Bhaskar Rao Ternary Designs and Applications

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Abstract

Generalized Bhaskar Rao n-ary Designs with elements from abelian groups are defined. This paper studies a special case of Generalized Bhaskar Rao n-ary Designs called Bhaskar Rao Ternary Designs. A Bhaskar Rao Ternary Design, X, is a v x b matrix of 0's, ± 1's and ± 2's such that the inner product of any two rows is 0 and the matrix obtained by replacing each entry of X by its absolute value is the incidence matrix of a Balanced Ternary Design. Applications of these designs to the construction of infinite families of Balanced Ternary Designs and Partially Balanced Ternary Designs are presented. Some construction methods and necessary conditions for the existence of Bhaskar Rao Ternary designs are given. A necessary condition for the existence of balanced ternary designs with even Λ and block size 4t is given.

1. Introduction

We shall assume that the reader is familiar with the concept of a **balanced incomplete block design (BIBD)** with parameters (v,b,r,k,λ), and the **incidence matrix** of a BIBD; for example see Street and Street(1987). A **balanced n-ary design**, introduced by Tocher(1952) in a slightly different form, is similar to BIBD except that its blocks are multisets and any point may appear in a block at most n-1 times. For an excellent survey on n-ary designs

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see Billington(1984). When n=3, these designs are called **balanced ternary designs (BTDs)**. A BTD on V points is a collection of B multisets (called blocks) of size K, where each element occurs singly in ρ_1 blocks and repeated in ρ_2 blocks , such that each pair of distinct elements occurs Λ times in the design. Clearly each element will occur a constant number, say R $= \rho_1 + 2\rho_2$ times. A block say, aabc, of size 4 with an element a occurring twice and elements b and c occurring singly is said to have the pairs (a,b) and (a,c) twice and the pair (b,c) once. We say that the BTD has parameters (V, B, ρ_1, ρ_2, R; K, Λ). The incidence matrix $N = (n_{ij})$ of a BTD is a V x B matrix and its (i,j)th entry n_{ij} is equal to the number of times the point i occurs in the jth block. Let $\Delta = \Sigma n_{ij}^2$ where the sum is over j. Saha(1975) has shown that Δ is independent of any row. In fact for BTDs, $\Delta = \rho_1 + 4\rho_2 = RK - \Lambda V + \Lambda$. The existence for these designs with block size three is proved by Billington(1985) and families of BTDs with block size four are obtained in Donovan(1986a,1986b, 1986c) but as in the case of BIBDs the general problem is still open. Billington and Robinson(1983) have a list of BTDs with $R \leq 15$ and several necessary conditions for the existence of BTDs. Dillon and Wertheimer(1985) have used other combinatorial designs for example group divisible designs and weighing designs, to obtain several of the designs which are listed in Billington and Robinson(1983) as unsolved or were obtained by computer search. Using **Bhaskar Rao Designs** Sarvate and Seberry(to appear) also have obtained designs which are nonisomorphic to the solutions given in Billington and Robinson(1983) or the solution was given by listing all the blocks. Many authors, for example Misra(1988), Patwardhan and Sharma(1988), Saha and Dey(1973) Sarvate(1989,1990), Sinha, Mathur and Nigam(1979), Sinha and Saha(1979) have obtained various partially balanced and balanced ternary designs whereas Vartak and Diwanji(1989) have constructed column-regular BTDs. Kageyama (1980) and Dillon and Wertheimer(1985) have obtained a characterization of certain balanced n-ary and ternary designs. Patwardhan, Dandwate and Vartak(1984) have used balanced orthogonal designs to obtain generalized partially balanced ternary designs with two associate classes and with triangular association scheme. Billington and Hoffman(to appear) have proved that a balanced ternary design with block size 3, index 2 and $\rho_2 = 2$ exists which contains exactly k pairs of repeated blocks if and only if $v \equiv 0$ or 2 modulo 3, $v \geq 5$, and $0 \leq k \leq v(v-5)/6$, $k+1 \neq v(v-5)/6$. In this paper we obtain some constructions of n-ary designs through a generalization of Bhaskar Rao designs.

Generalized Bhaskar Rao designs on binary designs are studied by a number of authors such as Bhaskar Rao(1966,1970), de Launey(1989), de Launey and Seberry(1984), Gibbons and Mathon(1987), Koukouvinos, Kounias, and Seberry (to appear), Lam and Seberry(1984), Palmer and Seberry(1989), Sarvate and Seberry(to appear), Seberry(1985), Singh(1982), Street and Rodger(1980), Vyas(1982) and the references therein. These authors have used GBRDs to construct BIBDs and PBIBDs. Curran and Vanstone(1988/89) constructed previously unknown resolvable BIBDs by using GBRDs. As mentioned earlier in Sarvate and Seberry (to appear) BRDs are used to construct n-ary designs. The aim of this note is to define GBRDs for n-ary designs, obtain some necessary conditions for the existence of Bhaskar Rao Ternary Designs (BRTDs), give some construction methods and then use these designs to construct n-ary designs, where our emphasis is on ternary designs. It is interesting to note that when we modified one of the methods of construction of PBIBDs by BRDs, using BRTDs we were able to construct balanced n-ary designs with little modification. This result encourages us to modify all the known methods of constructing block designs from matrices with group elements to the case where matrices are Generalized Bhaskar Rao n-ary Designs. Keeping with this spirit and also because we need these result for the construction of BRTDs and BTDs, in this paper several results from Seberry(1984) are being modified for the BRTD case.

Definition: Suppose we have a matrix W with elements as integral multiples of a finite group $G = \{h_1, h_2, \ldots, h_g\}$ where $W = h_1A_1 + h_2A_2 + \ldots + h_gA_g$

and $A_1, A_2, \ldots, A_g$ are $v \times b$ $(0,1,2,\ldots,n-1)$-matrices, (n is a positive integer greater than or equal to 2) and the Hadamard product $A_i * A_j$, $i \neq j$ is zero. Suppose $(t_1 a_{i1}, \ldots, t_b a_{ib})$ and $(s_1 b_{j1}, \ldots, s_b b_{jb})$ are the ith and jth rows of W then we define WW' by $(WW')_{ij} = (t_1 a_{i1}, \ldots, t_b a_{ib}) \cdot (s_1 b_{j1}^{-1}, \ldots, s_b b_{jb}^{-1})$ with $\cdot$ the scalar product. Then W is a **generalized Bhaskar Rao n-ary design** or GBRn-aryD if

(i) $WW' = \Delta I + \sum_{i=1}^{m} (c_i G) B_i$ and

(ii) $N = A_1 + \ldots + A_g$ satisfies $NN' = \Delta I + \sum_{i=1}^{m} \Lambda_i B_i$,

that is N is the incidence matrix of a partially balanced n-ary design, and $(c_i G)$ gives the number of times a complete copy of the group G occurs. In this paper we shall only be concerned with $n = 3$, $m = 1$, $c = \frac{\Lambda}{g}$ and $B_1 = J-I$. That is, in this case N is the incidence matrix of a balanced ternary design. So the above equations become:

(i) $WW' = \Delta I + \frac{\Lambda}{g} G\,(J-I)$;

(ii) $NN' = (RK-\Lambda V)I + \Lambda J$.

Such a matrix W is denoted by $GBRTD(V,B,\rho_1,\rho_2,R;K,\Lambda;G)$ or $GBRTD(V,K,\Lambda;G)$ when the values of ρ_1 and ρ_2 are clear from the context.

Example 1. $GBRTD(3,9;3,3,9;3,6;Z_3)$, X, is given below:

$$X = \begin{matrix} 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ w & 2 & 0 & w^2 & 2 & 0 & 1 & 2 & 0 \\ 0 & w & 2w & 0 & w^2 & 2w^2 & 0 & 1 & 2 \end{matrix}$$

One can check that $XX' = 15I + 2Z_3(J-I)$ as required.

Now consider the case when $n = 3$, that is, N is the incidence matrix of a ternary design and $G = Z_2 = \{1,-1\}$. In this case we will refer to it as a Bhaskar Rao Ternary Design, $BRTD(V,B;\rho_1,\rho_2,R;K,\Lambda)$.

Example 2. BRTD(3,6;2,2,6;3,4;Z_2) = BRTD(3,6;2,2,6;3,4):

$$\begin{bmatrix} 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & 0 & -1 & -2 \end{bmatrix}$$

As is common in the case of binary designs we refer to a BRTD as a signed matrix and the process of labelling + or - to the elements of the incidence matrix of a BTD (Balanced Ternary Design) as signing the BTD. Also notice that for a BRTD W, WW' = $(R+2\rho_2)I$. Now we will give one more example of BRTD with block size four as we will use it to construct families of BTDs and BRTDs. This BRTD is a member of a series of BRTDs obtained by cyclic difference sets in Francel and Sarvate(to appear).

Example 3. BRTD(6,12;4,2,8;4,4).

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & -1 \\ 1 & 2 & 0 & -1 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 & 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

2. Some Necessary Conditions

A trivial necessary condition for the existence of a BRTD is

Theorem 2.1. *A necessary condition for the existence of a BRTD is that* Λ *should be even.*

The following Theorem and its proof is based on Theorem 2.3 of Bhaskar Rao(1970). A similar result for Bhaskar Rao Designs can be found in Street and Rodger (1980, Theorem 5)

Theorem 2.2. *A necessary condition for the existence of BRTD* W *is that when* $K \equiv 3 \pmod 4$ *then* $\frac{1}{4}BK(K-1) - \frac{1}{2}V\rho_2$ *must be even, and when* K-1 *is 4 times an odd integer then* $2B - V\rho_2 \equiv 0 \pmod 4$.

Proof. Let $W = A - B$, where $A + B = N$ is the underlying BTD. Then

$$AA' + BB' = \frac{1}{2}(NN' + WW')$$
$$= \frac{1}{2}((RK-\Lambda V)I + \Lambda J + (R+2\rho_2)I)$$

Consider

$$J_{1,V}(A : B)(A : B)' J_{V,1} = J_{1,V}(AA'+BB')J_{V,1}$$
$$= \frac{1}{2}J_{1,V}((RK-\Lambda V)I + \Lambda J + (R+2\rho_2)I)J_{V,1}$$

The left hand side of the above equation is $\Sigma K_j^2 + (K-K_j)^2$, where K_j is the jth column sum of A. The right hand side of the above equation is

$$\frac{1}{2}VR(K+1) + V\rho_2 = \frac{1}{2}BK(K+1) + V\rho_2.$$

Now using $BK^2 = \Sigma(K_j+k-K_j)^2$ and the above equation, we get

$$\Sigma(K-K_j)K_j = \frac{1}{4}(BK(K-1) - 2V\rho_2).$$

Now if $K \equiv 3 \pmod 4$ or $K-1 = 4(2s+1)$, $s \geq 1$, the left hand side of the above equation is even. Arithmetical manipulation on the right hand side now gives the result.

$\square$

Corollary 2.3. *A necessary condition for the existence of a BRTD when* $K \equiv 3 \pmod 4$ *is* :

If $V\rho_2 \equiv 0 \pmod 4$ *then* $B \equiv 0 \pmod 4$;

If $V\rho_2 \equiv 1 \pmod 4$ *then either* $B \equiv 1 \pmod 4$ *and* $K(K-1) \equiv 1 \pmod 4$
or $B \equiv 3 \pmod 4$ *and* $K(K-1) \equiv 3 \pmod 4$;

If $V\rho_2 \equiv 2 \pmod 4$ *then* $B \equiv 2 \pmod 4$;

If $V\rho_2 \equiv 3 \pmod 4$ *then either* $B \equiv 1 \pmod 4$ *and* $K(K-1) \equiv 3 \pmod 4$
or $B \equiv 3 \pmod 4$ *and* $K(K-1) \equiv 1 \pmod 4$.

As one can see, Theorem 2.2 does not give any information about the nonexistence of BRTDs when K is even. The next theorem based on a result of Seberry(1984, Theorem 1) modifies the above result.

The following table gives some of the designs which do not satisfy the necessary condition in Theorem 2.2 and so BRTDs with these parameters do not exist.

No.	V	B	ρ_1	ρ_2	K	Λ	No in Billington and Robinson(1983)
1.	3	7	5	1	3	6	15
2.	3	8	4	2	3	6	31
3.	3	9	3	3	3	6	54
4.	3	11	9	1	3	10	89
5.	3	12	8	2	3	10	126
6.	3	14	6	4	3	10	242
7.	3	15	13	1	3	14	270
8.	3	15	5	5	3	10	312
9.	7	14	6	2	5	6	74
10.	10	28	12	1	5	6	208
11.	10	30	9	3	5	6	296
12.	6	18	5	5	5	10	315
13.	7	7	1	3	7	6	25
14.	7	11	5	3	7	10	103
15.	14	28	8	3	7	6	234
16.	13	26	2	6	7	6	258
17.	7	15	9	3	7	14	298
18.	14	30	3	6	7	6	334
19.	19	19	9	1	11	6	95
20.	18	18	3	4	11	6	109
21.	11	11	1	5	11	10	113
22.	22	26	9	2	11	6	182
23.	11	15	5	5	11	14	321
24.	26	26	7	3	13	6	74
25.	26	26	7	3	13	6	191
26.	15	15	1	7	15	14	344.

Table 1.

Theorem 2.4. *A Bhaskar Rao ternary design* $W = BRTD(V,B;\rho_1,\rho_2,R,K,\Lambda)$, *can only exist if the equations*

(i) $x_3 + 3x_5 + 6x_7 + \ldots + (\frac{1}{8}(K^2-1))x_K = \frac{1}{8}(B(K-1)+2\rho_2 V)$ for K odd,

(ii) $-x_0+3x_4+8x_6+ \ldots + (\frac{1}{4}(K^2-4))x_K = \frac{1}{4}(B(K-4)+2\rho_2 V)$ for K even,

have integral solutions. In particular, for $K = 3$, *a* BRTD *can only exists if 4 divides* $B + \rho_2 V$ *and for* $K = 4$, *a* BRTD *can exist only when we have an integral solution for the equation* $-x_0 + 3x_4 = \frac{1}{2}\rho_2 V$.

Proof. Suppose that $WW' = \Delta I$. Suppose that the column sum of the ith column is s_i. So we have

$$\Sigma s_i^2 = (1, \ldots, 1)WW'(1, \ldots, 1) = \Delta V \quad \ldots \quad \ldots \quad \ldots (2.1)$$

If K is odd then the column sums can only be $\pm 1, \pm 3, \ldots \pm K$ and if K is even then the sum can only be $\pm 0, \pm 2, \ldots \pm K$. Hence if there are x_i columns with column sum $\pm i$, then using (2.1) we have

$$x_1 + 9x_3 + \ldots + K^2 x_K = \Delta V,$$
$$x_1 + x_3 + \ldots + x_K = B \qquad \text{for K odd}$$

and

$$4x_2 + 16x_4 + \ldots + K^2 x_K = \Delta V,$$
$$x_0 + x_2 + x_4 + \ldots + x_K = B \quad \text{for K even.}$$

Now $VR = BK$ and $\Delta = R + 2\rho_2$, we have

$$8x_3 + 24x_5 + \ldots + (K^2-1)x_K = VR - B + 2\rho_2 V$$
$$= B(K-1) + 2\rho_2 V \qquad \text{for K odd}$$

and

$$-4x_0+12x_4 + \ldots + (K^2 - 4)x_K = VR - 4B + 2\rho_2 V$$
$$= B(K-4) + 2\rho_2 V \qquad \text{for K even.}$$

Hence for $K = 3$, we have 4 divides $B+2\rho_2 V$

and for $K = 4$, we have $-x_0 + 3x_4 = \frac{1}{2}\rho_2 V$.

□

Unfortunately for K= 4 the above theorem on its own can not directly give any result because the conditions $\Lambda(V-1) = R(K-1)-2P_2$ and $VR=BK$ can be used to obtain a necessary condition for the existence of a BTD with even Λ and block size 4t.

Theorem 2.5. *No BTD with even Λ and block size 4t exists if VP_2 is odd.*

3. Special Cases.

3.1. Block size 2: A ternary design with block size two is of no interest to us as it will be then only a binary design.

In this case Bhaskar Rao(1970) has shown that there always exist a BRDs with parameters (v, tv(v-1), 2t(v-1), 2, 2t).

From now on we will be concerned with ternary designs where $\rho_1\rho_2 > 0$. It is well known that any balanced ternary design is regular; that is, each element occurs i times in ρ_i blocks, i = 1,2, where ρ_1 and ρ_2 are constant for the design. (See Corollary 2.4 of Billington(1984)). Recall that the understanding is that balanced ternary designs means balanced equireplicate ternary designs.

3.2. Block size 3: Billington(1985) has proved the following result.

Theorem 3.1. *Necessary and sufficient conditions for the existence of a BTD with* K=3 *are*

(i) *V is congruent mod 6 to a value as given below,*

(mod 6)

		2	3	4	5	0	1
ρ_2	0	0,1,3,4	1,3,5	0,1,3,4	1,3	0,1,2,3,4,5	1,3
(mod 3)	1	0,3	3	0,2,3,5	3	0,3	3,5
	2	0,2,3,5	3	0,3	3,5	0,3	3

and

$$\text{(ii)}\ V \geq \begin{cases} \lfloor 4\rho_2 / \Lambda \rfloor + 1 & \text{if } \Lambda \text{ is even,} \\ \lfloor 4\rho_2 / (\Lambda-1) \rfloor + 1 & \text{if } \Lambda \text{ is odd.} \end{cases}$$

The following theorem is also from Billington(1985).

Theorem 3.2. *A* BTD *with* K=3, *any* Λ *and* ρ_2 *exists for all* $V \equiv 3 \pmod 6$ *satisfying Theorem 3.1(ii).*

Using the same designs constructed in the proof of the above Theorem in Billington(1985), we can prove :

Theorem 3.3. *A* BRTD *with* K = 3 *and* $V \equiv 3$ (mod 6) *exists for any even* $\Lambda \equiv 0$ (mod 4).

Proof. The construction for BTD with K=3 and V≡3(mod 6) given in Billington (1985) is reproduced in this proof for easy reference: We know that there exists a resolvable Steiner triple system (STS) on 6a+3, a ≥ 0 elements, for example see Ray-Chaudhuri and Wilson(1971). Take Λ identical copies of such a resolvable STS. Remove two identical copies of ρ_2 resolutions, and for each pair of blocks xyz , xyz that is removed, replace by the new blocks xxy, yyz, zzx. The remaining blocks are taken unaltered. Consider the incidence matrix N of the ternary design so obtained. We will sign N to produce the required BRTDs. Other than the blocks from the resolution classes which are changed, each block occurs $\Lambda = 4t$ times. Sign the corresponding entries by the rows of Hadamard matrix of size 4. The remaining blocks occur $\Lambda-2 = 4(t-1)+2$ times. Sign the 4(t-1) occurrences of each of the remaining blocks by the rows of Hadamard matrix of order 4. Keep the remaining two occurrences positive. Now each column corresponding to new block has a 2, sign it by -. We get the required BRTD.

□

Example 4. Consider the blocks, written as columns, of STS(9,3,1) with resolution classes:

1 4 7	1 2 3	1 2 3	1 2 3
2 5 8	4 5 6	6 4 5	5 6 4
3 6 9	7 8 9	8 9 7	9 7 8

We will construct BRTD(9,3,4;Z_2), ρ_2=2 as follows: first construct BTD(9,3,4); we take four copies of STS(9,3,1) and replace the two copies of the first two resolution classes by new blocks

1 2 3	4 5 6	7 8 9	1 4 7	2 5 8	3 6 9
1 2 3	4 5 6	7 8 9	1 4 7	2 5 8	3 6 9
2 3 1	5 6 4	8 9 7	4 7 1	5 8 2	6 9 3

The corresponding incidence matrix is given below, to save space let us denote the consecutive entries 2 0 1 by a_1, 1 2 0 by a_2 , 0 1 2 by a_3, and for i = 0 and 1, i i by i_2 , i i i by i_3, i i i i by i_4 ,

$$\begin{bmatrix}
a_1 & 0_3 & 0_3 & 1_2 & 0_2 & 0_2 & a_1 & 0_3 & 0_3 & 1_2 & 0_2 & 0_2 & 1_4 & 0_4 & 0_4 & 1_4 & 0_4 & 0_4 \\
a_2 & 0_3 & 0_3 & 1_2 & 0_2 & 0_2 & 0_3 & a_1 & 0_3 & 0_2 & 1_2 & 0_2 & 0_4 & 1_4 & 0_4 & 0_4 & 1_4 & 0_4 \\
a_3 & 0_3 & 0_3 & 1_2 & 0_2 & 0_2 & 0_3 & 0_3 & a_1 & 0_2 & 0_2 & 1_2 & 0_4 & 0_4 & 1_4 & 0_4 & 0_4 & 1_4 \\
0_3 & a_1 & 0_3 & 0_2 & 1_2 & 0_2 & a_2 & 0_3 & 0_3 & 1_2 & 0_2 & 0_2 & 0_4 & 1_4 & 0_4 & 0_4 & 0_4 & 1_4 \\
0_3 & a_2 & 0_3 & 0_2 & 1_2 & 0_2 & 0_3 & a_2 & 0_3 & 0_2 & 1_2 & 0_2 & 0_4 & 0_4 & 1_4 & 1_4 & 0_4 & 0_4 \\
0_3 & a_3 & 0_3 & 0_2 & 1_2 & 0_2 & 0_3 & 0_3 & a_2 & 0_2 & 0_2 & 1_2 & 1_4 & 0_4 & 0_4 & 0_4 & 1_4 & 0_4 \\
0_3 & 0_3 & a_1 & 0_2 & 0_2 & 1_2 & a_3 & 0_3 & 0_3 & 1_2 & 0_2 & 0_2 & 0_4 & 0_4 & 1_4 & 0_4 & 1_4 & 0_4 \\
0_3 & 0_3 & a_2 & 0_2 & 0_2 & 1_2 & 0_3 & a_3 & 0_3 & 0_2 & 1_2 & 0_2 & 1_4 & 0_4 & 0_4 & 0_4 & 0_4 & 1_4 \\
0_3 & 0_3 & a_3 & 0_2 & 0_2 & 1_2 & 0_3 & 0_3 & a_3 & 0_2 & 0_2 & 1_2 & 0_4 & 1_4 & 0_4 & 1_4 & 0_4 & 0_4
\end{bmatrix}$$

We sign the above incidence matrix by signing each occurrence of 2 by - and then sign each of the last six columns (where 1_4 occurs 3 times in each column) by using the first three rows of H_4, Hadamard matrix of size 4.

The rows of an Hadamard Matrix of size 4 can be used to obtain following special cases of Theorem 4.1:

Theorem 3.4. *If a* BTD(V,B;ρ_1,ρ_2,R;3,Λ)= X *exists then a* BRTD(V,4tB;4tρ_1,4tρ_2, 4tR;3,4tΛ) *exists for all positive integers t.*

Proof. Let X denotes the incidence matrix of X. Replace in a column the ith 1 by the ith row of the Hadamard matrix if that column consists of 1's only. and replace it by 1 -1 1 -1 if the column contains a 2. Now t copies of the resulting BRTD give the result.

□

Corollary 3.5. *If a* BTD(V,B;ρ_1,ρ_2,R;3,2)= X *exists then a* BRTD(V,4tB;4tρ_1,4tρ_2, 4tR;3,8t) *exists for all positive integers t.*

Now we can use Theorem 3.1 and give similar corollaries to construct various families of BRTDs for K=3.

Similarly by replacing the ith nonzero entry x of BTD by x times the ith row of H_4 we can prove:

Theorem 3.6. *If a* BTD(V,B;ρ_1,ρ_2,R;4,Λ)= X *exists then a* BRTD(V,4tB;4tρ_1,4tρ_2, 4tR;4,4tΛ) *exists for all positive integers t.*

3.2.1. Λ=2: When Λ=2 we have the following necessary and sufficient conditions for the existence of BTDs obtained from Theorem 3. 1:

When $\rho_2 \equiv 0 \pmod 3$ $V \equiv 0,1,3,4 \pmod 6$,

When $\rho_2 \equiv 1 \pmod 3$ $V \equiv 0,3 \pmod 6$,

When $\rho_2 \equiv 2 \pmod 6$ $V \equiv 0,2,3,5 \pmod 6$,

and $V \geq 2\rho_2 + 1$

Theorem 3.7. *A* BRTD(V,B;ρ_1,ρ_2>0,R;K,2) *does not exist.*

Proof. Recall we are concerned with ternary designs with $\rho_2 > 0$. Therefore the inner product of at least one pair of signed rows is either +2 or - 2.

$\square$

Corollary 3.8. *A* BRTD(V,B;ρ_1,ρ_2>0,R;3,2) *does not exist.*

An obvious generalization of Theorem 3.7 is

Theorem 3.9. *A necessary condition for the existence of a signed n-ary design* (ρ_{n-1}>0) *is that* Λ *be even and greater than or equal to* 2(n-1).

The existence problem for BRTD when the block size is three is under investigation in Francel and Sarvate(to appear).

4. General Constructions.

Theorem 3.4 can be generalized for any K, where we use a Hadamard matrix of order 4[K/4] provided it exists, where [x] is the least integer greater than or equal to x. The result is similar to Theorem 6 of Street and Rodger(1980).

Theorem 4.1. *Let* N *be a* BTD(V,B;ρ_1,ρ_2,R;K,Λ) *and* X=A-B be a BRTD(V,sB; sρ_1,sρ_2,sR;K,sΛ). *Then if s is as small as possible, s* $\leq$ 4[K/4] *assuming that a Hadamard matrix of order* 4[K/4] *exists.*

Proof. Let H be a Hadamard matrix of order 4[K/4]. In N replace the ith nonzero entry t of any column by t times the ith row of H and replace 0's by 4[K/4] 0's.

$\square$

Example 5. Let N be a BTD(3,4;2,1,4;3,3),

$$N = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad H = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array}$$

then

$$X = \begin{bmatrix} 2\ 2\ 2\ 2 & 1\ 1\ 1\ 1 & 0\ 0\ 0\ 0 & 1\ 1\ 1\ 1 \\ 1\ 1\ \text{-}1\ \text{-}1 & 0\ 0\ 0\ 0 & 2\ 2\ 2\ 2 & 1\ 1\ \text{-}1\ \text{-}1 \\ 0\ 0\ 0\ 0 & 2\ 2\ \text{-}2\ \text{-}2 & 1\ 1\ \text{-}1\ \text{-}1 & 1\ \text{-}1\ 1\ \text{-}1 \end{bmatrix} = \begin{bmatrix} 2r_1 & r_1 & 0 & r_1 \\ r_2 & 0 & 2r_1 & r_2 \\ 0 & r_2 & r_2 & r_3 \end{bmatrix}$$

The above result can be generalized to obtain a result similar to Theorem 4 of Seberry(1984):

Theorem 4.2. *Suppose we have a* BTD($V,B;\rho_1,\rho_2,R;K,\Lambda$) *and a* BRD($K,a,s,j,\lambda$). *Then there exist a* BRTD(V, $Ba;s\rho_1$, $s\rho_2$, sR; j, $\lambda\Lambda$).

Proof. Let B be the BTD and W be the BRD. Replace the jth non-zero element say t of each column of B by t times the j-th row of W to obtain the required BRTD.

□

Corollary 4.3. *If a* BTD($V,B;\rho_1,\rho_2,R;K,\Lambda$) *exists and* $K(K\text{-}1) \equiv 0 \pmod{12}$ *then a* BRTD(V, $BK(K\text{-}1)/3$; $(K\text{-}1)\rho_1$, $(K\text{-}1)\rho_2$, $(K\text{-}1)R$; 3, 2Λ) *exists.*

Proof. Seberry(1985) has proved that the condition $v(v\text{-}1) \equiv 0 \pmod{12}$ is necessary and sufficient for the existence of a BRD(v,3,2).

□

Corollary 4.4. *If a* BTD($V,B;\rho_1,\rho_2,R;K,\Lambda$) *exists and* $K \equiv 1 \pmod 6$ *then a* BRTD(V, $BK(K\text{-}1)/_6$; $2(K\text{-}1)\rho_{1/3}$, $2(K\text{-}1)\rho_{2/3}$, $2(K\text{-}1)R_{/3}$; 4, 2Λ) *exists.*

Proof. This follows since de Launey and Seberry(1984) have proved that the condition $v \equiv 1 \pmod 6$ is sufficient for the existence of BRD(v,4,2).

□

Several such corollaries can be given by using the results of existence of BRDs. For example, de Launey and Seberry(1984) have shown that $v \equiv 1 \pmod 3$ is necessary and sufficient for the existence of BRD(v,4,4) so we have

Corollary 4.5. *If a* BTD($V,B;\rho_1,\rho_2,R;K,\Lambda$) *exists and* $K \equiv 1 \pmod 3$ *then* *a* BRTD(V, BK(K-1)/3; 4(K-1)$\rho_{1/3}$, 4(K-1)$\rho_{2/3}$, 4(K-1)$R_{/3}$; 4 ,4Λ) *exists.*

Now we will give important construction methods for BRTDs using Latin squares. Form the auxiliary matrices M_{ij}'s from mutually orthogonal Latin squares as in Seberry(1984) and references therein, which satisfy the following conditions:

$$\sum_{j=1}^{t} M_{aj} M_{bj}' = J, \quad a \neq b,\ 0 \leq a, b \leq t$$

and

$$\sum_{j=1}^{t} M_{aj} M_{aj}' = tI \quad , \quad 0 \leq a \leq u.$$

Write

$$C = \begin{bmatrix} I & I & \cdots & I \\ M_{11} & M_{12} & \cdots & M_{1U} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \\ M_{K-1\,1} & M_{K-1\,2} & \cdots & M_{K-1\,U} \end{bmatrix}.$$

Let A = BRTD(V,B,R,K,Λ) and B = BRTD(U,A,S,K,Λ). Form D_i by replacing the sth nonzero entry say t of each column of A by tM_{si}, i.e. by t times the sth entry of the ith column of C. Then

$$E = [\underbrace{B \oplus B \oplus \ldots \oplus B}_{\text{<--v times-->}} : D_1 : D_2 : \ldots : D_u]$$

is a BRTD (UV, BU^2+ AV, UR+S,K,Λ). It is easy to check the first four entries of the parameters. We check the value of Λ by observing that the inner product of any two rows is zero in the $[D_1 : D_2 : \ldots : D_u]$ part of E. Consider the kth columns of D_i's. The inner product of any two rows g and h will be 0, ±J or ±2J depending on whether the kth column in A contributes 0, ±1, or ±2 in the inner product of the rows g and h. As the rows of A are orthogonal we have the result. Hence we have:

Theorem 4.6. *If* BRTD(V,B,R,K,Λ) *and* BRTD(U,A,S,K,Λ) *exist and if there are* K-1 *mutually orthogonal latin squares of order* U, *then there exists a*

BRTD *with parameters* BRTD(UV,BU^2+AV,UR+S,K,Λ).

Notice that we can use a BRD and a BRTD to get a BRTD.

Corollary 4.7. *If* BRD(v,b,r,K,Λ) *and* BRTD(U,A,S,K,Λ) *exist and if there are* K-1 *mutually orthogonal latin squares of order* U, *then there exists a* BRTD *with parameters* BRTD(Uv,bU^2+Av,Ur+S,K,Λ).

Corollary 4.8. *If* BRTD(V,B,R,K,Λ) *and* BRD(u,a,s,K,Λ) *exist and if there are* K-1 *mutually orthogonal latin squares of order* u, *then there exists a* BRTD *with parameters* BRTD(uV,Bu^2+aV,uR+s,K,Λ).

Example 6. (i) Use A = BRTD(3,6,6,3,4) and B = BRD(4,8,6,3,4), we get BRTD(12, 120,30,3,4). The parameters of the underlying BTD can be obtained by doubling the BTD listed as number 303 in Billington and Robinson.
(ii) Use A = BRD(4,8,6,3,4) and B =BRTD(3,6,6,3,4), we get BRTD(12,96,24,3,4). The parameters of the underlying BTD can be obtained by doubling the BTD listed as number 114 in Billington and Robinson.
(iii) Using BRTD(3,6,6,3,4) and BRD(3,4,4,3,4) we can get BRTD(9,66,22,3,4) and BRTD(9,54,18,3,4) which are multiples of already known designs (numbers 100 and 40 of Billington and Robinson(1983)).
(iv) Similarly we get BRTDs and hence BTDs if we use A = B = BRTD(6,12;4,2,8; 4,4) or BRTD(6,12;4,2,8;4,4) and BRD(4,4,4,4,4).

Corollary 4.9. *If* $v(v-1) \equiv 0 \pmod 3$ *and* $U\equiv 3 \pmod 6$ *then a* BRTD(Uv,3,4) *and hence a* BTD(Uv,3,4) *exists. In particular* BTD(3v,6v^2;6v-4,2,6v;3,4) *and* BTD(3v,8v^2-2v;4v-2,2v,8v-2;3,4) *exist for all* v *such that* $v(v-1) \equiv 0 \pmod 3$.

Proof. Seberry(1984) has proved that $v(v-1) \equiv 0 \pmod 3$ is necessary and sufficient condition for the existence of a BRD(v, 2v(v-1)/3, 2(v-1), 3, 4) and we have proved that a BRTD(U,3,4) exists where $U \equiv 3 \pmod 6$.

$\square$

Corollary 4.10. *If* $v \equiv 1 \pmod 3$ *then* a BRTD($6v, 12v(v+1); 8v-4, 2, 8v; 4, 4$) *and a* BRTD($6v, 2v(7v-1); 4(v-1)/3+4v, 2v, (28v-4)/3; 4, 4$) exist *and hence a* BTD($6v$, $12v(v+1)$; $8v-4$, 2, $8v$; 4, 4) *and a* BTD($6v$, $2v(7v-1)$; $4(v-1)/3+4v$, $2v$, $(28v-4)/3$; 4, 4) *exist.*

Proof. This follows since de Launey and Seberry(1984) have proved that $v \equiv 1 \pmod 3$ is sufficient condition for the existence of a BRD(v, $v(v-1)/3$, $4(v-1)/3$, 4, 4) and we have shown that a BRTD(6,12;4,2,8;4,4) exists.

$\square$

If there exists a BTD with $\Lambda = 2$, then we can obtain a BTD with $\Lambda = 4$ by a doubling construction, but if ρ_2 is odd our construction will give a non-isomorphic solution.

Now we will give a construction which does not give equireplicate BTD in general, but under certain condition stated in the theorem, it does give equireplicate BTD. The result generalizes Theorem 3 of Seberry(1984) and is stated without proof.

Theorem 4.11. *Suppose there exist* a BRTD(V, B; r_1, r_2, R; K, Λ), B, *and a* BRTD(U, A; s_1, s_2, S; K, Λ), A. *Further suppose there exist at least* $K-1$ *mutually orthogonal latin squares of order* $U-1$ and

$$(R(K-1)-2r_2)(U-1) = (S(K-1)-2s_2)(V-1).$$

Then there exists a BRTD($V(U-1)+1$, $B(U-1)^2+AV$, VS, K, Λ).

Corollary 4.12. *If a* BRTD(V, B, R, K, Λ) *exists and there are at least* $K-1$ *mutually orthogonal latin squares of order $V-1$, then there exists a* BRTD($V^2-V+1, B(V^2-V+1), VR, K, \Lambda$).

Example 7. BRTD(3,6,6,3,4) gives a BRTD(7,42,18,3,4) whose underlying

BTD is a double of the design listed as number 51 in Billington and Robinson(1983).

Remark: Theorem 3.4, 3.6 and Corollary 3.5 can be used recursively to obtain infinitely many families of BTDs.

5. Applications of BRn-aryDs in the construction of n-ary designs.

We have already seen some construction methods in the above section where we produced BRTDs from smaller BRTDs; by replacing each entry in the resulting BRTD by its absolute value we can construct BTDs. The following result gives another technique, which is well known for Generalized Bhaskar Rao Binary Designs.

Theorem 5.1. *Let* $X = A - B$ *be a* BRn-aryD(V,B,R,K,Λ). *Then*

$$M = \begin{matrix} A & B \\ B & A \end{matrix}$$

is the incidence matrix of a partially balanced n-ary design with parameters $V^* = 2V$, $B^* = 2B$, $R^* = R$, $K^* = K$, $\Lambda_1 = \Lambda/_2$ and $\Lambda_2 = 0$.

Proof. Let $N = A + B$. We know that

$$XX' = \Delta I,$$

$$AA' + BB' = (NN' + XX')/_2 = ((RK-\Lambda V+\Delta)I + \Lambda J)/_2$$

and

$$AB' + BA' = ((RK-\Lambda V-\Delta)I + \Lambda J)/_2$$

Hence as $\Delta = RK- \Lambda V + \Lambda$ we get the result.

$\square$

Example 8. Consider a Bhaskar Rao 4-ary design (3,12,16,4,16) :

$$X = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 2 & -2 & 0 & 0 & 3 & 3 & 1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 2 & -2 & 1 & 1 & -3 & -3 \end{bmatrix}.$$

Then

$$A=\begin{bmatrix}2&2&2&2&2&2&2&2&0&0&0&0\\1&1&0&0&2&0&0&0&3&3&1&1\\1&0&1&0&0&0&2&0&1&1&0&0\end{bmatrix}$$

and

$$B=\begin{bmatrix}0&0&0&0&0&0&0&0&0&0&0&0\\0&0&1&1&0&2&0&0&0&0&0&0\\0&1&0&1&0&0&0&2&0&0&3&3\end{bmatrix}.$$

So

$$\begin{bmatrix}A&B\\B&A\end{bmatrix}$$

is a partially balanced BT4-aryD(6,24,16,4,Λ_1=8,Λ_2=0)

But we observe an interesting property: if we augment this matrix by

$$[2\ 2]\times I_3$$
$$[2\ 2]\times I_3$$

then we get a balanced 4-ary design.

That is, we have the following 4-ary design:

$$\begin{bmatrix}A & B & \begin{matrix}2&2&0&0&0&0\\0&0&2&2&0&0\\0&0&0&0&2&2\end{matrix}\\ B & A & \begin{matrix}2&2&0&0&0&0\\0&0&2&2&0&0\\0&0&0&0&2&2\end{matrix}\end{bmatrix}.$$

This leads us to the following theorem:

Theorem 5.2. *If a* BRn-ary *design exists and* $\Lambda_{/2}$ = 2s((K-t)t) *for some positive integers t and s, then there is an n-ary design if K-t and t are both less than n. Otherwise we have a (K-t)- or t-ary design with parameters* V*=2V, B*= 2B+2sV, R*=R+s(K-t)+s(t), K*=K, *and* $\Lambda=\Lambda_{/2}$.

Proof. Augment the incidence matrix of the partially balanced n-ary design obtained in Theorem 4.1 by

$$[\text{K-t K-t}\ldots\text{K-t}\quad t\quad t\ \ldots\ t\,]\times I_V$$
$$[\ t\quad t\ldots\quad t\ \ \text{K-t K-t}\ldots\text{K-t}]\times I_V$$

where K-t and t occur s times.

∎

Corollary 5.3. *If a* BRD(v, b, r, 3, Λ=8t) *exists then* BTD(2v, 2b+2tv; ρ_1=r+t, ρ_2=t, R=r+3t; 3, 4t) *exists.*

Proof. The following matrix will give the required design, where A and B are as defined in Theorem 4.1.

$$\begin{bmatrix} A & B & [2\ldots 2\,1\ldots 1] \times I_v \\ B & A & [1\ldots 1\,2\ldots 2] \times I_v \end{bmatrix}$$

∎

Seberry(1984, Theorem 15) has proved that v(v-1) $\equiv$ 0(mod 3) is necessary and sufficient condition for the existence of BRD(v,3,8t), therefore we have

Corollary 5.4. *If* v(v-1) $\equiv$ 0(mod 3), *then there exist a series of* BTD(2v,2b+2tv, r+t, t, R=r+3t, 3, 4t) *for all positive integer t.*

For example for v = 4 and t = 1, a BRD(4,16,12,3,8) exists and therefore a BTD(8,40;13, 1,15;3,4) exist. This BTD is listed as number 268 in Billington and Robinson(1983) and is obtained by Billington(1985).

The following application is known for the case when we have a BRD, for example see Bhaskar Rao(1970).

Theorem 5.5. *Existence of a* BRTD(V,B;ρ_1,ρ_2,R; K, Λ) *and the* BIBD(v=2k, b = 2r, λ) *implies the existence of a partially balanced ternary design with the parameters* (vV, bB; rρ_1, rρ_2, rR; kK, Λ_1 = $\Delta\lambda$, Λ_2= $\Lambda r/2$).

Proof. Let N be the incidence matrix of the BIBD and N* be the incidence matrix of the complement of the BIBD. Replace each positive entry x of the BRTD by xN and each negative entry y by yN*. As usual 0 is replaced by a zero matrix. The resulting matrix gives the required PBTD.

∎

Example 9. (i) BRTD(3,6;2,2,6;3,4) and BIBD(4,6,3,2,1) give

PBTD(12,36;6,6,18; 6,10,6).

(ii) BRTD(3,6;2,2,6;3,4) and BIBD(6,10,5,3,2) give PBTD(18,60;10,10,30; 9,20,10).

(iii) BRTD(6,12;4,2,8;4,4) and BIBD(4,6,3,2,1) give PBTD(24,72;12,6,24; 8,12,6).

(iv) BRTD(6,12;4,2,8;4,4) and BIBD(6,10,5,3,2) give PBTD(36,120;20,10, 40;12,24,10).

We know that when v is a power of odd prime, then BIBD($v+1,2v,v,(v+1)/2,(v-1)/2$) exists and therefore we have:

Theorem 5.6. *Existence of a* BRTD($V,B;\ \rho_1,\rho_2,R;\ K,\ \Lambda$) *implies the existence of a partially balanced ternary design with the parameters* ($(v+1)V$, $2vB$; $v\rho_1$, $v\rho_2$, vR; $(v+1)K/2$, $\Lambda_1 = \Delta(v-1)/2$, $\Lambda_2 = \Lambda v/2$) *where v is any odd prime power.*

As we have proved that a BRTD with $K = 3$ and $V \equiv 3 \pmod 6$ exists for any $\Lambda = 4t$, we have:

Theorem 5.7. *For* $V \equiv 3 \pmod 6$, *a* PBTD ($(v+1)V, 2vB;\ v\rho_1, v\rho_2, vR;\ 3(v+1)/2,\ \Delta(v-1)/2,\ 2vt$) *exists.*

Corollary 5.8. *For* $V \equiv 3 \pmod 6$, *a* PBTD($4V,\ 6B;\ 3\rho_1,\ 3\rho_2,\ 3R;\ 6,\ \Delta,\ 6t$) *exists.*

Using BRTD(6,12;4,2,8;4,4) we get

Theorem 5.9. *If v is a prime power then there exists a* PBTD($6(v+1),24v;\ 4v,2v,\ 8v;\ 2(v+1),6(v-1),2v$).

Remark. Here again we can use Theorems 3.4 and 3.6 recursively to construct families of PBTDs.

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