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Minimal 1-saturating sets in $PG(2, q)$, $q \leq 16$

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Abstract

Minimal 1-saturating sets in the projective plane $PG(2, q)$ are considered. The classification of all the minimal 1-saturating sets in $PG(2, q)$ for $q \leq 8$, the classification of the smallest minimal 1-saturating sets in $PG(2, q)$, $9 \leq q \leq 13$ and the determination of the smallest size of minimal 1-saturating sets in $PG(2, 16)$ are given. These results have been found using a computer-based exhaustive search that exploits projective equivalence properties.

1 Introduction

Let $PG(n, q)$ be the n-dimensional projective space over the Galois field $GF(q)$. For an introduction to such spaces and the geometrical objects therein, see [6]–[9].

Definition 1 *A point set S in the space $PG(n, q)$ is ϱ-saturating if ϱ is the least integer such that for any point $x \in PG(n, q)$ there exist $\varrho + 1$ points in S generating a subspace of $PG(n, q)$ in which x lies.*

Definition 2 [13] *A ϱ-saturating set of l points is called minimal if it does not contain a ϱ-saturating set of $l - 1$ points.*

A q-ary linear code with codimension r has covering radius R if every r-positional q-ary column is equal to a linear combination of R columns of a parity check matrix of this code and R is the smallest value with such property. For an introduction to coverings of vector spaces over finite fields and to the concept of code covering radius, see [1].

The points of a ϱ-saturating set in $PG(n, q)$ can be considered as columns of a parity check matrix of a q-ary linear code with codimension $n + 1$. So, in terms of coding theory, a ϱ-saturating l-set in $PG(n, q)$ corresponds to a parity check matrix of a q-ary linear code with length l, codimension $n+1$, and covering radius $\varrho+1$ ([2], [5],[10]). Such a code is denoted by an $[l, l-(n+1)]_q(\varrho+1)$ code.

Note that a ϱ-saturating set in $PG(n,q)$, $\varrho+1 \leq n$, can generate an infinite family of ϱ-saturating sets in $PG(N,q)$ with $N = n + (\varrho+1)m$, $m = 1, 2, 3, \ldots$; see [1, Chapter 5.4], [2], [3, Example 6] and references therein, where such infinite families are considered as linear codes with covering radius $\varrho+1$.

This paper deals with the minimal 1-saturating sets in $PG(2,q)$. We give the classification of the minimal 1-saturating sets in $PG(2,q)$, $q \leq 8$, the classification of the smallest minimal 1-saturating sets in $PG(2,q)$, $9 \leq q \leq 13$ and we determine the size of smallest minimal 1-saturating sets in $PG(2,16)$. These results have been found using a computer-based exhaustive search that exploits projective equivalence properties among sets of points. Properties of the ϱ-saturating sets in $PG(n,q)$ are presented in [4].

In the projective plane $PG(2,q)$ over the Galois field $GF(q)$, an n-arc is a set of n points no 3 of which are collinear. An n-arc is called complete if it is not contained in an $(n+1)$-arc of the same projective plane. The complete arcs of $PG(2,q)$ are examples of minimal 1-saturating sets, but there are minimal 1-saturating sets that are not complete arcs.

We use the following notations in $PG(2,q)$: $m(2,q,1)$ is the size of the largest minimal 1-saturating sets, $m'(2,q,1)$ is the size of the second largest minimal 1-saturating sets and $l(2,q,1)$ is the size of the smallest minimal 1-saturating sets.

The values of $m(2,q,1)$ and $m'(2,q,1)$ have been determined in [4]. These results and some constructions of minimal 1-saturating sets of such sizes have been reported in Section 2. Section 3 contains the description of the algorithm we used to classify the minimal 1-saturating sets. Section 4 contains the classification of all the minimal 1-saturating sets in $PG(2,q)$ for $q \leq 8$, the classification of the smallest minimal 1-saturating sets in $PG(2,q)$, $9 \leq q \leq 13$ and the determination of the value of $l(2,16,1)$.

2 The values of $m(2,q,1)$, and $m'(2,q,1)$

In this section we recall some theorems from [4] that allow us to determine the values of $m(2,q,1)$ and $m'(2,q,1)$ and give constructions of minimal 1-saturating sets of such sizes. Let $\theta(n,q) = (q^{n+1}-1)/(q-1) = |PG(n,q)|$.

Theorem 1 *In the space $PG(n,q)$, let S_A be a $(\theta(n-1,q)+1)$-set consisting of a whole hyperplane V of $\theta(n-1,q)$ points, plus one point P not belonging to V. The point set S_A is a minimal 1-saturating $(\theta(n-1,q)+1)$-set in the space $PG(n,q)$.*

Remark 1 *Theorem 1 can be considered as an example of using [13, Lemma 10]. This lemma is treated as the "direct sum" construction in covering codes theory [1, Section 3.2].*

Theorem 2 *Any $\theta(n-1,q)+1$ points in the space $PG(n,q)$ are a 1-saturating set.*

Corollary 1 *The greatest cardinality of a minimal* 1*-saturating set in a space* $PG(n,q)$ *is equal to* $\theta(n-1,q)+1$*, i.e.,* $m(n,q,1) = \theta(n-1,q)+1$ *for all* q*.*

Corollary 2 *In the plane* $PG(2,q)$*,* $m(2,q,1) = q+2$ *and a* $(q+2)$*-set containing a whole line* l *of* $q+1$ *points and one point* $P \notin l$ *is a largest minimal* 1*-saturating set.*

Example 1 *For* q *even, in the plane* $PG(2,q)$ *a hyperoval of* $q+2$ *points is another example of a largest minimal* 1*-saturating set.*

Theorem 3 *Let* $l = \{L_1, L_2, \ldots, L_{q+1}\}$ *be a line in the plane* $PG(2,q)$ *consisting of the points* L_i*. Denote by* P *an external point for* l*. Let* T *be a point on the line through the points* L_1 *and* P *and* $P \neq T \neq L_1$*. Let us consider a* $(q+1)$*-set* $S_B = \{L_3, L_4, \ldots, L_{q+1}, P, T\}$*. Then the point set* S_B *is a minimal* 1*-saturating* $(q+1)$*-set in a plane* $PG(2,q)$*,* $q \geq 3$*.*

Corollary 3 *In* $PG(2,q)$*,* $q \geq 3$*,* $m'(2,q,1) = q+1$*.*

Remark 2 *For* q *odd, in the plane* $PG(2,q)$ *an oval of* $q+1$ *points is another example of a minimal* 1*-saturating* $(q+1)$*-set.*

Remark 3 *Since in the plane* $PG(2,q)$ *a* q*-arc is always incomplete* [6]*, the minimal* 1*-saturating sets of size* q *cannot be arcs.*

3 The computer search for the non-equivalent minimal 1-saturating sets

The program that computes the classes of the minimal 1-saturating sets has been written using MAGMA, a system for symbolic computation developed at the University of Sydney.

The program performs a breadth-first search to construct all the non-equivalent minimal 1-saturating sets of size belonging to the interval $[2, M]$. The program maintains two lists: the non-equivalent minimal 1-saturating sets of size k and the non-equivalent sets of size k that are not 1-saturating, $k \in [2, M]$. For $k = 2$ the first list is empty, while the second list contains one set of two points, as all the sets of two points are equivalent.

The non-equivalent sets of size k are obtained by expanding all the non-equivalent sets of points of size $k-1$ that are not 1-saturating; let them be S_i^{k-1}, $i \in I_{k-1}$. Each S_i^{k-1} is expanded in the following way. The orbits of the stabilizer group of S_i^{k-1} are considered. As the sets $S_i^{k-1} \cup \{P\}$ and $S_i^{k-1} \cup \{Q\}$ are equivalent if P and Q belong to the same orbit, it is sufficient to consider just one expansion of size k for each orbit. Let E_j^k, $j \in J_k$ be the sets of size k obtained by extending all the $S_i^{k-1}, i \in I_{k-1}$. The first non-equivalent set of size k is E_1^k. The other non-equivalent sets are obtained by comparing each E_j^k with the non-equivalent E_l^k already selected: if E_j^k is equivalent to an E_l^k already selected it is neglected, otherwise E_j^k is selected as a representative of another class of non-equivalent sets of size k.

To reduce the computational complexity of this phase a pre-classification strategy is adopted. For each E_j^k the stabilizer group G_j^k is computed and the projective equivalence is tested only between E_j^k and the non-equivalent sets already found with stabilizers of the same cardinality as G_j^k. In this way, at the cost of computing the order of the stabilizer group for each E_j^k, the number of computations of projective equivalence between pairs of sets of points is decreased. This is convenient because computing the order of the stabilizer is less expensive than computing whether two sets of points are equivalent and the number of computations of the stabilizers is $|J_k|$, while the number of tests of equivalence is of order $O(|J_k| \times |I_k|)$.

When the non-equivalent sets of size k have been computed they are tested to check if they are 1-saturating and in this case they are tested to check if they are minimal.

4 The non-equivalent minimal 1-saturating sets

This section contains the classification of the minimal 1-saturating sets in $PG(2, q)$ for $q \leq 8$, the classification of the smallest minimal 1-saturating sets in $PG(2, q)$, $9 \leq q \leq 13$ and the value of $l(2, 16, 1)$. The first two theorems deal with the case $q = 2, 3$.

Theorem 4 $m(2, 2, 1) = l(2, 2, 1) = 4$ *and there are two minimal* 1*-saturating sets of size* 4 *up to projective equivalence.*

Proof. In $PG(2, 2)$ there exists only one complete arc [6]. It is the hyperoval and has size 4. Another example of minimal 1-saturating set is given by Theorem 1. In $PG(2, q)$ all the arcs of size 4 are equivalent up to projective equivalence. Also the sets consisting of a line and an external point are equivalent in $PG(2, q)$, therefore the two examples are unique. □

Theorem 5 $l(2, 3, 1) = 4$ *and* $m(2, 3, 1) = 5$*. The minimal* 1*-saturating sets of both sizes are unique up to projective equivalence.*

Proof. In $PG(2, 3)$ the minimum size of a complete arc is four [6], therefore l(2,3,1)= 4. Theorem 1 gives a minimal 1-saturating set of size 5. A set of 5 points consisting of three collinear points and two other points contains a complete arc of size 4, therefore it is not minimal. The two examples are unique as in the previous theorem. □

The other cases have been solved using the program described in the previous section. The following table presents the classification of the minimal 1-saturating l-sets in $PG(2, q)$, $4 \leq q \leq 8$. The asterisk * denotes that the 1-saturating sets of the smallest size are complete arcs, while the subscripts indicate the number of non-equivalent minimal 1-saturating sets.

q	$l(2,q,1)$	Sizes l of minimal 1-saturating sets with $l(2,q,1) < l \leq q$	$m'(2,q,1)$ $= q+1$	$m(2,q,1)$ $= q+2$
3	4_1^*		4_1	5_1
4	5_1		5_1	6_3
5	6_6		6_6	7_1
7	6_3	7_7	8_{31}	9_3
8	6_1^*	$7_2, 8_{60}$	9_{18}	10_5

Sizes of the minimal 1-saturating l-sets in $PG(2,q)$, $3 \leq q \leq 8$

The following tables give the classification in $PG(2,q)$ of the minimal 1-saturating sets for $q = 4, 5$ and of the smallest and the largest minimal 1-saturating sets for $q = 7, 8$. For the complete classification see [12].

In the examples we represent the elements of the Galois fields as follows. If q is prime, the elements are $GF(q) = \{0, 1, \ldots, q-1\}$ and we operate on these modulo q. If q is a power of a prime, we denote $GF(q) = \{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, q-1 = \alpha^{q-2}\}$ where α is a primitive element. This defines multiplication. For addition we use a primitive polynomial generating the field. For example, we can design the table of Zech logarithms. In this work the primitive polynomials are $x^2 + x + 1$ for $q = 4$, $x^3 + x^2 + 1$ for $q = 8$ and $x^2 + x + 2$ for $q = 9$ [11]. All the examples contain the points $(0,0,1)$, $(0,1,0)$, $(1,0,0)$.

In the tables, the column "Group" describes the stabiliser group of the minimal 1-saturating set up to $PGL(3,q)$ if q is prime, up to $P\Gamma L(3,q)$ otherwise. With the symbol G_i we denote a group of order i. For the meaning of the other symbols see [14]. The columns "l_i" contain the number of lines intersecting the minimal 1-saturating set in i points.

Size		Group	l_0	l_1	l_2	l_3	l_5
5	$(1,3,3), (1,2,0)$	D_6	5	8	7	1	
6	$(0,1,2), (1,2,0), (1,0,3)$	G_{48}	2	12	3	4	
6	$(1,2,1), (1,3,3), (1,1,2)$	G_{720}	6		15		
6	$(1,1,0), (1,2,0), (1,3,0)$	G_{360}		15	5		1

The minimal 1-saturating sets in $PG(2,4)$

Size		Group	l_0	l_1	l_2	l_3	l_4	l_6
6	$(0,1,1),(1,1,3),(1,2,1)$	D_4	8	12	9	2		
6	$(1,2,2),(1,1,3),(1,4,1)$	G_{120}	10	6	15			
6	$(1,1,0),(1,1,3),(1,3,4)$	S_3	7	15	6	3		
6	$(1,1,0),(1,1,3),(1,4,1)$	Z_2	8	12	9	2		
6	$(1,1,2),(1,1,3),(1,4,1)$	S_3	9	9	12	1		
6	$(0,1,4),(0,1,1),(1,1,3)$	$Z_2 \times Z_4$	7	14	9		1	
7	$(1,1,0),(1,2,0),$ $(1,3,0),(1,4,0)$	G_{480}		26	6			1

The minimal 1-saturating sets in $PG(2,5)$

Size		Group	l_0	l_1	l_2	l_3	l_4
6	$(1,3,2),(1,5,3),(1,1,5)$	G_{36}	24	18	15		
6	$(1,1,5),(1,4,4),(1,1,5)$	A_4	24	18	15		
6	$(1,5,3),(1,1,5),(1,2,0)$	S_3	23	21	12	1	

The minimal 1-saturating sets in $PG(2,7)$ of smallest size

Size		Group	l_0	l_1	l_2	l_3	l_4	l_8
9	$(105),(125),(115),$ $(110),(116),(120)$	S_4	8	36	6	4	3	
9	$(110),(120),(130),$ $(140),(150),(160)$	G_{2016}		48	8			1
9	$(105),(115),(110),$ $(120),(103),(113)$	Z_3	9	33	9	3	3	

The minimal 1-saturating sets in $PG(2,7)$ of largest size

Size		Group	l_0	l_1	l_2	l_3
6	$(1,7,1),(1,4,7),(1,6,5)$	S_4	34	24	15	

The minimal 1-saturating set in $PG(2,8)$ of smallest size

Size		Group	l_0	l_1	l_2	l_3	l_4	l_5	l_6	l_9
10	$(1,7,0)$, $(1,7,1),(1,7,7)$, $(1,7,4),(1,2,0)$, $(0,1,2),(1,4,0)$	Z_6	12	42	13	4		2		
10	$(0,1,3)$, $(1,7,1),(1,4,5)$, $(1,1,2),(1,1,5)$, $(0,1,2),(1,4,0)$	Z_2	12	43	11	4	2	1		
10	$(1,3,0)$, $(1,7,0),(1,6,0)$, $(1,1,0),(1,2,0)$, $(1,4,0),(1,5,0)$	G_{10584}	63	9						1
10	$(1,7,0)$, $(1,7,1),(1,4,6)$, $(1,1,0),(1,2,0)$, $(0,1,2),(1,4,0)$	Z_3	12	42	12	6			1	
10	$(1,5,4)$, $(1,2,2),(1,7,1)$, $(1,3,5),(1,1,6)$, $(1,6,3),(1,4,7)$	G_{1512}	28		45					

The minimal 1-saturating sets in $PG(2,8)$ of largest size

For $9 \leq q \leq 16$, the complete classification of the minimal 1-saturating sets, using the program of Section 2, would take too long. However we determined the values of $l(2,1,q)$ and also classified the smallest minimal 1-saturating sets in $PG(2,q)$, $9 \leq q \leq 13$. These results are presented in the next theorem.

Theorem 6 *The following hold:*
$l(2,1,9) = 6$ *and only one minimal* 1*-saturating set of size* 6 *exists up to* $P\Gamma L(3,9)$;
$l(2,1,11) = 7$ *and only one minimal* 1*-saturating set of size* 7 *exists up to* $PGL(3,11)$;
$l(2,1,13) = 8$ *and two minimal* 1*-saturating sets of size* 8 *exist up to* $PGL(3,13)$;
$l(2,1,16) = 9$.

The following table describes the smallest minimal 1-saturating sets in $PG(2,q)$, $9 \leq q \leq 13$.

q	Size		Group	l_0	l_1	l_2
9	6	$(1,3,3),(1,8,6),(1,5,8)$	G_{120}	46	30	15
11	7	$(1,7,10),(1,1,1),$ $(1,2,3),(1,10,5)$	$Z_7 \rtimes Z_3$	70	42	21
13	8	$(1,4,10),(1,8,11),$ $(1,12,6),(1,10,3),(1,1,1)$	D_7	99	56	28
13	8	$(1,9,10),(1,2,11),$ $(1,12,6),(1,10,4),(1,1,1)$	S_3	99	56	28

The smallest minimal 1-saturating sets in $PG(2,q)$, $q = 9, 11, 13$

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