choose a vertex w such that

$$\varphi(w) = \varphi(v) - 6.$$

It is necessarily the case that v and w are in different constituents of X, and so every shortest path in X from v to w uses only horizontal edges, starting with the edge vu. Therefore in $X - u$, these paths are no longer present, and the distance from v to w is at least seven. (In fact, it is exactly seven, as we can find a path from v using vertices labelled $\varphi(v) + 1$, $\varphi(v) + 2$, …, $\varphi(v) + 7$.)

We conclude that X is vertex diameter critical with diameter six. ∎

This proof can easily be extended to the general case, under some fairly modest restrictions on the parameters n_1, n_2, …, n_s.

2.2 Theorem. *Let n_1, n_2, …, n_s be a sequence of $s > 1$ integers such that*

(1) *each $n_i \geq 4$,*
(2) *$n = n_1 + n_2 + \cdots + n_s$ is odd, and*
(3) *$n_i \leq (n+3)/2$, for all i.*

Then the graph $X = \mathcal{L}(n_1, n_2, \ldots, n_s)$ is vertex diameter critical with diameter $d = (n-1)/2$.

Proof. It is immediate that we can define a labelling

$$\varphi : V(X) \mapsto \{0, 1, \ldots, n-1\}$$

analogously to that defined in the previous proof. This labelling has the analogous property that the distance between two vertices in different constituents is equal to the difference between their labels modulo n.

Therefore we can find two vertices in different constituents at distance $d = (n-1)/2$, but at no greater distance. Two vertices in the *same* constituent $L(n_i)$ have distance at most $n_i - 2$, and because $n_i \leq (n+3)/2$, it follows that $n_i - 2 \leq d$. Therefore the graph X has diameter d.

It is immediate that removing a head or tail vertex leaves a graph with diameter $n-2$, and so once more it suffices to consider the remaining vertices. Suppose that we remove a vertex u that is neither a head or tail vertex, and consider the diameter of $X - u$. Because each $n_i \geq 4$, there is a vertex v, adjacent to u, not a head or tail vertex, for which we may assume that $\varphi(v) = \varphi(u) + 1$.

If we now pick any vertex w with label

$$\varphi(w) = \varphi(v) - d$$

then w is in a different constituent to v (this follows from the third condition). Because n is odd, every shortest path from v to w in X uses the edge vu, and so the distance between these two vertices in $X - v$ is $d + 1$.

Hence X is vertex diameter critical with diameter $(n-1)/2$. ■

3 Improving the bound

This shows that we can construct cubic vertex diameter critical graphs of any desired diameter $d \geq 4$ simply by selecting a sequence of numbers, each at least 4, that sum to $n = 2d + 1$. For example, each of the graphs

$$\mathcal{L}(4,4,7),\ \mathcal{L}(4,5,6),\ \mathcal{L}(7,8) \text{ and } \mathcal{L}(6,9)$$

is a vertex diameter critical graph with diameter 7.

Each of the constituents $L(n_i)$ has $2n_i - 2$ vertices, and therefore the graph $\mathcal{L}(n_1, n_2, \ldots, n_s)$ has

$$2(n_1 + n_2 + \ldots + n_s) - 2s = 2n - 2s = 4d + 2 - 2s$$

vertices.

Therefore, for each fixed diameter d, the graphs in this family with the fewest vertices are those with the *greatest* number of constituents.

3.3 Theorem. *If $d \geq 4$ is even, then the graph*

$$\mathcal{L}(4, 4, \ldots, 4, 5)$$

with $d/2-1$ constituents equal to $L(4)$, and one constituent equal to $L(5)$, is a cubic vertex diameter critical graph with diameter d and $3d+2$ vertices.

If $d \geq 5$ is odd, then the graph

$$\mathcal{L}(4, 4, \ldots, 4, 7)$$

with $(d-1)/2-1$ constituents equal to $L(4)$, and one constituent equal to $L(7)$, is a cubic vertex diameter critical graph with diameter d and $3d+3$ vertices.

Proof. Follows directly from the main theorem, and simple arithmetic. ■

When d is even, there is a unique graph in this family with $3d+2$ vertices. When d is odd, this is not the case, because the graphs $\mathcal{L}(4, 4, \ldots, 4, 5, 6)$, $\mathcal{L}(4, 4, \ldots, 5, 4, 6)$ (and so on), also have $3d+3$ vertices.

A computer search has been performed for all vertex diameter critical cubic graphs on up to 24 vertices. The results are summarized in Table 1.

$n\backslash d$	2	3	4	5	6	7	Total
10	1	1					2
12		10					10
14		26	3				29
16		14	18				32
18		1	398	6			405
20		1	7101	26	1		7129
22			81459	509	16		81984
24			285396	12918	84	5	298403

Table 1: Numbers of cubic vertex diameter critical graphs

This table shows that $f(3,5)$, $f(3,6)$ and $f(3,7)$ are all realized by the graphs found in this paper. This leads naturally to the following conjecture:

3.4 Conjecture. *The smallest number of vertices $f(3,d)$, for which there exists a cubic vertex diameter critical graph of diameter $d \geq 3$ is given by*

$$f(3,d) = \begin{cases} 10, & d = 3; \\ 3d+2, & d \text{ even}; \\ 3d+3, & d \text{ odd}. \end{cases}$$

We further conjecture that this bound is realized uniquely by $\mathcal{L}(4, 4, \ldots, 4, 5)$ for even diameter $d \geq 6$; this has been confirmed for $d = 6$ and $d = 8$.

4 Higher Valencies

The construction of the preceding sections can easily be generalized to produce vertex diameter critical graphs of any valency k. Let $L_k(n)$ be the graph obtained from $L(n)$ by replacing each vertical edge — viewed as a complete graph on 2 vertices — with a $(k-1)$-clique. Two such $(k-1)$-cliques are joined by a matching if and only if the corresponding vertical edges in $L(n)$ were also joined by a matching. Once again, this definition is considerably clarified by a diagram, given as Figure 5.

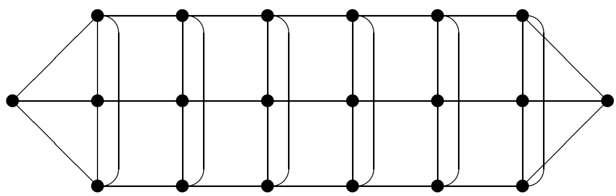

Figure 5: The graph $L_4(8)$

Now define the graph $\mathcal{L}_k(n_1, n_2, \ldots, n_s)$ to be the k-regular graph obtained by joining the constituents $L_k(n_1)$, $L_k(n_2)$, …, $L_k(n_s)$ "head-to-tail" in a circular chain. (So $\mathcal{L}_3(n_1, n_2, \ldots, n_s)$ is just the original cubic graph $\mathcal{L}(n_1, n_2, \ldots, n_s)$.)

We quickly obtain a result analogous to the theorem for cubic graphs.

4.5 Theorem. *Let n_1, n_2, …, n_s be a sequence of $s > 1$ integers such that*

(1) *each $n_i \geq 4$,*
(2) *$n = n_1 + n_2 + \cdots + n_s$ is odd, and*
(3) *$n_i \leq (n+3)/2$, for all i.*

Then $\mathcal{L}_k(n_1, n_2, \ldots, n_s)$ is a k-regular vertex diameter critical graph with diameter $d = (n-1)/2$.

Proof. The proof is identical to the proof of the theorem for cubic graphs. ■

For every even diameter $d \geq 4$, the graph

$$\mathcal{L}_k(4, 4, \ldots, 4, 5)$$

with $d/2 - 1$ constituents equal to $L_k(4)$ and one constituent equal to $L_k(5)$ is the unique smallest k-regular member of this family with diameter d. Each component $L_k(4)$ has $2(k-1)+2$ vertices, while $L_k(5)$ has $3(k-1)+2$ vertices. Hence this graph has

$$(d/2 - 1)(2(k-1)+2) + 3(k-1) + 2 = kd + (k-1)$$

vertices.

For odd diameter $d \geq 5$, the graph

$$\mathcal{L}_k(4, 4, \ldots, 4, 7)$$

with $(d-1)/2-1$ constituents equal to $L_k(4)$ and one constituent equal to $L_k(7)$ is one of the smallest k-regular members of this family with diameter d. Each component $L_k(4)$ has $2(k-1)+2$ vertices, while $L_k(7)$ has $5(k-1)+2$ vertices. Hence this graph has

$$((d-1)/2-1)(2(k-1)+2) + 5(k-1) + 2 = kd + (2k-3)$$

vertices.

5 Conclusion

For even valencies, the graphs constructed in this paper are always larger than the previously known smallest graphs, but for odd valencies the situation is quite different.

When $k \equiv 1 \bmod 4$, they are the smallest currently known family of k-regular vertex diameter critical graphs, and so provide the entire upper bound for $f(k,d)$. For $k \equiv 3 \bmod 4$, the graphs found by Cacetta and El-Batanouny [1] are smaller for low diameters, but for each fixed k, there is a threshold for the diameter above which the graphs in the $\mathcal{L}_k$ family are smaller. More precisely,

$$\begin{aligned} (k+1)d \leq kd + (k-1) \quad &\text{if and only if} \quad d \leq k-1, \\ (k+1)d \leq kd + (2k-3) \quad &\text{if and only if} \quad d \leq 2k-3. \end{aligned}$$

Combining all of this information, we end up with the following somewhat unwieldy bound on the function $f(k,d)$.

5.6 Theorem. *For valency $k \geq 3$ and diameter $d \geq 4$, we have the following bound:*

$$f(k,d) \leq \begin{cases} 14, & \text{if } k = 3 \text{ and } d = 4; \\ kd+1, & \text{if } k \text{ is even;} \\ kd+(k-1), & \text{if } k \equiv 1 \bmod 4 \text{ and } d \text{ is even;} \\ kd+(k-1), & \text{if } k \equiv 3 \bmod 4 \text{ and } d > k-1 \text{ is even;} \\ kd+(2k-3), & \text{if } k \equiv 1 \bmod 4 \text{ and } d \text{ is odd;} \\ kd+(2k-3), & \text{if } k \equiv 3 \bmod 4 \text{ and } d > 2k-3 \text{ is odd;} \\ (k+1)d, & \text{otherwise.} \end{cases}$$

The exact values of $f(k,d)$ are known only when $k=2$, $d=2$ or according to the following list:

f(3,3) = 10, f(3,4) = 14, f(3,5) = 18,
f(3,6) = 20, f(3,7) = 24, f(3,8) = 26,
f(4,3) = 13, f(4,4) = 17.

We finish with two open questions regarding this problem that may be relatively amenable to attack.

Firstly, for each fixed valency k, the bound has asymptotic form $kd + c(k)$, where $c(k)$ is a constant depending only on k, and there are some small-diameter exceptions. For cubic graphs, we can perform exhaustive computations for much higher diameters (up to $d = 8$) than for other valencies, and so have greater confidence that we have found the correct values of $c(k)$. This leads to the natural question:

5.7 Question. *Prove that $f(3,d) = 3d + 2$ for even $d \geq 4$ and that $f(3,d) = 3d + 3$ for odd $d \geq 5$.*

Secondly, note that the graphs in this paper all diameter at least four, and that Caccetta and El-Batanouny [1] do not cover valencies $k \equiv 1 \bmod 4$. Therefore, there is a significant gap in our knowledge when the diameter $d = 3$. The diameter two case has been completely solved, producing an exact result, and so one might hope that the techniques used there could somehow be extended. In any case, there is ample motivation to study the following question:

5.8 Question. *Study the situation when the diameter is 3, and either find exact values for $f(k,3)$ or at least provide a good upper bound for all valencies.*

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