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Minimal (n) and (n, h, k) Completely Separating Systems

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Abstract

$R(n)$ denotes the minimum possible size of a completely separating system $\mathcal{C}$ on an n-set. $R(n, h, k)$ denotes the minimum possible size of a completely separating system $\mathcal{C}$ on an n-set with $h \leq |A| \leq k$ for each $A \in \mathcal{C}$. In this paper a catalogue of non-isomorphic systems which achieve $R(n)$ for $n \leq 10$ is given. Values of $R(n, h, k)$ are determined for $n \leq 10$ and for $n > \frac{k^2}{2}$.

1 Introduction

This paper catalogues completely separating systems (CSSs) which achieve $R(n)$ for $n \leq 10$ and determines $R(n, h, k)$ for $n \leq 10$ or $n > \frac{k^2}{2}$. This also gives the minimum and maximum volume of various classes of CSSs, something that has been found to be important in related work by one of the authors (Roberts).

Throughout this paper $h \leq k \leq n$ are positive integers and $[n] = \{1, 2, \ldots, n\}$. A (n)**Completely Separating System (or** (n)**CSS)** $\mathcal{C}$ on $[n]$ is a collection of subsets of $[n]$, called blocks, such that for each $a, b \in [n]$ there are blocks $A, B \in \mathcal{C}$ with $a \in A - B$ and $b \in B - A$. If $h \leq |A| \leq k$ for all $A \in \mathcal{C}$ then $\mathcal{C}$ is a (n, h, k)**CSS**, and if $h = k$ then $\mathcal{C}$ is a (n,k)**CSS**. In this paper $\mathcal{C}$ will always denote a CSS. The **volume** of $\mathcal{C}$ is $V(\mathcal{C}) = \sum_{A\in\mathcal{C}} |A|$.

The integers **R(n), R(n,h,k), R(n,k)** are defined by:
$R(n) = \min\{|\mathcal{C}| : \mathcal{C}$ is a $(n)CSS\}$; $R(n,h,k) = \min\{|\mathcal{C}| : \mathcal{C}$ is a $(n,h,k)CSS\}$; and $R(n,k) = \min\{|\mathcal{C}| : \mathcal{C}$ is a $(n,k)CSS\}$. An (n,h,k)CSS $\mathcal{C}$ for which $|\mathcal{C}| = R(n,h,k)$ is a **minimal** (n,h,k)CSS. Similarly, an (n)CSS for which $|\mathcal{C}| = R(n)$ is a **minimal** (n)CSS. Let $m(\mathcal{C}) = \max_{A\in\mathcal{C}}\{|A|\}$. Clearly $\lceil \frac{V(\mathcal{C})}{|\mathcal{C}|} \rceil \le m(\mathcal{C}) < n$ if $\mathcal{C}$ is minimal and $n > 1$.

Two CSSs are said to be **isomorphic** if one can be obtained from the other by permuting $1,\ldots,n$. A p**-element** in a CSS is an element which occurs in exactly p blocks of the CSS. The **complementary CSS** of an (n)CSS $\mathcal{C}$ is $\mathcal{C}' = \{A \subseteq [n] : A' \in \mathcal{C}\}$.

CSSs were introduced by Dickson [2]. They were defined as an extension of a Separating System as defined by Renyi [7]. In this paper CSSs and all derivations are treated directly from the combinatorial design perspective. It should be noted that the same material could be presented in the language of hypergraphs as the dual of a CSS is an antichain (see Cai [1]).

Spencer [9] showed that

Lemma 1.1.

$$R(n) = min\{t : \binom{t}{\lceil \frac{t}{2} \rceil} \ge n\}.$$

Explicit constructions of collections which achieve $R(n)$ were not supplied by Spencer. Cai [1] notes that for $n > \frac{k^2}{2}$ and $R = \lceil \frac{2n}{k} \rceil$, it is easy to construct a simple graph with n edges and R vertices with each vertex of degree k or less. Labelling the edges $1,\ldots,n$ and taking a block to be the set of edges incident with a vertex, one obtains an $(n,1,k)$CSS. Cai also shows that for an $(n,1,k)$CSS, the number of blocks is at least $\lceil \frac{2n}{k} \rceil$. Hence

Theorem 1.1. *$R(n,1,k) = \lceil \frac{2n}{k} \rceil$, $\forall n > \frac{k^2}{2} \ge 2$.*

2 Basic results

Lemma 2.1. *For all positive n*
(i) $R(n,a,b) \ge R(n,h,k)$ whenever $a \ge h$ and $b \le k$.
(ii) For all h and k, $R(n) \le R(n,h,k) \le R(n,k)$.
(iii) $R(n) \le R(n+1) \le R(n) + 1$.
(iv) $R(n,1,k) \le R(n+1,1,k) \le R(n,1,k) + 1$.
(v) If $n \ge 3$ then $R(n+1,2,k) \le R(n,2,k) + 1$.
(vi) For $h > 1$, $R(n,h,k) \ge \lceil \frac{2n}{k} \rceil$.

(vii) For $k \geq 2$ there is a minimal (n)CSS and a minimal $(n, 1, k)$CSS, each containing at most two singleton blocks.
(viii) $R(n, h, k) = R(n, n - k, n - h)$.

Proof. (iii) & (iv) In each case the second inequality follows from the fact that the addition of the block $\{n + 1\}$ to a CSS on $[n]$ yields a CSS on $[n + 1]$.
(v) Let $\mathcal{C}$ be a $(n, 2, k)$CSS which achieves $R(n, 2, k)$. Replace one occurrence of the element 1 by the element $n+1$. Add the 2-block $\{1, n+1\}$ to the collection to obtain an $(n + 1, 2, k)$CSS.
(vi) Let $\mathcal{C}$ be an $(n, h, k)CSS$. As $h > 1$, $V(\mathcal{C}) \geq 2n$. Since there are at most k elements in each block in $\mathcal{C}$, $|\mathcal{C}| \geq 2n/k$.
(vii) If $\mathcal{C}$ is minimal and contains $\{a\}$, $\{b\}$ and $\{c\}$ then replace these three blocks by the three 2-blocks $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$.
$(viii)$ Complementary CSSs must both be minimal if one of them is minimal. □

Lemma 2.2. *Let $\mathcal{C}$ be a minimal CSS on $[n]$. Assume $R(m) = R(n)$, $m < n$. Then $n - m < |A| < m$ for each $A \in \mathcal{C}$.*

Proof. Assume $A \in \mathcal{C}$, $|A| \geq m$. Then, as $R(m) = R(n)$, any m elements of A cannot be completely separated in the remaining blocks of $\mathcal{C}$. Hence $|A| < m$. Applying this to the complementary CSS $\mathcal{C}'$ yields $n - m < |A|$. □

Theorem 2.1. *Let $\mathcal{C}$ be a minimal (n)CSS, $n \geq 5$. Then $\mathcal{C}$ contains at most one singleton block and $2n - 1 \leq V(\mathcal{C}) \leq |\mathcal{C}|n - 2n + 1$. If there are no singleton blocks then $2n \leq V(\mathcal{C}) \leq |\mathcal{C}|n - 2n$.*

Proof. Assume that $\mathcal{C}$ contains more than one singleton. By Lemma 2.1 we may assume that $\mathcal{C}$ contains at most 2 singletons, so that $\mathcal{C} = \{\{1\}, \{2\}, A_1, A_2, \ldots, A_l\}$. The blocks $A_1, A_2, \ldots, A_l$ must completely separate at least 3 elements and have size 2 or more. Clearly $l \geq 2$. Then $\{\{1, 2\}, A_1 \cup \{1\}, A_2 \cup \{2\}, A_3, \ldots, A_l\}$ is an (n)CSS with fewer blocks than $\mathcal{C}$, contradicting the minimality of $\mathcal{C}$.

The minimum possible volume of $\mathcal{C}$ is thus $2n - 1$; $2n$ if there are no singleton blocks. Consideration of complementary CSSs gives the remaining inequalities. □

3 $R(n)$ for $n \leq 10$

Hereafter the blocks in a CSS are shown as rows in an array. In some of these representations extra spaces are left in some rows to help highlight some structures of the CSS.

Theorem 3.1. (i) *For each $n \leq 10$, $R(n)$ has the values as shown in the row labelled R in the table below.*
(ii) *For each $n \leq 10$, the number of non-isomorphic CSSs which achieve $R(n)$ is shown in the row labelled d in the table below.*

(iii) The minimum and maximum volumes of (n)CSS which achieve $R(n)$ are shown in the rows labelled Min and Max in the table below.

n:	1	2	3	4	5	6	7	8	9	10
R:	1	2	3	4	4	4	5	5	5	5
d:	1	1	2	6	1	1	18	7	2	2
Min:	1	2	3	4	10	12	13	16	18	20
Max:	1	2	6	12	10	12	22	24	27	30

Proof. The values of $R(n)$ for $n \leq 10$ are given by Lemma 1.1. All non-isomorphic (n)CSSs for $n \leq 10$ are catalogued below. It should be noted that the cases $n = 1, 2, 3, 6, 10$ follow directly from Sperner's Theorem (see [3]).

The first block in each CSS can always be assumed to be $1, 2, \ldots, m(\mathcal{C})$. The remaining blocks must completely separate $1, 2, \ldots, m(\mathcal{C})$ as well as $m(\mathcal{C}) + 1, \ldots, n$. This reduces the problem to using CSSs for smaller n which are usually, but not always, minimal. This yields a computationally feasible exhaustive construction of all non-isomorphic minimal CSSs with the given parameters.

The cases when $n \leq 3$ are simple and can be checked by exhaustion.

n = 1:

1

n = 2:

1
2

n = 3:

1
2
3
,
1 2
1 3
2 3

In the following cases $\mathcal{C}$ denotes a minimal (n)CSS and A a block of $\mathcal{C}$.

n = 4: There are 3 possible values for $m(\mathcal{C})$.

1. $m(\mathcal{C}) = 1$. There is one CSS:

1
2
3
4

2. $m(\mathcal{C}) = 2$. There are two CSSs:

1 2
1 3
2 3
4
,
1 2
1 3
2 4
3 4

3. $m(\mathcal{C}) = 3$. It can be assumed that $A = \{1, 2, 3\} \in \mathcal{C}$. There are two ways of completely separating 1, 2 and 3 using three blocks as shown in the case $n = 3$.

1 2 3
1 4
2 4
3
,
1 2 3
1 4
2 4
3 4
,
1 2 3
1 2 4
1 3 4
2 3 4

n = 5 and n = 6: By Lemma 2.2, $1 < |A| < 4$. In particular, $m(\mathcal{C}) < 4$ and $\mathcal{C}$ contains no singletons. As $|\mathcal{C}| = 4$, $V(\mathcal{C}) = 2n$ (Theorem 2.1) and so $m(\mathcal{C}) \geq \lceil \frac{10}{4} \rceil = 3$. Hence for each of $n = 5$ and $n = 6$ there is one CSS:

1 2 3
1 4 5
2 4
3 5

1 2 3
1 4 5
2 4 6
3 5 6

n = 7: Theorem 2.1 implies $13 \le V(\mathcal{C}) \le 22$, so $m(\mathcal{C}) \ge \lceil \frac{13}{5} \rceil = 3$.

1. $m(\mathcal{C}) = 3$. The three possible CSSs are:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 6 \\ 7 & & \end{array}, \quad \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \\ 4 & 7 & \end{array}, \quad \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \\ 1 & 6 & 7 \end{array}$$

2. $m(\mathcal{C}) = 4$. The nine possible CSSs are:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & & \\ 3 & 6 & & \\ 4 & 7 & & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & \\ 2 & 5 & 7 & \\ 3 & 6 & 7 & \\ 4 & & & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & & 5 & 6 \\ 2 & & 5 & 7 \\ 3 & & 6 & \\ 4 & & 7 & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & & 5 & 6 \\ 2 & & 5 & 7 \\ 3 & & 6 & 7 \\ 4 & & 7 & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & \\ 2 & 5 & 7 & \\ 3 & 6 & 7 & \\ 4 & 5 & 6 & \end{array},$$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & 6 & \\ 3 & 5 & 7 & \\ 4 & 6 & 7 & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 7 \\ 1 & 3 & 6 & 7 \\ 2 & 3 & 7 & \\ 4 & 5 & 6 & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 7 \\ 1 & 3 & 6 & 7 \\ 2 & 3 & 4 & 7 \\ 4 & 5 & 6 & \end{array}, \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ 1 & 2 & & 5 & 6 \\ 1 & 3 & & 5 & 7 \\ 2 & 3 & & 6 & 7 \\ 4 & & 5 & 6 & 7 \end{array}$$

3. $m(\mathcal{C}) = 5$. The four possible CSSs are:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & & 6 \\ 1 & 4 & 5 & & 6 \\ 2 & 4 & & & 7 \\ 3 & 5 & & 6 & 7 \end{array}, \quad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & & 6 & 7 \\ 1 & 4 & 5 & & 6 & 7 \\ 2 & 4 & & & 6 & \\ 3 & 5 & & & 7 & \end{array}, \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & & 6 \\ 1 & 4 & 5 & & 7 \\ 2 & 4 & & 6 & 7 \\ 3 & 5 & & 6 & 7 \end{array}, \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 6 & 7 \\ 1 & 4 & 5 & & 6 \\ 2 & 4 & & & 7 \\ 3 & 5 & & 6 & 7 \end{array}$$

4. $m(\mathcal{C}) = 6$. The two possible CSSs are:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & & 7 & \\ 1 & 4 & 5 & & 7 & \\ 2 & 4 & 6 & & 7 & \\ 3 & 5 & 6 & & & \end{array}, \quad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & & 7 & \\ 1 & 4 & 5 & & 7 & \\ 2 & 4 & 6 & & 7 & \\ 3 & 5 & 6 & & 7 & \end{array}$$

n = 8: The results of Section 2 imply that $16 \le V(\mathcal{C}) \le 24$ and $4 \le m(\mathcal{C}) \le 6$.

1. $m(\mathcal{C}) = 4$. The four possible CSSs are:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & 8 & \\ 3 & 6 & 8 & \\ 4 & 7 & & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & \\ 2 & 5 & 7 & \\ 3 & 6 & 8 & \\ 4 & 7 & 8 & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & 8 & \\ 3 & 6 & 8 & \\ 4 & 7 & 8 & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 8 \\ 1 & 3 & 6 & 8 \\ 2 & 3 & 7 & 8 \\ 4 & 5 & 6 & 7 \end{array}$$

2. $m(\mathcal{C}) = 5$. The two possible CSSs are:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & & 6 & 7 \\ 1 & 4 & 5 & & 6 & 7 \\ 2 & 4 & & & 6 & 8 \\ 3 & 5 & & & 7 & 8 \end{array}, \quad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & & 6 & 7 \\ 1 & 4 & 5 & & 6 & 8 \\ 2 & 4 & & & 7 & 8 \\ 3 & 5 & & 6 & 7 & 8 \end{array}$$

3. $m(\mathcal{C}) = 6$. The only possible CSS is

1	2	3	4	5	6
1	2	3		7	8
1	4	5		7	8
2	4	6		7	
3	5	6		8	

n = 9: The results of Section 2 imply that $18 \leq V(\mathcal{C}) \leq 27$ and $4 \leq m(\mathcal{C}) \leq 6$.

1. $m(\mathcal{C}) = 4$. The only possible CSS is

1	2	3	4
1	5	6	7
2	5	8	9
3	6	8	
4	7	9	

2. $m(\mathcal{C}) \neq 5$ can be seen by assuming $A = \{1, 2, 3, 4, 5\} \in \mathcal{C}$, applying the unique construction for completely separating 5 elements in 4 blocks and then trying to add four more elements whilst maintaining complete separation.

3. $m(\mathcal{C}) = 6$. There is just one CSS

1	2	3	4	5	6
1	2	3	7	8	9
1	4	5		7	8
2	4	6		7	9
3	5	6		8	9

n = 10: The results of Section 2 imply that $20 \leq V(\mathcal{C}) \leq 30$ and $4 \leq m(\mathcal{C}) \leq 6$.

1. $m(\mathcal{C}) = 4$. There is only one way to completely separate the 6 elements not in the first block, and then only one way to complete the design.

1	2	3	4
1	5	6	7
2	5	8	9
3	6	8	10
4	7	9	10

2. $m(\mathcal{C}) \neq 5$ for similar reasons to the case $n = 9$ and $m(\mathcal{C}) = 5$.

3. $m(\mathcal{C}) = 6$. There is only one way to completely separate 6 elements, and only one way of completely separating the remaining 3 elements to complete the design.

1	2	3	4	5	6	
1	2	3		7	8	9
1	4	5		7	8	10
2	4	6		7	9	10
3	5	6		8	9	10

The bounds on the volume, Min and Max, shown in the table are now clear. □

4 $R(n, 1, k)$ for $n \leq 10$

Theorem 4.1. *For $n \leq 10$, the values of $R(n, 1, k)$ are*

	k								
n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
2	**2**								
3	**3**	**3**							
4	**4**	**4**	*4*						
5	**5**	**5**	**4**	*4*					
6	**6**	**6**	**4**	*4*	*4*				
7	**7**	**7**	**5**	*5*	*5*	*5*			
8	**8**	**8**	**6**	*5*	*5*	*5*	*5*		
9	**9**	**9**	**6**	**5**	*5*	*5*	*5*	*5*	
10	**10**	**10**	**7**	**5**	*5*	*5*	*5*	*5*	*5*

Proof. The cases in bold are given by Theorem 1.1 or because $R(n, 1, 1) = n$. In the other cases the CSSs in the proof of Theorem 3.1 are $(n, 1, k)CSS$s. □

5 $R(n, h, k)$ for $h \geq 2$ and $n \leq 10$

The cases $R(n, h, k)$ with $h = k$, that is the $R(n, k)$ cases, are dealt with in [5], hence are not included.

Theorem 5.1. *The values of $R(n, h, k)$ with $2 \leq h < k < n \leq 7$ are*
(i) $R(4, 2, 3) = 4$*,*
(ii) $R(5, 2, k) = 4$ *for* $k \leq 4$*,*
(iii) $R(5, 3, 4) = 5$*,*
(iv) $R(6, h, k) = 4$ *for* $h \leq 3$*,*
(v) $R(6, 4, 5) = 6$*,*
(vi) $R(7, h, k) = 5$ *for* $h \leq 4$*,*
(vii) $R(7, 5, 6) = 7$.

Proof. Parts (i), (ii), (iv) and (vi) follow from Theorem 3.1. By Lemma 2.1 $(viii)$, $R(5, 3, 4) = R(5, 1, 2)$, $R(6, 4, 5) = R(6, 1, 2)$ and $R(7, 5, 6) = R(7, 1, 2)$. The results for parts (iii), (v) and (vii) now follow from Theorem 4.1. □

Theorem 5.2. *The values of $R(8, h, k)$ with $2 \leq h < k < 8$ are*
(i) $R(8, 2, 3) = 6$*,*
(ii) $R(8, 2, k) = 5$ *for* $k \geq 4$*,*
(iii) $R(8, 3, k) = 5$ *for all* k*,*
(iv) $R(8, 4, k) = 5$ *for all* k*,*
(v) $R(8, 5, k) = 6$ *for all* k*,*
(vi) $R(8, 6, 7) = 8$.

Proof. Parts (ii), (iii) and (iv) follow from Theorem 3.1. Part (vi) follows from Lemma 2.1 $(viii)$ and Theorem 4.1. Theorem 3.1 does not give CSSs with the parameters of (i) or (v) so in these cases $R(8, h, k) > R(8) = 5$.

(i), (v) To see that $R(8,2,3) = 6$ and $R(8,5,k) = 6$ consider

1	2	3		1	2	3	4	5	6
1	4	5		1	2	3	4	5	7
2	6	7		1	2	3	6	8	
3	6	8		1	4	5	7	8	
4	7			2	4	6	7	8	
5	8			3	5	6	7	8	

□

Theorem 5.3. *The values of $R(9,h,k)$ with $2 \le h < k < 9$ are*
(i) $R(9,2,3) = 6$,
(ii) $R(9,2,k) = 5$ *for* $k \ge 4$,
(iii) $R(9,3,k) = 5$ *for all* k,
(iv) $R(9,4,5) = 6$,
(v) $R(9,4,k) = 5$ *for all* $k \ge 6$,
(vi) $R(9,5,k) = 5$ *for all* k,
(vii) $R(9,6,k) = 6$ *for all* k,
$(viii)$ $R(9,7,8) = 9$.

Proof. Parts (ii), (iii), (v) and (vi) follow from Theorem 3.1. Part $(viii)$ follows from Lemma 2.1 $(viii)$ and Theorem 4.1. Theorem 3.1 does not give CSSs with the parameters of (i), (iv) or (vii), so in these cases $R(9,h,k) > R(9) = 5$.

(i), (iv) To see that $R(9,2,3) = 6$ and $R(9,4,5) = 6$ consider

1	2	3		1	2	3	4	5
1	4	5		1	2	3	6	7
2	6	7		1	4	5	6	8
3	6	8		2	4	7	9	
4	7	9		3	5	8	9	
5	8	9		6	7	8	9	

(vii) It is shown in [6] that $R(9,6) = 6$, hence $R(9,6,k) = 6$. □

Theorem 5.4. *The values of $R(n,h,k)$ with $2 \le h < k < n = 10$ are*
(i) $R(10,2,3) = 7$,
(ii) $R(10,2,k) = 5$ *for* $k \ge 4$,
(iii) $R(10,h,k) = 5$ *for* $3 \le h \le 6$,
(iv) $R(10,7,8) = 7$,
(v) $R(10,7,9) = 7$,
(vi) $R(10,8,9) = 10$.

Proof. (i) $R(10,2,3) > 6$ as a minimal $(10,2,3)$CSS must have volume at least 20.

To see that $R(10,2,3) = 7$ consider

1	2	3
1	4	5
2	6	7
3	7	8
4	8	9
5	9	10
6	10	

Parts (ii) and (iii) follow from Theorem 3.1. Part (iv) follows from (i). Parts (v) and (vi) follows from Theorem 4.1. □

6 $R(n,h,k)$ for $n > \frac{k^2}{2}$

The following theorem is a part of Theorem 2 in [5]. Together with Theorem 1.1, it is used to determine $R(n,h,k)$ for sufficiently large n, as expressed in Theorem 6.2.

Theorem 6.1. *If $n \geq \binom{k+1}{2}$ then $R(n,k) = \lceil \frac{2n}{k} \rceil$ for $2 \leq k < n$. If $n = \binom{k+1}{2} - 1$ then $R(n,k) = \lceil \frac{2n}{k} \rceil + 1$ for $3 \leq k < n$. If $\frac{k^2}{2} < n < \binom{k+1}{2} - 1$ then $R(n,k) = \lceil \frac{2n}{k} \rceil$ for $k \geq 5$.*

Theorem 6.2. *Let $1 \leq h \leq k$, $k \geq 2$ and $n > \frac{k^2}{2}$. Then*

$$R(n,h,k) = \begin{cases} \lceil \frac{2n}{k} \rceil + 1 & \text{if } h = k \text{ and } n = \binom{k+1}{2} - 1 \\ \lceil \frac{2n}{k} \rceil & \text{otherwise} \end{cases}.$$

Proof. By Theorem 1.1, if $h = 1$ then $R(n,h,k) = \lceil \frac{2n}{k} \rceil$.

Assume that $h \geq 2$. By Lemma 2.1 (ii) and (vi), $\lceil \frac{2n}{k} \rceil \leq R(n,h,k) \leq R(n,k)$. Thus, by Theorem 6.1, $R(n,h,k) = \lceil \frac{2n}{k} \rceil$ for $n \geq \binom{k+1}{2}$, or for $\frac{k^2}{2} < n < \binom{k+1}{2} - 1$.

For $n = \binom{k+1}{2} - 1$, $k \geq 3$ begin with the construction of a minimal $(\binom{k+1}{2}, k)$CSS $\mathcal{C}$ as described in [5, p.135]: start with a $(k+1) \times k$ array of zeros. For each value of m from 1 up to n in turn set $c_{ij} = m$ for the two ordered pairs (i,j) defined by

$$min_j min_i \{c_{ij} : c_{ij} = 0\},$$
$$min_i min_j \{c_{ij} : c_{ij} = 0\}.$$

The rows of this array are the blocks of $\mathcal{C}$. The removal of any one of the elements provides a $(\binom{k+1}{2} - 1, h, k)$CSS in $\lceil \frac{2n}{k} \rceil$ blocks for $1 < h < k$.

This covers all the required cases. □

7 Final Comments

There are many open CSS problems. These problems may be of interest as finite designs or in terms of asymptotic results. These include:

1. What are the bounds on the volumes of minimal (n)CSSs and (n, h, k)CSSs for each n and how are they achieved?
2. For each n, how many non-isomorphic minimal (n)CSSs $(n > 10)$, and how many (n, h, k)CSSs are there?
3. For a fixed number of blocks R, is the number of non-isomorphic minimal (n)CSSs with $R(n) = R$ monotonic in n?

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