Progressions of squares

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Abstract

Some generalizations of arithmetic progressions are: quasi-progressions, combinatorial progressions, semi-progressions, and descending waves. (The definitions are given below.) We study the occurrence of these progressions in the set of squares of integers.

1 Introduction

It is well known that there is no four-term arithmetic progression (AP) consisting of squares. We have not found a really lucid demonstration of this fact (first proved by Fermat), but one can work through the proof in Chapter 4 of [3]. However, three-term arithmetic progressions occur in abundance among the squares: take any Pythagorean triple, $a^2 + b^2 = c^2$; then $(b-a)^2$, c^2, $(b+a)^2$ is clearly a 3-term AP with common difference $2ab$. It is also easy to show that every 3-term AP of squares has this form.

In [1] and [2] several generalizations of arithmetic progressions have been introduced and their properties investigated. For instance, since $\frac{(n+1)^2}{n^2} \to 1$, Corollary 4, page 94 of [1] shows that the set of squares contains arbitrarily long *descending waves*. A

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descending wave is a set $\{a_1 < a_2 < \cdots < a_n\}$ such that $a_{j+1} - a_j \geq a_{j+2} - a_{j+1}$, $1 \leq j \leq n-2$.

Thus the problems concerning the existence of long progressions in the set of squares is completely solved for APs and descending waves.

For other types of progressions studied in the above mentioned papers (quasi-progressions (QP), combinatorial progressions (CP) and semi-progressions (SP)) very little is known about the existence or non-existence of long progressions of these types among the squares.

In this note we relate what we have found regarding APs, QPs, CPs and SPs occurring in the set of squares. We give the definitions as we go along, and we use some notation consistent with [1] and [2].

2 Arithmetic Progressions and Combinatorial Progressions

In Theorem 1 below, an n-CP$\left(\frac{n-1}{2}\right)$ is a set $\{b_1, b_2, \cdots, b_n\}$ such that $|\{b_2 - b_1, b_3 - b_2, \cdots, b_n - b_{n-1}\}| \leq \frac{n-1}{2}$. Consider the sequence $\{a_n\} = \{1, 5, 7, 13, 17, 25, \cdots\}$ where a_n is defined by

$$a_n = \left\{ \begin{array}{l} \frac{(n+1)^2-2}{2}, \text{ if } n \text{ is odd} \\ \frac{(n+1)^2+1}{2}, \text{ if } n \text{ is even} \end{array} \right\}.$$

A simple calculation shows that, if n is odd, then a_n^2, a_{n+1}^2, a_{n+2}^2 is a 3-term AP of squares with common difference $(n+1)(n+2)(n+3)$. Using this we get the following result concerning "combinatorial progressions."

Theorem 1. *For each odd $n \geq 1$ there exists an n-CP$\left(\frac{n-1}{2}\right)$ among the squares.*

Proof. Using the first n terms of the sequence $\{a_n\} = \{1, 5, 7, 13, 17, 25, \cdots\}$ defined above, the sequence of the $n-1$ differences of $a_1^2, a_2^2, a_3^2, \cdots, a_n^2$ is $24, 24, 120, 120, 336, 336, ..., N, N$, where $N = (n-1)n(n+1)$. Here, the number of distinct differences is clearly $\frac{n-1}{2}$.

Hence, the set of squares contains arbitrarily long progressions, P, with the property

$$\frac{\text{cardinality of the difference set of } P}{\text{cardinality of } P} < \frac{1}{2}.$$

We do not know whether or not the value $\frac{1}{2}$ in this statement can be improved. Also, this result can be compared with that of Theorem 3 below.

Another proof of Theorem 1 can be obtained from the sequence $\{g_n\}$ formed as follows: We start with $g_1 = 1$, $g_2 = a^2$, $g_3 = b^2$, where 1, a^2, b^2 is a 3-term AP. (The smallest such 3-term AP is 1, 25, 49. The most general 3-term AP of this form is 1, a^2, b^2, where $\frac{b}{a}$ is any even convergent of the simple continued fraction of $\sqrt{2}$; see below.) Then we define $g_i = b^{i-1}$ if i is odd and $g_i = a^2 b^{i-2}$ if i is even. The sequence $\{g_n\}$ has properties similar to those of the sequence $\{a_n\}$.

In passing we note two interesting facts regarding APs in the squares.

The first is that the even terms of the sequence $\{a_n\}$ above are exactly the hypotenuses of all the Pythagorean triples of the form $A^2 + B^2 = (B+1)^2$.

Proof. This equation holds if and only if $B + 1 = (A^2 + 1)/2$. Thus A must be odd, say $A = 2k + 1$, and so $B + 1 = a_{2k}$.

The second fact is this: 1, a^2, b^2 (where $a > 0$, $b > 0$) is a 3-term AP if and only if $\frac{b}{a}$ is an even convergent of the simple continued fraction of $\sqrt{2}$.

Proof. If 1, a^2, b^2 is a 3-term AP, then $1 + b^2 = 2a^2$, which gives $2 - \frac{b^2}{a^2} = \frac{1}{a^2}$, or

$$(\sqrt{2} - \frac{b}{a})(\sqrt{2} + \frac{b}{a}) = \frac{1}{a^2},$$

$$0 < \sqrt{2} - \frac{b}{a} = \frac{1}{(\sqrt{2} + \frac{b}{a})a^2} < \frac{1}{2a^2}$$

which yields the result. (See [4].) One can also prove this starting with the equation $b^2 - 2a^2 = -1$. For the converse, one easily shows by induction on n that if $\frac{p_{2n}}{q_{2n}}$ is the $(2n)$th convergent of the simple continued fraction of $\sqrt{2}$, then 1, q_{2n}^2, p_{2n}^2 is a 3-term AP.

3 Quasi-progressions

We now turn our attention to quasi-progressions of squares. While no 4-AP of squares exists, we can nevertheless construct infinitely many 4-term *quasi-progressions* of squares, where the sets of consecutive differences each have *diameter* 1. That is, we can construct sequences $a^2 < b^2 < c^2 < d^2$ where $\{b^2 - a^2, c^2 - b^2, d^2 - c^2\} = \{T, T+1\}$ for some T. Such a sequence is called a 4-QP(1).

Theorem 2. *There are infinitely many* 4*-QP*(1)*s among the squares.*

Proof. Let (a, b, c) be any Pythagorean triple. Recall that $(b-a)^2, c^2, (b+a)^2$ is a 3-term AP. We note that $(b+a)^2 + 2ab$ is not a square lest we get the 4-AP of squares $(b-a)^2, c^2, (b+a)^2, (b+a)^2 + 2ab$. Let (x, y) be any one of the infinitely many solutions to the Pellian equation $x^2 - [(b+a)^2 + 2ab]y^2 = 1$.

Then $(b-a)^2y^2, c^2y^2, (b+a)^2y^2, x^2$ is a 4-QP(1) since the first two differences are $2aby^2$ and the last difference is $x^2 - (b+a)^2y^2 = [(b+a)^2 + 2ab]y^2 + 1 - (b+a)^2y^2 = 2aby^2 + 1$.

Similar 4-QP(1)'s, where the third difference is one less than the first two differences, may be found in the same way from solutions to the equation $x^2 - [(b+a)^2 +$

$2ab]y^2 = -1$ when they exist. Furthermore, the x^2 may be made the first term of the 4-progression when (x, y) is a solution to $x^2 - [(b-a)^2 - 2ab]y^2 = 1$ or $x^2 - [(b-a)^2 - 2ab]y^2 = -1$ (provided $(b-a)^2 > 2ab$).

The simplest example is $(a, b, c) = (3, 4, 5)$. Then $(b+a)^2 + 2ab = 73$ and a solution to $x^2 - 73y^2 = -1$ is $x = 1068, y = 125$. The 4-QP(1) produced is $125^2, (5 \cdot 125)^2, (7 \cdot 125)^2, 1068^2$ with difference sequence $24 \cdot 125^2, 24 \cdot 125^2, 24 \cdot 125^2 - 1$. Examples with very large numbers are also easy to produce provided the Pellian equation can be solved. In fact, most 4-QP(1)'s in the squares consist of very large numbers.

A 5-QP(1) is a sequence $\{x_1 < x_2 < \cdots < x_5\}$ such that $\{x_2 - x_1, x_3 - x_2, x_4 - x_3, x_5 - x_4\} \subseteq \{T, T+1\}$ for some T. More generally, an n-QP(K) is a sequence $\{x_1 < x_2 < \cdots x_n\}$ such that $\{x_{i+1} - x_i : 1 \leq i \leq n-1\} \subseteq \{T, T+1, \cdots, T+K\}$ for some T.

We do not know whether or not there exists any 5-QP(1) among the squares. In fact, we have not found a 5-QP(5) among the squares. Here is a 5-term progression of squares with small difference set diameter: $1^2, 41^2, 58^2, 71^2, 82^2$. The differences are 1680, 1683, 1677, 1683 so the progression is a 5-QP(6). Another 5-QP(6) is: $10^2, 25^2, 34^2, 41^2, 47^2$. Although these examples are curious, they are not very significant in the present context since it happens that any progression of five consecutive squares is a 5-QP(6).

We offer the following conjecture: for each $K \geq 0$, there is an N such that any n-progression of squares, with $n \geq N$, is not an n-QP(K). (For $K = 0$, we have $N = 4$, but this is all we know.)

4 Semi-progressions

For a given function g, an n-SP(g) is a set $\{b_1, b_2, \cdots, b_n\}$ such that the diameter of the set $\{b_2 - b_1, b_3 - b_2, \cdots, b_n - b_{n-1}\}$ is less than or equal to $g(n)$. If a set X contains n-SP(g)s for arbitrarily large n, we say that X has *property* SP(g)

We remark that, if $g(k)$ is bounded above by a polynomial, property SP(g) is stronger than the property of containing arbitrarily long descending waves. (See [2]).

As was remarked in [2], the set of squares has property SP($2n$). (Just consider the progression consisting of the first n squares. The diameter of the difference sequence is $(n^2 - (n-1)^2) - (2^2 - 1^2) < 2n$.)

Our purpose here is to improve this result by replacing $2n$ with $(3/2)n$.

Theorem 3. *The set of squares has property SP($\frac{3}{2}n$).*

Proof. We wish to prove that there are arbitrarily long progressions of squares $a_1^2 < a_2^2 < \cdots < a_n^2$ such that the diameter D of the difference set, $D = \max\{a_{i+1}^2 - a_i^2\} - \min\{a_{i+1}^2 - a_i^2\}$, does not exceed $\frac{3}{2}n$, that is, $\frac{D}{n} \leq \frac{3}{2}$.

We will refer to a positive integer a as the *base* of the square a^2. Let K be a positive integer and let $A = 2^K$. We construct the following progression, B, of squares: The progression B will be the union of $K+1$ blocks of squares $B_0, B_1, \cdots, B_K$ where

$B_0 = \{(2A)^2\}$,

$B_1 = \{(2A + \frac{A}{2})^2, (2A + 2\frac{A}{2})^2, (2A + 3\frac{A}{2})^2, (2A + 4\frac{A}{2})^2\}$,

$\vdots$

$B_i = \{(2^i A + \frac{A}{2^i})^2, (2^i A + 2\frac{A}{2^i})^2, (2^i A + 3\frac{A}{2^i})^2, \cdots, (2^i A + 4^i \frac{A}{2^i})^2\}$,

$\vdots$

$B_K = \{(2^K A + 1)^2, (2^K A + 2)^2, (2^K A + 3)^2, \cdots, (2^K A + 4^K)^2\}$.

We observe that $|B_i| = 4^i$ for $0 \le i \le K$. Hence the number of terms in B is

$$n = 1 + 4 + 4^2 + \cdots + 4^K = \frac{1}{3}(4^{K+1} - 1).$$

Also note that the base of the last term of B_i is $2^{i+1}A$. Since the bases of the squares, starting from the last term of a given block and continuing through all the terms of the next block, increase by a constant amount ($= \frac{A}{2^i}$ in block B_i), the difference sequence of the squares is increasing for these terms. In order to calculate the diameter of the difference set, we need only check the K largest differences (which occur between the last two terms of the blocks) and the K smallest differences (which occur between the last term of a block and the first term of the succeeding block). Finding the difference between the largest and smallest among these differences will produce our D. The largest difference in block B_i is

$$(2^{i+1}A)^2 - (2^{i+1}A - \frac{A}{2^i})^2 = \frac{A}{2^i}(2^{i+2}A - \frac{A}{2^i}) = 4A^2 - \frac{A^2}{4^i},$$

and the maximum of these occurs when $i = K$ and is equal to $4A^2 - 1 = H$. The difference from the last term of B_i to the first term of B_{i+1} is

$$(2^{i+1}A + \frac{A}{2^{i+1}})^2 - (2^{i+1}A)^2 = \frac{A}{2^{i+1}}(2^{i+2}A + \frac{A}{2^{i+1}}) = 2A^2 + \frac{A^2}{4^{i+1}},$$

and the minimum of these occurs at $i + 1 = K$ with value $2A^2 + 1 = L$. Hence

$$D = H - L = 2A^2 - 2 = 2^{2K+1} - 2.$$

Finally we have

$$\frac{D}{n} = \frac{2^{2K+1} - 2}{\frac{1}{3}(4^{K+1} - 1)} = \frac{3(1 - \frac{1}{2^{2K}})}{2(1 - \frac{1}{2^{2K+2}})} < \frac{3}{2}.$$

We do not know whether or not this theorem is best possible. Perhaps the set of squares possesses property $\mathrm{SP}((1+\varepsilon)n)$ for every $\varepsilon > 0$, or even property $\mathrm{SP}(n)$.

References

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