AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **29** (2004), Pages 323–336

Labelings of unions of up to four uniform cycles

Diane Donovan

Centre for Discrete Mathematics and Computing
Department of Mathematics
The University of Queensland
Queensland 4072, Australia

Saad I. El-Zanati   Charles Vanden Eynden

4520 Mathematics Department
Illinois State University
Normal, Illinois 61790–4520, U.S.A.

Somporn Sutinuntopas

Department of Mathematics
Faculty of Science
Ramkhamhaeng University
Hua-Mark, Bangkok 10240, Thailand

Abstract

We show that every 2-regular graph consisting of at most four uniform components has a ρ-labeling (or a more restricted labeling). This has an application in the cyclic decomposition of certain complete graphs into the disjoint unions of cycles.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a < b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. Let $V(K_v) = \mathbb{Z}_v$ and let G be a subgraph of K_v. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Let K and G be graphs such that G is a subgraph of K. A *G-decomposition* of K is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that the edge sets of the graphs G_i form a partition of the edge set of K. If K is K_v, a G-decomposition Γ of K is *cyclic* if clicking is a permutation of Γ. If G is a graph and r is a positive integer, rG denotes the vertex disjoint union of r copies of G.

For any graph G, an injective function $h : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [14], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let h be a labeling of G. Let $h(V(G)) = \{h(u) : u \in V(G)\}$. Define a function $\bar{h} : E(G) \to \mathbb{Z}^+$ by $\bar{h}(e) = |h(u) - h(v)|$, where $e = \{u, v\} \in E(G)$. Let $\bar{E}(G) = \{\bar{h}(e) : e \in E(G)\}$. Consider the following conditions:

(a) $h(V(G)) \subseteq [0, 2n]$,

(b) $h(V(G)) \subseteq [0, n]$,

(c) $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n+1-i$,

(d) $\bar{E}(G) = [1, n]$.

If in addition G is bipartite, then there exists a bipartition (A, B) of $V(G)$ (with every edge in G having one endvertex in A and the other in B) such that

(e) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $h(a) < h(b)$,

(f) there exists an integer λ such that $h(a) \leq \lambda$ for all $a \in A$ and $h(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:
(a), (c) is called a *ρ-labeling*;
(a), (d) is called a *σ-labeling*;
(b), (d) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies (e), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (f) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [14].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results by Rosa [14].

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2 *Let G be a graph with n edges that has an α-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

Clearly if G is bipartite, then an α-labeling of G is the most desired labeling. However, there exist numerous classes of bipartite graphs (including some classes of trees) which do not admit α-labelings (see [14]). Hence the need to introduce the variations on the theme of α-labelings. In [6] it was shown that Theorem 2 extends to graphs with ρ^+-labelings.

Theorem 3 *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

Let G be a graph with n edges and Eulerian components and let h be a β-labeling of G. It is well-known (see [14]) that we must have $n \equiv 0$ or $3 \pmod 4$. Moreover, if such a G is bipartite, then $n \equiv 0 \pmod 4$. These conditions hold since for such a G, $\sum_{e \in E(G)} \bar{h}(e) = n(n+1)/2$. This sum must in turn be even, since each vertex is incident with an even number of edges and $\bar{h}(e) = |h(u) - h(v)|$, where u and v are the endvertices of e. Thus we must have $4|n(n+1)$. Clearly, the same will hold if such a G admits a σ-labeling. We shall refer to this restriction as the *parity condition*. There are no such restrictions on $|E(G)|$ if h is a ρ-labeling.

Theorem 4 (Parity Condition) *If a graph G with Eulerian components and n edges has a σ-labeling, then $n \equiv 0$ or $3 \pmod 4$. If such a G is bipartite, then $n \equiv 0 \pmod 4$.*

In [14], Rosa presented α- and β-labelings of C_{4m} and of C_{4m+3}, respectively. It is also known that both C_{4m+1} and C_{4m+2} admit ρ-labelings. It was also shown in [6] that there exists a ρ^+-labeling of C_{4m+2}, for all positive integers m. It can be easily checked that this labeling is actually a ρ^{++}-labeling.

In this manuscript, we will focus on labelings of 2-regular graphs (i.e., the vertex-disjoint union of cycles). If a 2-regular graph G is bipartite, then it is shown in [3] that G necessarily admits a ρ^{++}-labeling. Such a G need not admit an α-labeling, even if the parity condition is satisfied. It is well-known for example that $3C_4$ does not have an α-labeling (see [11]). Similarly, if G is not bipartite, then G need not admit a β-labeling even if the parity condition is satisfied. For example, it is shown in [12] that rC_3 does not admit a β-labeling for all $r > 1$ and rC_5 never admits a β-labeling. It is thus reasonable to focus on labelings that are less restrictive than β-labelings when studying 2-regular graphs.

Here, we shall show that every 2-regular graph consisting of at most four uniform components has a ρ-labeling (or a more restricted labeling). This has an application in the cyclic decomposition of certain complete graphs into the disjoint unions of cycles. Moreover, it provides further evidence in support of a conjecture that every 2-regular graph admits a ρ-labeling.

2 Summary of Some of the Known Results

As stated in the previous section, the following is known for cycles (see [13], [14] and [6]).

Theorem 5 *Let $m \geq 3$ be an integer. Then, C_m admits an α-labeling if $m \equiv 0 \pmod 4$, a ρ-labeling if $m \equiv 1 \pmod 4$, a ρ^{++}-labeling if $m \equiv 2 \pmod 4$, and a β-labeling if $m \equiv 3 \pmod 4$.*

For 2-regular graphs with two components, we have the following from Abrham and Kotzig [2].

Theorem 6 *Let $m \geq 3$ and $n \geq 3$ be integers. Then the graph $C_m \cup C_n$ has a β-labeling if and only if $m+n \equiv 0$ or $3 \pmod 4$. Moreover, $C_m \cup C_n$ has an α-labeling if and only if both m and n are even and $m+n \equiv 0 \pmod 4$.*

Thus $2C_m$ has an α-labeling if $m \geq 4$ is even. In the next section, we show that $2C_m$ admits a ρ-labeling if $m \geq 3$ is odd.

For 2-regular graphs with more than two components, the following is known. In [11], Kotzig shows that if $r > 1$, then rC_3 does not admit a β-labeling. Similarly, he shows that rC_5 does not admit a β-labeling for any r. In [12], Kotzig shows that $3C_{4k+1}$ admits a β-labeling for all $k \geq 2$. In [5], it is shown that rC_3 admits a ρ-labeling for all $r \geq 1$. In [8], Eshghi shows that $C_{2m} \cup C_{2n} \cup C_{2k}$ has an α-labeling for all m, n, and $k \geq 2$ with $m+n+k \equiv 0 \pmod 2$ except when $m = n = k = 2$. Thus $3C_{4m}$ has an α-labeling for all $m > 1$. In [1], Abrham and Kotzig show that rC_4 has an α-labeling for all positive integers $r \neq 3$. An additional result follows by combining results from [6] and from [3].

Theorem 7 *Let G be a 2-regular bipartite graph of order n. Then G has a σ^{++}-labeling if $n \equiv 0 \pmod 4$ and a ρ^{++}-labeling if $n \equiv 2 \pmod 4$.*

3 Main results

We shall show that $2C_m$ has a ρ-labeling when m is odd, $3C_5$ has a σ-labeling, $3C_m$ has a ρ-labeling when $m \equiv 3 \pmod 4$, and $4C_m$ has a σ-labeling when m is odd. This along with some of the known results shows that rC_m has a ρ-labeling (or a more restricted labeling) when $r \leq 4$. Some additional definitions and notational conventions are necessary.

We denote the path with consecutive vertices $a_1, a_2, \ldots, a_k$ by $(a_1, a_2, \ldots, a_k)$. By $(a_1, a_2, \ldots, a_k) + (b_1, b_2, \ldots, b_j)$, where $a_k = b_1$, we mean the path $(a_1, \ldots, a_k, b_2, \ldots, b_j)$.

To simplify our consideration of various labelings, we will sometimes consider graphs whose vertices are named by distinct nonnegative integers, which are also their labels.

Let a, b, and h be integers with $0 \leq a \leq b$ and $h > 0$. Set $d = b - a$. We define the path

$$P(a, h, b) = (a, a+h+2d-1, a+1, a+h+2d-2, a+2, \ldots, b-1, b+h, b).$$

It is easily checked that $P(a, h, b)$ is simple and

$$V(P(a, h, b)) = [a, b] \cup [b+h, b+h+d-1].$$

Furthermore, the edge labels of $P(a, h, b)$ are distinct and

$$\bar{E}(P(a, h, b)) = [h, h+2d-1].$$

These formulas will be used extensively in the proofs that follow.

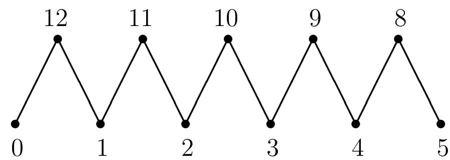

Figure 1: The path $P(0, 3, 5)$.

Theorem 8 *Let the graph G consist of two vertex-disjoint cycles, each of the same odd length. Then G has a ρ-labeling.*

Proof. First we consider cycles of length $4x+1$, x a positive integer. The two cycles will be G_1 and G_2, defined as follows:

$$\begin{aligned} G_1 &= P(0, 6x+4, x-1) + P(x-1, 4x+3, 2x-1) + (2x-1, 2x, 8x+3, 0), \\ G_2 &= P(8x+4, 2x+2, 9x+4) + P(9x+4, 3, 10x+3) \\ &+ (10x+3, 10x+5, 12x+6, 8x+4). \end{aligned}$$

Now we compute

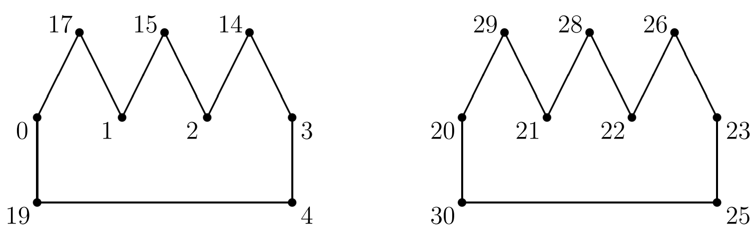

Figure 2: A ρ-labeling of $C_9 \cup C_9$.

$$\begin{aligned} V(G_1) &= [0, 2x-1] \cup [7x+3, 8x+1] \cup [6x+2, 7x+1] \cup \{2x, 8x+3\} \\ V(G_2) &= [8x+4, 10x+3] \cup [11x+6, 12x+5] \cup [10x+6, 11x+4] \\ &\quad \cup \{10x+5, 12x+6\}. \end{aligned}$$

We can order these as

$$[0, 2x-1], 2x, [6x+2, 7x+1], [7x+3, 8x+1], 8x+3$$

from G_1, and

$$[8x+4, 10x+3], 10x+5, [10x+6, 11x+4], [11x+6, 12x+5], 12x+6$$

from G_2. We see that the vertices of the two cycles are distinct and contained in $[0, 2(8x+2)] = [0, 16x+4]$. (If $x = 1$ the sets $[7x+3, 8x+1]$ and $[10x+6, 11x+4]$ are empty, but this does not change the proof.)

Likewise we compute

$$\begin{aligned}\bar{E}(G_1) &= [6x+4, 8x+1] \cup [4x+3, 6x+2] \cup \{1, 6x+3, 8x+3\},\\ \bar{E}(G_2) &= [2x+2, 4x+1] \cup [3, 2x] \cup \{2, 2x+1, 4x+2\}.\end{aligned}$$

We can order these as the edge label 1 from G_1,

$$2, [3, 2x], 2x+1, [2x+2, 4x+1], 4x+2$$

from G_2, and

$$[4x+3, 6x+2], 6x+3, [6x+4, 8x+1], 8x+3$$

from G_1. Thus $\bar{E}(G) = [1, 8x+1] \cup \{8x+3\}$. Since $2(8x+2)+1-(8x+3) = 8x+2$, we have a ρ-labeling. (If $x = 1$ the sets $[3, 2x]$ and $[6x+4, 8x+1]$ are empty, but this does not change the proof.)

Now suppose the cycles have length $4x+3$, x a nonnegative integer. The two cycles will be defined as follows:

$$\begin{aligned}G_1 &= P(0, 6x+6, x) + P(x, 4x+5, 2x) + (2x, 2x+2, 8x+7, 0),\\ G_2 &= P(8x+8, 2x+4, 9x+8) + P(9x+8, 3, 10x+8)\\ &+ (10x+8, 10x+9, 12x+12, 8x+8).\end{aligned}$$

Now we compute

$$\begin{aligned}V(G_1) &= [0, 2x] \cup [7x+6, 8x+5] \cup [6x+5, 7x+4] \cup \{2x+2, 8x+7\}\\ V(G_2) &= [8x+8, 10x+8] \cup [11x+12, 12x+11] \cup [10x+11, 11x+10]\\ &\quad \cup \{10x+9, 12x+12\}.\end{aligned}$$

We can order these as

$$[0, 2x], 2x+2, [6x+5, 7x+4], [7x+6, 8x+5], 8x+7$$

from G_1, and

$$[8x+8, 10x+8], 10x+9, [10x+11, 11x+10], [11x+12, 12x+11], 12x+12$$

from G_2. We see that the vertices of the two cycles are distinct and contained in $[0, 2(8x+6)] = [0, 16x+12]$. (If $x = 0$ the sets $[6x+5, 7x+4]$, $[7x+6, 8x+5]$, $[10x+11, 11x+10]$ and $[11x+12, 12x+11]$ are empty, but this does not change the proof.) Likewise we compute

$$\begin{aligned}\bar{E}(G_1) &= [6x+6, 8x+5] \cup [4x+5, 6x+4] \cup \{2, 6x+5, 8x+7\},\\ \bar{E}(G_2) &= [2x+4, 4x+3] \cup [3, 2x+2] \cup \{1, 2x+3, 4x+4\}.\end{aligned}$$

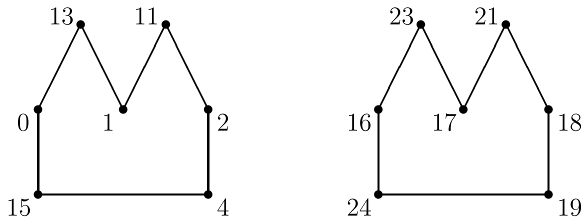

Figure 3: A ρ-labeling of $C_7 \cup C_7$.

We can order these as the edge label 1 from G_2, 2 from G_1,

$$[3, 2x+2], 2x+3, [2x+4, 4x+3], 4x+4$$

from G_2, and

$$[4x+5, 6x+4], 6x+5, [6x+6, 8x+5], 8x+7$$

from G_1. Thus $\bar{E}(G) = [1, 8x+5] \cup \{8x+7\}$. Since $2(8x+6)+1-(8x+7) = 8x+6$, we have a ρ-labeling. (If $x = 0$ the sets $[3, 2x+2], [2x+4, 4x+3], [4x+5, 6x+4]$ and $[6x+6, 8x+5]$ are empty, but this does not change the proof.) $\square$

It is known that $3C_5$ does not have a β-labeling (see [11]) and that $3C_{4x+1}$ has a β-labeling for $x \geq 2$ (see [12]).

Lemma 9 *The graph consisting of the vertex-disjoint union of three C_5's has a σ-labeling.*

Proof. Take the cycles $(0, 14, 1, 4, 15, 0)$, $(16, 25, 17, 18, 28, 16)$ and $(5, 11, 6, 8, 12, 5)$. $\square$

Theorem 10 *Let x be a nonnegative integer, and let a graph consist of three vertex-disjoint cycles, each of length $4x+3$. Then the graph has a ρ-labeling.*

Proof. First assume $x > 0$. The three cycles will be G_1, G_2, and G_3, defined as follows:

$$\begin{aligned}
G_1 &= P(0, 10x+9, x) + P(x, 8x+7, 2x) + (2x, 2x+3, 12x+10, 0),\\
G_2 &= P(12x+11, 4x+6, 14x+11) + (14x+11, 14x+13, 22x+19, 12x+11),\\
G_3 &= P(2x+4, 2x+5, 3x+4) + P(3x+4, 4, 4x+4)\\
&+ (4x+4, 4x+5, 6x+9, 2x+4).
\end{aligned}$$

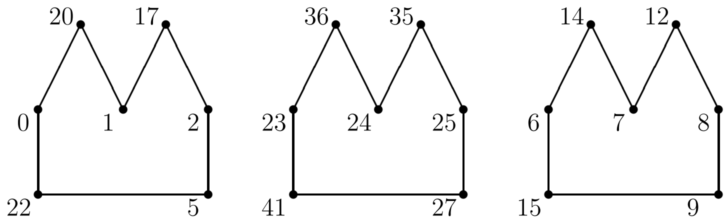

Figure 4: A ρ-labeling of $C_7 \cup C_7 \cup C_7$.

Now we compute

$$\begin{aligned}
V(G_1) &= [0, 2x] \cup [11x+9, 12x+8] \cup [10x+7, 11x+6] \cup \{2x+3, 12x+10\}, \\
V(G_2) &= [12x+11, 14x+11] \cup [18x+17, 20x+16] \cup \{14x+13, 22x+19\}, \\
V(G_3) &= [2x+4, 4x+4] \cup [5x+9, 6x+8] \cup [4x+8, 5x+7] \cup \{4x+5, 6x+9\}.
\end{aligned}$$

We can order these as

$$[0, 2x], 2x+3$$

from G_1,

$$[2x+4, 4x+4], 4x+5, [4x+8, 5x+7], [5x+9, 6x+8], 6x+9$$

from G_3,

$$[10x+7, 11x+6], [11x+9, 12x+8], 12x+10$$

from G_1, and

$$[12x+11, 14x+11], 14x+13, [18x+17, 20x+16], 22x+19$$

from G_2. We see that for $x > 0$ the vertices of the three cycles are distinct and contained in $[0, 2(12x+9)] = [0, 24x+18]$.

Likewise we compute

$$\begin{aligned}
\bar{E}(G_1) &= [10x+9, 12x+8] \cup [8x+7, 10x+6] \cup \{3, 10x+7, 12x+10\}, \\
\bar{E}(G_2) &= [4x+6, 8x+5] \cup \{2, 8x+6, 10x+8\}, \\
\bar{E}(G_3) &= [2x+5, 4x+4] \cup [4, 2x+3] \cup \{1, 2x+4, 4x+5\}.
\end{aligned}$$

We can order these as the edge label 1 from G_3, 2 from G_2, 3 from G_1,

$$[4, 2x+3], 2x+4, [2x+5, 4x+4], 4x+5$$

from G_3,

$$[4x+6, 8x+5], 8x+6$$

from G_2,

$$[8x+7, 10x+6], 10x+7$$

from G_1, $10x+8$ from G_2, and

$$[10x+9, 12x+8], 12x+10$$

from G_1. Thus $E(G) = [1, 12x+8] \cup \{12x+10\}$. Since $2(12x+9)+1-(12x+10) = 12x+9$, we have a ρ-labeling.

Finally if $x = 0$ we take our cycles to be $(0, 3, 10, 0)$, $(1, 2, 6, 1)$, and $(5, 7, 13, 5)$. □

Theorem 11 *Let the graph G consist of four vertex-disjoint cycles, each of the same odd length. Then G has a σ-labeling.*

Proof. First we consider the case when the cycles have length $4x+1$, where x is a positive integer. Our four cycles will be G_1, G_2, G_2, and G_4, defined as follows:

$$\begin{aligned}
G_1 &= P(0, 12x+6, 2x-1) + (2x-1, 4x-1, 16x+4, 0), \\
G_2 &= P(16x+5, 8x+6, 18x+4) + (18x+4, 20x+5, 28x+9, 16x+5), \\
G_3 &= P(4x, 4x+6, 6x-1) + (6x-1, 8x+1, 12x+5, 4x), \\
G_4 &= P(20x+6, 2x+6, 21x+5) + (21x+5, 23x+9, 21x+6) \\
&+ P(21x+6, 2, 22x+5) + (22x+5, 22x+6, 24x+11, 20x+6)
\end{aligned}$$

Now we compute

$$\begin{aligned}
V(G_1) &= [0, 2x-1] \cup [14x+5, 16x+3] \cup \{4x-1, 16x+4\}, \\
V(G_2) &= [16x+5, 18x+4] \cup [26x+10, 28x+8] \cup \{20x+5, 28x+9\}, \\
V(G_3) &= [4x, 6x-1] \cup [10x+5, 12x+3] \cup \{8x+1, 12x+5\}, \\
V(G_4) &= [20x+6, 21x+5] \cup [23x+11, 24x+9] \cup \{23x+9\} \\
&\cup [21x+6, 22x+5] \cup [22x+7, 23x+5] \cup \{22x+6, 24x+11\}.
\end{aligned}$$

We can order these sets as follows.

cycle	vertices
G_1	$[0, 2x-1]$
G_1	$4x-1$
G_3	$[4x, 6x-1]$
G_3	$8x+1$
G_3	$[10x+5, 12x+3]$
G_3	$12x+5$
G_1	$[14x+5, 16x+3]$
G_1	$16x+4$
G_2	$[16x+5, 18x+4]$
G_2	$20x+5$

cycle	vertices
G_4	$[20x+6, 21x+5]$
G_4	$[21x+6, 22x+5]$
G_4	$22x+6$
G_4	$[22x+7, 23x+5]$
G_4	$23x+9$
G_4	$[23x+11, 24x+9]$
G_4	$24x+11$
G_2	$[26x+10, 28x+8]$
G_2	$28x+9$

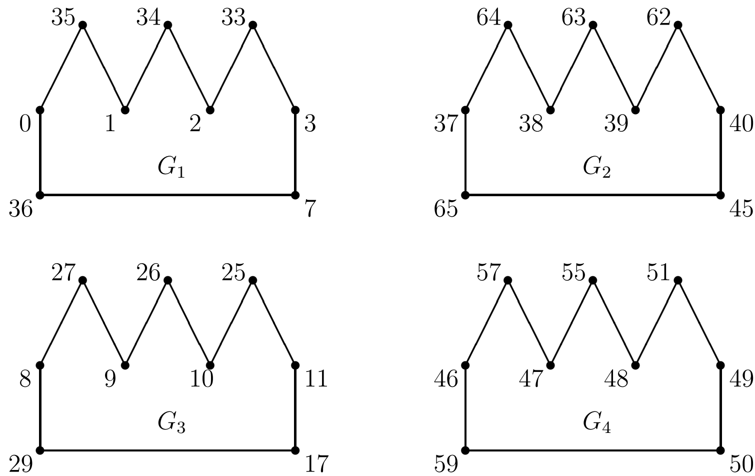

Figure 5: A σ-labeling of $4C_9$.

From this we can see that the vertices of the four cycles are distinct and contained in $[0, 2(16x+4)] = [0, 32x+8]$. (If $x = 1$ the sets $[22x+7, 23x+5]$ and $[23x+11, 24x+9]$ are empty, but this does not change the proof.)

Likewise we compute

$$\begin{aligned}
\bar{E}(G_1) &= [12x+6, 16x+3] \cup \{2x, 12x+5, 16x+4\}, \\
\bar{E}(G_2) &= [8x+6, 12x+3] \cup \{2x+1, 8x+4, 12x+4\}, \\
\bar{E}(G_3) &= [4x+6, 8x+3] \cup \{2x+2, 4x+4, 8x+5\}, \\
\bar{E}(G_4) &= [2x+6, 4x+3] \cup \{2x+4, 2x+3\} \cup [2, 2x-1] \cup \{1, 2x+5, 4x+5\}.
\end{aligned}$$

We can order these sets as follows.

cycle	edge labels
G_4	1
G_4	$[2, 2x-1]$
G_1	$2x$
G_2	$2x+1$
G_3	$2x+2$
G_4	$2x+3$
G_4	$2x+4$
G_4	$2x+5$
G_4	$[2x+6, 4x+3]$
G_3	$4x+4$

cycle	edge labels
G_4	$4x+5$
G_3	$[4x+6, 8x+3]$
G_2	$8x+4$
G_3	$8x+5$
G_2	$[8x+6, 12x+3]$
G_2	$12x+4$
G_1	$12x+5$
G_1	$[12x+6, 16x+3]$
G_1	$16x+4$

We see that the edge labels are exactly the set $[1, 16x+4]$. (If $x = 1$, the sets $[2, 2x-1]$ and $[2x+6, 4x+3]$ are empty, but this does not change the proof.)

Now we consider the case when the cycles have length $4x+3$, where x is a positive integer. The case $x = 0$ will be considered separately. Our four cycles will be defined as follows:

$$
\begin{aligned}
G_1 &= P(0, 14x+12, x) + P(x, 12x+11, 2x) + (2x, 2x+1, 16x+12, 0),\\
G_2 &= P(16x+13, 10x+8, 17x+14) + P(17x+14, 8x+9, 18x+13)\\
&+ (18x+13, 18x+16, 28x+23, 16x+13),\\
G_3 &= P(2x+2, 6x+8, 3x+2) + P(3x+2, 4x+6, 4x+2)\\
&+ (4x+2, 4x+4, 10x+10, 2x+2),\\
G_4 &= (18x+17, 22x+22, 18x+18) + P(18x+18, 5, 20x+17)\\
&+ (20x+17, 20x+21, 24x+24, 18x+17).
\end{aligned}
$$

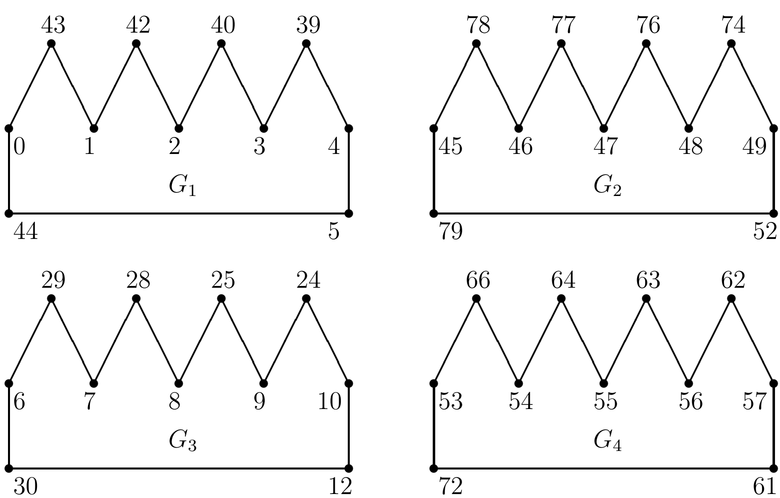

Figure 6: A ρ-labeling of $4C_{11}$.

Now we compute

$$
\begin{aligned}
V(G_1) &= [0, 2x] \cup [15x+12, 16x+11] \cup [14x+11, 15x+10] \cup \{2x+1, 16x+12\},\\
V(G_2) &= [16x+13, 18x+13] \cup [27x+22, 28x+22] \cup [26x+22, 27x+20]\\
&\quad \cup \{18x+16, 28x+23\},\\
V(G_3) &= [2x+2, 4x+2] \cup [9x+10, 10x+9] \cup [8x+8, 9x+7] \cup \{4x+4, 10x+10\},\\
V(G_4) &= \{18x+17, 22x+22\} \cup [18x+18, 20x+17] \cup [20x+22, 22x+20]\\
&\quad \cup \{20x+21, 24x+24\}.
\end{aligned}
$$

We can order these sets as follows.

cycle	vertices	cycle	vertices
G_1	$[0, 2x]$	G_2	$18x + 16$
G_1	$2x + 1$	G_4	$18x + 17$
G_3	$[2x + 2, 4x + 2]$	G_4	$[18x + 18, 20x + 17]$
G_3	$4x + 4$	G_4	$20x + 21$
G_3	$[8x + 8, 9x + 7]$	G_4	$[20x + 22, 22x + 20]$
G_3	$[9x + 10, 10x + 9]$	G_4	$22x + 22$
G_3	$10x + 10$	G_4	$24x + 24$
G_1	$[14x + 11, 15x + 10]$	G_2	$[26x + 22, 27x + 20]$
G_1	$[15x + 12, 16x + 11]$	G_2	$[27x + 22, 28x + 22]$
G_1	$16x + 12$	G_2	$28x + 23$
G_2	$[16x + 13, 18x + 13]$		

From this we can see that the vertices of the four cycles are distinct and contained in $[0, 2(16x + 12)] = [0, 32x + 24]$ for $x > 0$. (If $x = 1$ the set $[26x + 22, 27x + 20]$ is empty, but this does not change the proof.)

Likewise we compute

$$
\begin{aligned}
\bar{E}(G_1) &= [14x + 12, 16x + 11] \cup [12x + 11, 14x + 10] \cup \{1, 14x + 11, 16x + 12\}, \\
\bar{E}(G_2) &= [10x + 8, 12x + 9] \cup [8x + 9, 10x + 6] \cup \{3, 10x + 7, 12x + 10\}, \\
\bar{E}(G_3) &= [6x + 8, 8x + 7] \cup [4x + 6, 6x + 5] \cup \{2, 6x + 6, 8x + 8\}, \\
\bar{E}(G_4) &= \{4x + 5, 4x + 4\} \cup [5, 4x + 2] \cup \{4, 4x + 3, 6x + 7\}.
\end{aligned}
$$

We can order these sets as follows.

cycle	edge labels	cycle	edge labels
G_1	1	G_3	$[6x + 8, 8x + 7]$
G_2	3	G_3	$8x + 8$
G_3	2	G_2	$[8x + 9, 10x + 6]$
G_4	4	G_2	$10x + 7$
G_4	$[5, 4x + 2]$	G_2	$[10x + 8, 12x + 9]$
G_4	$4x + 3$	G_2	$12x + 10$
G_4	$4x + 4$	G_1	$[12x + 11, 14x + 10]$
G_4	$4x + 5$	G_1	$14x + 11$
G_3	$[4x + 6, 6x + 5]$	G_1	$[14x + 12, 16x + 11]$
G_3	$6x + 6$	G_1	$16x + 12$
G_4	$6x + 7$		

We see that for $x \geq 2$ the edge labels are exactly the set $[1, 16x + 12]$. (If $x = 1$ the set $[8x + 9, 10x + 6]$ is empty, but this does not change the proof.)

Finally, if $x = 0$ we take $G_1 = (0, 1, 12, 0)$, $G_2 = (13, 16, 23, 13)$, $G_3 = (2, 4, 10, 2)$, and $G_4 = (5, 9, 14, 5)$. □

The results for labelings of rC_{4x+k}, $1 \leq r \leq 4$, $3 \leq k \leq 6$ and $x \geq 0$ are summarized in the table below.

<table>
<tr><th></th><th>$k=3$</th><th>$k=4$</th><th>$k=5$</th><th>$k=6$</th></tr>
<tr><td>$r=1$</td><td>β</td><td>α</td><td>ρ</td><td>ρ^{++}</td></tr>
<tr><td>$r=2$</td><td>ρ</td><td>α</td><td>ρ</td><td>α</td></tr>
<tr><td rowspan="2">$r=3$</td><td rowspan="2">ρ</td><td>σ^{++} if $x=0$</td><td>σ if $x=0$</td><td rowspan="2">ρ^{++}</td></tr>
<tr><td>α if $x>0$</td><td>β if $x>0$</td></tr>
<tr><td rowspan="2">$r=4$</td><td rowspan="2">σ</td><td>α if $x=0$</td><td rowspan="2">σ</td><td rowspan="2">σ^{++}</td></tr>
<tr><td>σ^{++} if $x>0$</td></tr>
</table>

Table 1. Labelings of rC_{4x+k}, $1 \leq r \leq 4$, $3 \leq k \leq 6$ and $x \geq 0$.

We can offer the following corollary.

Corollary 12 *Let G be a 2-regular graph with n edges and at most 4 components. Then there exists a cyclic G-decompositions of K_{2n+1}.*

4 Concluding Remarks

The study of graph decompositions is a popular branch of modern combinatorial design theory (see [4] for an overview). In particular, the study of G-decompositions of K_{2n+1} (and of K_{2nx+1}) when G is a graph with n edges (and x is a positive integer) has attracted considerable attention. The study of graph labelings is also quite popular (see Gallian [9] for a dynamic survey). Theorem 1 provides a powerful link between the two areas. Much of the attention on labelings has been on graceful labelings (i.e., β-labelings). Unfortunately, the parity condition "disqualifies" large classes of graphs from admitting graceful labelings. This difficulty is compounded by the fact that certain classes of graphs with ρ-labelings meet the parity condition, yet fail to be graceful.

In conclusion, we note that our results here, along with results from [5] and [10], provide further evidence in support of the following conjecture which is presented in a forthcoming survey [7].

Conjecture 13 *Every 2-regular graph has a ρ-labeling.*

Evidence suggests that the above conjecture can be strengthened to predict a σ-labeling if the parity condition is satisfied.

Acknowledgement

The first two authors wish to thank the Mathematics Department at Ramkhamhaeng University for the tremendous hospitality shown during a recent workshop. This work was initiated at the said workshop.

References

[1] J. Abrham and A. Kotzig, All 2-regular graphs consisting of 4-cycles are graceful, *Discrete Math.* **135** (1994), 1–14.

[2] J. Abrham and A. Kotzig, Graceful valuations of 2-regular graphs with two components, *Discrete Math.* **150** (1996), 3–15.

[3] A. Blinco and S.I. El-Zanati, A note on the cyclic decomposition of complete graphs into bipartite graphs, *Bull. Inst. Combin. Appl.*, to appear.

[4] J. Bosák, *Decompositions of Graphs*, Kluwer Academic Publishers Group, Dordrecht, 1990.

[5] J.H. Dinitz and P. Rodney, Disjoint difference families with block size 3, *Util. Math.* **52** (1997), 153–160.

[6] S.I. El-Zanati, C. Vanden Eynden and N. Punnim, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.* **24** (2001), 209–219.

[7] S.I. El-Zanati and C. Vanden Eynden, Graph labelings and graph decompositions, a survey, forthcoming.

[8] K. Eshghi, The existence and construction of α-valuations of 2-regular graphs with 3 components, Ph.D. Thesis, Industrial Engineering Dept., University of Toronto, 1997.

[9] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, Dynamic Survey **6**, 106 pp.

[10] H. Hevia and S. Ruiz, Decompositions of complete graphs into caterpillars, *Rev. Mat. Apl.* **9** (1987), 55–62.

[11] A. Kotzig, β-valuations of quadratic graphs with isomorphic components, *Util. Math.*, **7** (1975), 263–279.

[12] A. Kotzig, Recent results and open problems in graceful graphs, *Congress. Numer.* **44** (1984), 197–219.

[13] A. Rosa, On the cyclic decomposition of the complete graph into polygons with odd number of edges, *Časopis Pěst. Mat.* **91** (1966), 53–63.

[14] A. Rosa, On certain valuations of the vertices of a graph, in: Théorie des graphes, journées internationales d'études, Rome 1966 (Dunod, Paris, 1967), 349–355.

(Received 13 Jan 2003)