MINIMAL GRAPHS WITH PRESCRIBED VERTEX INDEPENDENCE AND CLIQUE NUMBERS

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ABSTRACT: The **vertex independence number** of a graph G is the maximal number of independent vertices in G. The **clique number** of G is the size of the largest complete subgraph of G. Let $\mathcal{G}(\nu,n,r)$ denote the class of simple graphs on ν vertices having vertex independence number n and clique number r. Let

$$f(\nu,n,r) = \min \{\varepsilon(G) : G \in \mathcal{G}(\nu,n,r)\},$$

where $\varepsilon(G)$ denotes the number of edges in G. In this paper we study the class $\mathcal{G}(\nu,n,r)$ and in particular, consider the problem of determining the function $f(\nu,n,r)$.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [1]. Thus G is a simple graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges and minimum degree $\delta(G)$. However, we find it convenient to denote the complement of G by $\overline{G}$. K_n denotes the complete graph on n vertices and G[X] the subgraph of G induced by the vertex set X.

The **vertex independence number** $\alpha(G)$ of G is the maximum number of independent vertices of G. The **clique number** $c\ell(G)$ of G is the size of the largest complete subgraph of G. Let $\mathcal{G}(\nu,n,r)$ denote the class of graphs on ν vertices having independence number n and clique number r. The problem that arises is that of characterizing this class of graphs. Observe that if $G \in \mathcal{G}(\nu,n,r)$ then $\overline{G} \in \mathcal{G}(\nu,r,n)$. Further, $\mathcal{G}(\nu,n,r) = \phi$ for $\nu < n + r - 1$. So we may assume that $\nu \geq n + r - 1$.

In this paper we consider the particular problem of determining the edge-minimal numbers of $\mathcal{G}(\nu,n,r)$. A special case of this problem ($r = 2$) was posed to us in a personal communication by P. Erdös. Let

$$f(\nu,n,r) = \min\{\varepsilon(G):G \in \mathcal{G}(\nu,n,r)\}.$$

In Section 3, we determine $f(\nu,n,2)$ for $\nu < \frac{5}{2}n$ and establish that the extremal graphs are unique. In Section 4, $f(\nu,2,r)$ is determined for $\nu \le \frac{5}{2}r$ and for $\nu = 3r$. A number of properties of the class $\mathcal{G}(\nu,n,r)$ are established in Section 2.

2. PROPERTIES OF THE CLASS $\mathcal{G}(\nu,n,r)$.

Throughout this section $G \in \mathcal{G}(\nu,n,r)$. For $A \subseteq V(G)$, we let $N(A)$ denote the neighbour set of A in G and $\overline{N}(A)$ denote the vertices of $V(G)\setminus\{N(A) \cup A\}$. We sometimes refer to $\overline{N}(A)$ as the non-neighbours of A. Observe that for any vertex u of G, the subgraph $H = G[\overline{N}(u)]$ has $\alpha(H) \le n - 1$. For $X \subset V(G)$ and $A \subseteq V(G)\setminus X$ we let $N_X(A) = N(A) \cap X$. The following is a consequence of the definition of independence number.

Lemma 2.1. Let X be an independent set of n vertices in a graph $G \in \mathcal{G}(\nu,n,r)$. If Y is an independent set of vertices in $V(G)\setminus X$, then $|N_X(A)| \ge |A|$, for every $A \subseteq Y$. □

A consequence of Lemma 2.1 is that the subgraph $G[X \cup Y]$ has a maximum matching that saturates every vertex of Y.

We say two vertices x and y are **equivalent**, written $x \sim y$, if $x \cup N(x) = y \cup N(y)$. By altering the edges of G we can make any two non-equivalent vertices u and v equivalent. When we make v equivalent to u we mean that the edges incident to v are altered so that v is adjacent to every vertex in $u \cup N(u)$.

Suppose that we make $v \sim u$. Then the resulting graph G' will clearly have $c\ell(G') \le r + 1$ and

$$\varepsilon(G') = \varepsilon(G) + \begin{cases} d_G(u) - d_G(v), & \text{if } v \in N(u) \\ d_G(u) - d_G(v) + 1, & \text{otherwise.} \end{cases} \qquad (2.1)$$

Further, $\alpha(G') \leq n$ since otherwise in G' v must be in a maximum independent set S which cannot contain u and hence $S' = u \cup (S \backslash v)$ is an independent set of size greater than n in G, a contradiction. If $\alpha(G') < n$, then by deleting edges we obtain a graph G'' with $\alpha(G'') = n$. Now observing that $c\ell(G') = r + 1$ only if u is in an r-clique, and $c\ell(G') < r$ only if v is in every r-clique of G we have:

Lemma 2.2. Let G be a graph in $\mathcal{G}(\nu,n,r)$ having a vertex u not contained in an r-clique and a vertex v such that $c\ell(G - v) = r$. Then the graph G' obtained from G by making $v \sim u$ is in $\mathcal{G}(\nu,n',r)$, $n' \leq n$, and $\varepsilon(G')$ is given by (2.1). □

If the G in Lemma 2.2 is edge-minimal, then $n' = n$ (if $n' < n$ then, as noted above, we can delete edges) and from (2.1) it is clear that when $v \in N(u)$ $d_G(u) \geq d_G(v)$, and when $v \notin N(u)$ $d_G(u) \geq d_G(v) - 1$. In particular, if G is edge-minimal and $d_G(u) = \delta$ then $d_G(v) \leq \delta + 1$ with strict inequality holding whenever $v \in N(u)$. Our next two lemmas establish some useful properties of edge-minimal graphs.

Lemma 2.3. There exists an edge-minimal graph $G \in \mathcal{G}(\nu,n,r)$ in which every vertex of minimum degree is contained in a clique of size $t = \min\{\delta + 1, r\}$.

Proof: Suppose the lemma is false. From the edge-minimal members of $\mathcal{G}(\nu,n,r)$ consider those which have the smallest minimum degree. From these choose a G which has the fewest number of vertices of degree $\delta(G)$ which are not contained in a t-clique.

Let u be a vertex of G having degree δ and not contained in a t-clique. Take a vertex $v \in N(u)$ that is not equivalent to u. If $c\ell(G - v) = r$, then according to Lemma 2.2 we can make $v \sim u$. In this case, since G is edge-minimal $d_G(u) = \delta$. Consequently we may assume that every vertex v of $N(u)$ that is not equivalent to u is contained in every r-clique of G. Thus the set of vertices of N(u) not equivalent to u form a clique. But then $G[u \cup N(u)]$ is a clique. This contradicts the choice of G and thus establishes the lemma. □

Theorem 2.1. Let G be an edge-minimal graph of $\mathcal{G}(\nu,n,r)$. Then for $\nu \geq nr$, $\delta(G) \geq r - 1$.

Proof: Suppose to the contrary that $\delta \leq r - 2$. Let K_r denote an r-clique of G and let $H = G[V(G)\setminus V(K_r)]$. Let u be a vertex of H having degree δ in G. Such a vertex clearly exists. Moreover, if N(u) has vertices of degree greater than δ in G then such vertices are in the K_r. If H contains a vertex v such that $d_G(v) > \delta + 1$, then by Lemma 2.2 making $v \sim u$ produces a graph $G' \in \mathcal{G}(\nu,n,r)$ with fewer edges than G, a contradiction. Hence the vertices of H have degree at most $\delta + 1$ in G.

The subgraph H has $\alpha(H) = n$ or $\alpha(H) \leq n - 1$. If $\alpha(H) = n$, then u is in a maximum independent set. Hence

$$\nu(G) \leq \delta + 1 + (n - 1)(\delta + 2)$$

and so, since $\nu(G) \geq nr$, $r \leq \delta + 1$ as required. Now suppose that $\alpha(H) \leq n - 1$. Then, since G is edge minimal, $\alpha(H) = n - 1$ and no vertex of H is joined, in G, to any vertex of K_r. Consequently, the vertices of N(u) all have degree δ in G. Thus u is contained in an independent set of size $n - 1$ in H. Hence

$$\begin{aligned} \nu(H) &\leq \delta + 1 + (n - 2)(\delta + 2) \\ &< (n - 1)(\delta + 2) \end{aligned}$$

and so

$$\begin{aligned} \nu(G) &= r + \nu(H) \\ &< (n - 1)(\delta + 2) + r \\ &< n(\delta + 2). \end{aligned}$$

Since $\nu(G) \geq nr$ we must have $\delta \geq r - 1$, as required. This completes the proof of the theorem. □

As a corollary we have:

Corollary 1. For $\nu \geq nr$ there exists an edge-minimal graph $G \in \mathcal{G}(\nu,n,r)$ in which every vertex of minimum degree is contained in a clique of size r. □

Given a graph H we let qH denote the graph consisting of q disjoint copies of H. We can now state the following corollary to Theorem 2.1.

Corollary 2. Let $G \in \mathcal{G}(nr,n,r)$. Then G is edge-minimal if and only if $G \cong nK_r$. □

For $\nu < nr$ we have the following characterization of edge-minimal graphs.

Theorem 2.2. For $n + r - 1 \leq \nu < nr$, there exists an edge-minimal graph $G \in \mathcal{G}(\nu,n,r)$ consisting of n disjoint cliques, one of size r and the rest of size $\lfloor(\nu - r)/(n - 1)\rfloor$ or $\lceil(\nu - r)/(n - 1)\rceil$.

Proof: Let K_r denote an r-clique of G and let $H = G[V(G)\setminus V(K_r)]$. Suppose $\nu - r = (n - 1)t + \lambda$, where $0 \leq \lambda \leq n - 2$. Since the graph

$$\hat{G} = K_r \cup \lambda K_{t+1} \cup (n - 1 - \lambda)K_t$$

belongs to the class $\mathcal{G}(\nu,n,r)$ and has $\Delta(\hat{G}) = r - 1$ and $\delta(\hat{G}) = t - 1 < r - 1$, we may assume that $\delta(G) < r - 1$. In view of Lemma 2.3 we may further assume that the vertices of G having degree δ are contained in cliques of size $\delta + 1$.

Let $d_G(u) = \delta$. Then $u \in H$ and the subgraph $G[u \cup N(u)]$ is a $(\delta + 1)$-clique. As in the proof of Theorem 2.1, if N(u) has vertices of degree greater than δ then such vertices are in the K_r. Further, the vertices of H all have degree δ or $\delta + 1$ in G. Consider a vertex v of H having degree $\delta + 1$ in G. We may suppose that v is in a clique of size $\delta + 2$, for otherwise there exists a vertex $w \in N(v) \cap V(H)$ that is not equivalent to v. Making $w \sim v$ does not increase the edge count. Consequently our H consists of a disjoint union of cliques. Vertices of these cliques may be joined to vertices of K_r.

If $\alpha(H) = n - 1$, then since G is edge-minimal no vertex of K_r is joined to a vertex of H. Since the cliques of H are of size $\delta + 1$ or $\delta + 2$, we have proved the theorem. Hence we need only consider the case $\alpha(H) = n$. If a vertex of K_r is joined to vertices in different cliques of H, then we can delete some edges and still have a graph in $\mathcal{G}(\nu,n,r)$. Hence no vertex of K_r is joined to vertices in different cliques of H. Consequently our G is the union of n disjoint cliques $Q_1,Q_2,\ldots,Q_n$.

Suppose $|V(Q_i) \cap V(K_r)| = a_i$, $1 \le i \le n$. Without any loss of generality suppose that $a_1 \ge a_2 \ge \ldots \ge a_n$. Since $\alpha(H) = n$, we have $1 < a_1 < r$. Hence $a_2 > 0$. Consider Q_1 and Q_2. There exist vertices $x \in V(Q_1)\backslash V(K_r)$, $y \in V(Q_1) \cap V(K_r)$, $v \in V(Q_2)\backslash V(K_r)$ and $w \in V(Q_2) \cap V(K_r)$. Clearly x and y are not equivalent nor are v and w. Now the graph G′ formed from G by making $v \sim w$ and then making $y \sim x$ belongs to $\mathcal{G}(\nu,n,r)$ and has $a_1 - a_2 + 1$ (> 0) fewer edges than G, a contradiction. Thus $\alpha(H) = n - 1$, and G consists of disjoint cliques one of size r and the rest of size $\delta + 1$ or $\delta + 2$. Since $\nu(H) = \nu - r$, we must have $\delta = \lfloor(v - r)/(n - 1)\rfloor$. This completes the proof of the theorem. □

We conclude this section by noting that the δ in the proof of Theorem 2.2 could be r and so the edge-minimal graph may consist of a number of cliques of size r.

3. THE CLASS $\mathcal{G}(\nu,n,2)$

For the purposes of this section we define the graphs G_1 and G_2 as follows:

$$G_1 = (\nu - n)K_2 \cup n\,K_1,\ n \le \nu \le 2n,$$

and

$$G_2 = (\nu - 2n)C_5 \cup (5n - 2\nu)K_2,\ 2n \le \nu \le \frac{5}{2}\,n.$$

Observe that G_1 and G_2 are both in $\mathscr{G}(\nu,n,2)$. We shall prove that these graphs are edge-minimal. The edge-minimality of G_1 is easily established. In fact, when $n < \nu \le 2n$, we can easily determine the possible values of $\varepsilon(G)$, $G \in \mathscr{G}(\nu,n,2)$. This is done in the following lemma.

Lemma 3.1. Let $G \in \mathscr{G}(\nu,n,2)$. Then for $n < \nu \le 2n$,

$$\nu - n \le \varepsilon(G) \le n(\nu - n)$$

Moreover, every value in this range is realizable.

Proof: Let X be an independent set of size n and let $\overline{X} = V(G)\backslash X$. Then the lower bound follows since every vertex of $\overline{X}$ must be joined to at least one vertex of X.

Suppose G is edge-maximal. We claim that $d_G(u) \le n$ for every vertex $u \in \overline{X}$. If this is not the case, then u has neighbours in X and in $\overline{X}$. Let $A = N(u) \cap \overline{X}$ and $B = N(u) \cap X$. Since $c\ell(G) = 2$, the vertices of A form an independent set and hence, by Lemma 2.1, $|N_X(A)| \ge |A|$. So there are at least $|A|$ vertices in X not joined to u. Hence $d_G(u) \le n$. Simple counting now establishes the upper bound which is achieved by the graph $K_{n,\nu-n}$. The realizability problem is easily established by construction. This completes the proof of the lemma. □

We now turn our attention to the case $\nu \ge 2n$. Theorem 2.1 implies that $\delta(G) \ge 1$. In fact, we shall prove that G has vertices of degree 1 or 2 only. The arguments we use to establish that G_2 is the unique edge-minimal graph is slightly more complicated than those used in proving the edge-minimality of G_1. We begin with a simple lemma.

Lemma 3.2. Let G be any edge-minimal graph of $\mathscr{G}(\nu,n,2)$. Let S denote the vertices of G having degree 1. Then G[S] is 1-regular.

Proof: Suppose that G[S] is not 1-regular. Then there exist a vertex $u \in S$ that is adjacent to a vertex v with $d_G(v) \geq 2$. But the subgraph $G-u-v \in \mathcal{G}(\nu-2,n-1,2)$ and hence the graph $G' = (G-u-v) \cup uv \in \mathcal{G}(\nu,,n,2)$ and has less edges than G, a contradiction. This proves the lemma. □

Lemma 3.3. Let G be any edge-minimal graph of $\mathcal{G}(\nu,n,2)$. Then for $2n \leq \nu < \frac{5}{2}n$ every vertex of G has degree 1 or 2.

Proof: Suppose to the contrary that G has vertices of degree ≥ 3. Let n_1 and n_2 denote the number of vertices of degree 1 and 2, respectively and let $n_3 = \nu - n_1 - n_2$. Then

$$n_1 + 2n_2 + 3n_3 \leq 2\varepsilon$$

and so

$$n_1 \geq 2(\nu - \varepsilon) + n_3,$$

with strict inequality holding if G has vertices of degree greater than 3.

Since $G_2 \in \mathcal{G}(\nu,n,2)$ and has $3\nu - 5n$ edges, we can assume that $\varepsilon(G) \leq 3\nu - 5n$. Hence $n_1 \geq 2 + 2n_3$. Now in view of Lemma 3.2 we must have $n_1 \geq 4$. Let a,b,c and d be four vertices of G having degree 1 with ab and cd $\in E(G)$. Let x be a vertex of maximum degree in G. Consider the graph G′ formed from G by deleting the edges of G incident to x and adding the edges xa,xd and bc. If G′ contains an independent set of size greater than n, then it has one which does not contain x implying that $\alpha(G) > n$. Hence $\alpha(G') \leq n$. In fact, since G is edge-minimal and $\varepsilon(G) \leq \varepsilon(G')$ we must have $\alpha(G') = n$ and $d_G(x) = 3$. Hence G′ is also edge-minimal. Now if x is joined, in G, to vertices of degree 2 then G′ would have a vertex of degree 1 adjacent to a vertex of degree ≥ 2, contradicting Lemma 3.2 Consequently, we conclude that the set of vertices of G having degree 3 form a 3-regular subgraph of G. But then G′ has a vertex of degree 3 joined to a vertex of degree 2. Applying the above transformation on G′ yields the necessary contradiction, thus proving the lemma. □

we can state the main theorem of this section.

Theorem 3.1. For $\nu \geq n + 1$

$$f(\nu,n,2) = \begin{cases} \nu - n, & \nu \leq 2n \\ 3\nu - 5n, & 2n \leq \nu < \frac{5}{2}\, n. \end{cases}$$

Moreover G_1 is the unique edge-minimal graph for $\nu \leq 2n$ and G_2 is the unique edge-minimal graph for $2n \leq \nu < \frac{5}{2}\, n$.

Proof: The case $\nu \leq 2n$ was established in Lemma 3.1, so we need only consider the case $2n \leq \nu < \frac{5}{2}\, n$. According to Lemma 3.3 every edge-minimal graph contains only vertices of degree 1 or 2. Further, the vertices of degree 1 form an independent set of edges.

Let G be an edge-minimal graph with n_1 vertices of degree 1 and $\nu - n_1$ vetices of degree 2. The subgraph H of G induced by the vertices of degree 2 is a union of cycles. If H has an even cycle C, then we can delete an edge from C and our resulting graph $G' \in \mathcal{G}(\nu,n,2)$. Hence H consists of a union of t odd cycles. Since a cycle of length $2p + 1$ contributes $p + 1$ independent vertices, we have

$$n = \frac{1}{2}\,\nu - t.$$

Therefore, $t = \frac{1}{2}\,\nu - n$ is fixed. Now

$$\begin{aligned} \varepsilon &= \frac{1}{2}\, n_1 + (\nu - n_1) \\ &= \nu - \frac{1}{2}\, n_1 \,. \end{aligned}$$

Since t is fixed, ε is minimum when n_1 is maximum (i.e. when $\nu(H)$ is minimum). This happens when the cycles are of the smallest possible length, namely 5. This proves that G_2 is the unique edge-minimal graph and completes the proof of the theorem. □

Theorem 3.1 answers the original question posed by P. Erdös.

Let $G \in \mathcal{G}(\nu,2,r)$. For every vertex $u \in G$, $G[\overline{N}(u)]$ is a clique as otherwise $\alpha(G) > 2$. Note that, as in Section 2, $\overline{N}(u)$ denotes the set of vertices of G not joined to u. A consequence of the fact that $G[\overline{N}(u)]$ is a clique is that

$$\delta(G) \geq \nu - r - 1. \tag{4.1}$$

Further, for $r + 1 \leq \nu \leq 2r$, the unique edge-minimal graph is

$$G = K_r \cup K_{\nu-r}.$$

So our concern is with the case $\nu > 2r$. We begin with the following simmple lemma.

Lemma 4.1. For $\nu > 2r$ there exists an edge-minimal graph $G \in \mathcal{G}(\nu,2,r)$ containing two disjoint r-cliques.

Proof: Let K_r denote an r-clique in G and let $H = G[V(G)\setminus V(K_r)]$. Suppose that $c\ell(H) < r$. Let u be a vertex of minimum degree in H. Since H is not a clique, $d_H(u) < \nu - r - 1$. Hence, by (4.1), u is joined in G to vertices in K_r. Consider the graph G' formed from G by deleting the edges joining u to the vertices of K_r and then adding edges to make u adjacent to every vertex in H. Observe that $G' \in \mathcal{G}(\nu,2,r)$ and $\varepsilon(G') \leq \varepsilon(G)$ since $d_G'(u) = \nu - r - 1 \leq \delta(G)$. We can repeat this operation until we get an r-clique disjoint from K_r. This completes the proof of the lemma. □

For $2r < \nu \leq \frac{5}{2} r$ consider the graph G_3 defined in Figure 4.1. Note that the "double line" between two graphs represents the join, i.e. denotes all possible edges between the vertices of the two graphs. It is clear that $G_3 \in \mathcal{G}(\nu,2,r)$, and

$$\varepsilon(G_3) = r(r - 1) + \frac{1}{2}(\nu - 2r)(6r - \nu - 1).$$

We now prove that G_3 is edge-minimal.

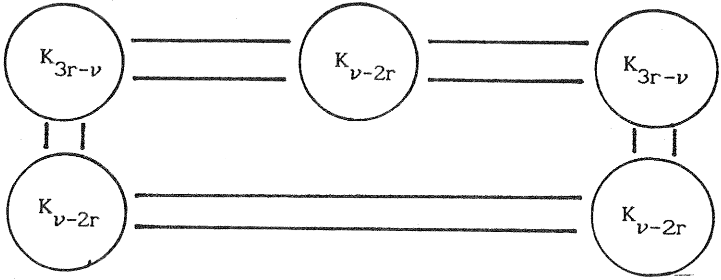

Figure 4.1 The graph G_3

Theorem 4.1. For $2r < \nu \le \frac{5}{2}r$

$$f(\nu,2,r) = r(r-1) + \frac{1}{2}(\nu - 2r)(6r - \nu - 1). \qquad (4.2)$$

Proof: Let G be an edge-minimal graph of $\mathcal{G}(\nu,2,r)$ containing two disjoint r-cliques. Such a G exists by Lemma 4.1. Let A and C denote the vertices of the two r-cliques and let B denote the remaining vertices of G. We now examine in detail the subgraph $G[A \cup B]$.

Since G[A] is a maximum clique, each vertex of B is joined in $\overline{G}$ to some vertex of A. Let $\overline{M}$ be a maximum matching in $\overline{G}$ between the vertices of A and the vertices of B. Denote the $\overline{M}$-saturated vertices of A and B by R and X, respectively. Let $S = A\backslash R$ and $Y = B\backslash X$. Note that in G every vertex of Y is joined to every vertex of S. Figure 4.2 illustrates our notation. Note that we are drawing G, so the "dashed lines" indicate edges in $\overline{G}$.

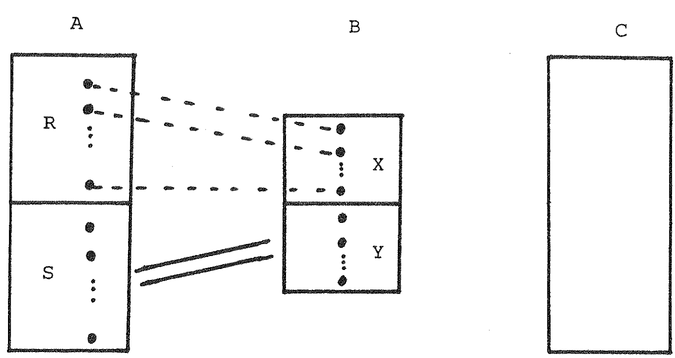

Figure 4.2

Consider the subgraph $H = G[R \cup X \cup Y]$. We will prove that H contains a maximum clique of size $|R|$. Suppose Q is a maximum clique in H. Let $T = V(Q) \cap B$ and denote by T' those vertices of T that are adjacent to every vertex of S. If $T' = \phi$, then $V(Q) \cap Y = \phi$ and hence the size of the clique cannot exceed $|R|$. We may therefore assume that $T' \neq \phi$.

Let W denote the set of vertices in R that are not joined to at least one vertex in T'. $|W| \geq |T'|$, as otherwise the subgraph $G[S \cup T' \cup (R\setminus W)]$ contains a clique of size greater than r. Every vertex of W is $\overline{M}$-saturated in $\overline{G}$. Let U denote the mates of W in B under the matching $\overline{M}$. We will show that $U \cap (T\setminus T') = \phi$.

Observe that if $U \cap T = \phi$, then $|V(Q)| < |R|$. So $U \cap T \neq \phi$. Let $x_0 \in U \cap (T\setminus T')$. We find it convenient to consider the edges of $\overline{G}$ as being coloured. In particular, suppose the edges of $\overline{M}$ are coloured blue and those edges joining vertices of W to vertices of T' are coloured red. We will construct a path P starting at x_0 and ending at some vertex x_t in $T' \cap Y$ whose edges alternate in colour beginning with blue and ending with red.

By definition there is a blue edge in $\overline{G}$ incident to x_0. Let x_1 be the other end of this edge. Clearly $x_1 \in W$. Now by the definition of W and the fact that $\overline{M}$ is a matching, x_1 has a red edge incident to it. Let x_2 be the other end of this edge. Then $x_2 \in T'$. If $x_2 \notin Y$, then there is a blue edge incident to x_2 whose other end x_3 is in W. Moreover, the definition of W and $\overline{M}$ imply that a red edge is incident to x_3 whose other end x_4 is in T'. Continuing in this way we get a path $P = x_0,x_1,x_2,\ldots,x_t$ whose edges alternate in colour, vertices $x_1,x_3,x_5,\ldots,x_{t-1}$ belonging to W, vertices $x_2,x_4,x_6,\ldots,x_t$ belonging to T' and $x_t \in Y$. Now since x_0 is not in S' there must be a vertex, u say, in S such that $x_0u \notin E(G)$. But then

$$\overline{M}' = (\overline{M}\setminus\{x_0x_1,x_2x_3,x_4x_5,\ldots x_{t-2}x_{t-1}\})$$
$$\cup\{x_0u,x_1x_2,x_3x_4,\ldots,x_{t-1}x_t\}$$

is a matching in G having more edges than M, a contradiction. This proves that $U \cap (T \setminus T') = \phi$. It thus follows that H contains a clique of size $|R|$.

Consider a vertex $u \in S$. The graph G′ obtained from G by making u adjacent to every vertex of B and non-adjacent to any vertex of C is in the class $\mathcal{G}(\nu,2,r)$. Furthermore, since $d_G{'}(u) = \nu - v - 1 = \delta(G)$, $\varepsilon(G') \le \varepsilon(G)$ and hence G′ is edge-minimal. Consequently we can assume that every vertex of S is joined to every vertex of B and to no vertex of C.

The above arguments can be applied to the subgraph $G[B \cup C]$. Thus we can identify four sets X', Y', R' and S', and a subgraph $H' = G[R' \cup X' \cup Y']$. Note that the sets X' and R' are determined by a maximum matching $\overline{M}'$ in $\overline{G}$ between the vertices of B and C. Of course, $S' = C \setminus R'$ and $Y' = B \setminus X'$. We can conclude, from the above discussion, that $c\ell(H') = |R'|$ and without any loss of generality every vertex of S' is joined to every vertex of B and to no vertex of A.

Now consider a vertex u of B and the subgraph $G' = G[R \cup B \cup R']$. Suppose $N_1 = N(u) \cap R$, $N_2 = N(u) \cap B$ and $N_3 = N(u) \cap R'$, and let $n_i = |N_i|$. So $d_G{'}(u) = n_1 + n_2 + n_3$. Since the subgraphs H and H′ induced by the vertices of $R \cup B$ and $R' \cup B$ have clique numbers of $|R|$ and $|R'|$, respectively we must have (since $\alpha(G) = 2$):

$$|B| \le \min(n_1 + n_2 + 1,\ n_2 + n_3 + 1).$$

Consequently

$$\begin{aligned} d_G{'}(u) &= n_1 + n_2 + n_3 \\ &\ge |B| - 1 + \tfrac{1}{2}(n_1 + n_3). \end{aligned}$$

Thus $d_G{'}(\cup) \ge |B| - 1$ with equality possible only if $n_1 = n_3 = 0$. Now in G we have

$$
\begin{aligned}
d_G(u) &= |S| + |S'| + d_{G'}(u) \\
&\geq |S| + |S'| + |B| - 1 \\
&= 2r - |R| - |R'| + |B| - 1 \\
&\geq 2r - |B| - 1 \qquad (|R| \leq |B|,\ |R'| \leq |B|) \\
&= 4r - \nu - 1.
\end{aligned}
$$

Therefore, since $\delta(G) \geq \nu - r - 1$, we have

$$
\begin{aligned}
\varepsilon(G) &\geq r(\nu - r - 1) + \frac{1}{2}(\nu - 2r)(4r - \nu - 1) \\
&= r(r - 1) + \frac{1}{2}(\nu - 2r)(6r - \nu - 1).
\end{aligned}
$$

Now we have a graph, namely G_3, which is in the class $\mathcal{G}(\nu,2,r)$ and $\varepsilon(G_3)$ satisfies (4.2). This completes the proof of the Theorem. □

We remark that G_3 is not the unique extremal graph.

The above theorem gives no information when $\nu > \frac{5}{2}r$. We now describe a construction for the particular case $\nu = 3r$. Our basic building block is the graph H drawn in Figure 4.3. We first

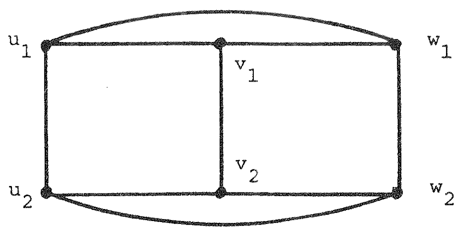

Figure 4.3 The Graph H

give the construction for r even.

Let $r = 2t$, $t \geq 2$. We construct the graph G_{2t} as follows. Take t disjoint copies $H_1, H_2, \ldots, H_t$ of H. The vertices of H_i are identified by assigning a superscript i to the vertex labels of H. Thus $V(H_i) = \{u_1^i, u_2^i, v_1^i, v_2^i, w_1^i, w_2^i\}$. Join the vertices of H_i to the vertices of each H_j, $1 \le j \le i - 1$ as follows: vertices u_1^i and u_2^i are joined to u_1^j, u_2^j, v_1^j and v_2^j; vertices v_1^i and v_2^i are joined to v_1^j, v_2^j, w_1^j and w_2^j; vertices w_1^i and w_2^i are joined to w_1^j, w_2^j, u_1^j and u_2^j. Call the resulting graph G_{2t}. We will now prove that $G_{2t} \in \mathcal{G}(6t,2,2t)$.

Lemma 4.2. For $t \ge 2$, $G_{2t} \in \mathcal{G}(6t,2,2t)$.

Proof: We can consider the vetices of G_{2t} as being on an r x 3 grid with columns $C_1 = \{u_1^i, u_2^i : 1 \le i \le t\}$, $C_2 = \{v_1^i, v_2^i : 1 \le i \le t\}$ and $C_3 = \{w_1^i, w_2^i : 1 \le i \le t\}$. The vertices of H_i are in rows $2i - 1$ and $2i$. Observe that the vertices in any column form an r-clique and the vertices in any row form a triangle. Further, a vertex x of H_i belonging to column C_j ($j = 2i - 1$ or $2i$) is joined to every vertex of G_{2t} except: two vertices of H_i one in column C_{j-1} and one in column C_{j+1} (the subscripts are written modulo 3); any vertex of column C_{j-1} not in H_i lying in any row above x; any vertex of column C_{j+1} not in H_i lying in any row below x. Note that the vertices of C_{j-1} and C_{j+1} not joined to x form a clique in G_{2t} of size r. As x is an arbitrary vertex of G_{2t} we can conclude that G_{2t} is $(2r - 1)$-regular and the non-neighbours of any vertex form a clique of size r.

Now we show that $c\ell(G_{2t}) = r$. Let Q be the largest clique in G_{2t}. Consider a vertex x say, occurring in the highest row. Suppose, without any loss of generality that $x \in C_1$. Since x is not joined to any vertex of C_2 belonging to any row below the row in which x is, the vertices of Q are contained in columns C_1 and C_3. Now observing that in any H_i a vertex of C_1 is joined to only one vertex of C_3, we can conclude that Q has size r. This completes the proof of the lemma. □

We consider now the case when r is odd. Let $r = 2t + 1$, $t \ge 2$. We form the graph G_{2t+1} as follows. First we take the graph G_{2t} defined above. We add three new vertices u_1^{2t+1}, v_1^{2t+1}, and w_1^{2t+1}.

Join u_1 to every vertex of G_{2t} in columns C_1 and C_2, and to the vertex w_1^{2t+1}. Join v_1^{2t+1} to every vertex of G_{2t} in columns C_2 and C_3, and to the vertex u_2^{2t}. Join w_1^{2t+1} to every vertex of G_{2t} in columns C_1 and C_3 except u_2^{2t}, and to the vertex v_1^{2t}. Observe that G_{2t+1} has one vertex of degree 2r (namely v_1^{2t}) and every other vertex has degree 2r − 1. The non-neighbours of every vertex of G_{2t+1} form a clique. Further, it is not too difficult to establish that $c\ell(G_{2t+1}) = 2t + 1$. We thus have:

Lemma 4.3 For $t \geq 2$, $G_{2t+1} \in \mathcal{G}(6t + 3,2,2t + 1)$. □

We have noted above that G_{2t} is (2r − 1)-regular and G_{2t+1} has all vertices except one having degree 2r − 1; the exceptional vertex has degree 2r. Thus (4.1) together with lemmas 4.2 and (4.3) yield:

Theorem 4.2. For $r \geq 4$.

$$f(3r,2,r) = \left\lceil \frac{1}{2}(3r)(2r - 1) \right\rceil . \qquad \square$$

Theorems 4.1 and 4.2 leave unresolved the range $\frac{5}{2}r < \nu < 3r$. The constructions for this range are somewhat complex and the arguments needed to establish their minimality are lengthy. We intend to describe these constructions and report on a number of other results in a subsequent paper.

REFERENCE

1. J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, North Holland (1976).