AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **34** (2006), Pages 17–21

Multipartite tournaments with small number of cycles

Gregory Gutin and Arash Rafiey

Department of Computer Science
Royal Holloway, University of London
Egham, Surrey, TW20 0EX, UK
{Gutin,Arash}@cs.rhul.ac.uk

Abstract

L. Volkmann, Discrete Math. 245 (2002), 19–53, posed the following question. Let $4 \leq m \leq n$. Are there strong n-partite tournaments, which are not themselves tournaments, with exactly $n-m+1$ cycles of length m? We answer this question in the affirmative. We raise the following problem. Given $m \in \{3, 4, \ldots, n\}$, find a characterization of strong n-partite tournaments having exactly $n-m+1$ cycles of length m.

1 Introduction

We use terminology and notation of [1]; all necessary notation and a large part of terminology used in this paper are provided in the next section.

A very informative paper [11] of L. Volkmann is the latest survey on cycles in an important class of digraphs, multipartite tournaments. Cycles in multipartite tournaments were earlier overviewed in [2, 6, 8]. Along with description of a large number of results on cycles in multipartite tournaments, L. Volkmann [11] poses several open problems. In this paper, we solve one of them.

Problem 1.1 *(Problem 2.27 in [11]) Let $4 \leq m \leq n$. Are there strong n-partite tournaments, which are not themselves tournaments, with exactly $n-m+1$ cycles of length m?*

This problem is a natural question due to the following reasons:

(i) According to Theorem 2.24 in [11], every strong n-partite tournament, $n \geq 3$, has at least $n-m+1$ cycles of length m for $3 \leq m \leq n$.

(ii) By reversing the arcs of the unique Hamilton path of the transitive tournament on n vertices, we obtain a strong tournament with exactly $n-m+1$ cycles of length m for every $3 \leq m \leq n$ (see [9]).

(iii) For every odd $n \geq 3$, there exists a strong n-partite tournament with $n-2$ cycles of length 3 (see [5] or Theorem 2.26 in [11]).

One may wish to strengthen Problem 1.1 as follows.

Problem 1.2 *Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong n-partite tournaments, which are not themselves tournaments, with exactly $n-m+1$ cycles of length m for two values of m?*

In Section 3, we solve Problem 1.1 in the affirmative. We do it by exhibiting a simple family of multipartite tournaments. We also show that such multipartite tournaments cannot have m-cycles with a pair of vertices from the same partite set. This result might well be of interest for solving the following open problem: Given $m \in \{3, 4, \ldots, n\}$, find a characterization of strong n-partite tournaments having exactly $n - m + 1$ cycles of length m. In Section 4 we show that Problem 1.2 has a negative answer for $m \in \{n-1, n\}$.

2 Terminology, notation and known results

A digraph obtained from an undirected graph G by replacing every edge of G with a directed edge (arc) with the same end-vertices is called an *orientation* of G. An *oriented graph* is an orientation of some undirected graph. A *tournament* is an orientation of a complete graph, and an *n-partite tournament* is an orientation of a complete n-partite graph. Partite sets of complete graphs become *partite sets* of n-partite tournaments.

The terms *cycles* and *paths* mean simple directed cycles and paths. A cycle of length k is a *k-cycle.* A digraph D is *strongly connected* (or *strong*) if for every ordered pair x, y of vertices in D there exist paths from x to y. For a set X of vertices of a digraph D, $D\langle X\rangle$ denotes the subdigraph of D induced by X.

For sets T, S of vertices of a digraph $D = (V, A)$, $T{\rightarrow}S$ means that for every vertex $t \in T$ and for every vertex $s \in S$, we have $ts \in A$, and $T{\Rightarrow}S$ means that for no pair $s \in S$, $t \in T$, we have $st \in A$. While for oriented graphs $T{\rightarrow}S$ implies $T{\Rightarrow}S$, this is not always true for general digraphs. If $u{\rightarrow}v$ (i.e., $uv \in A$), we say that u *dominates* v and v *is dominated* by u.

The following three results on cycles in strong n-partite tournaments are of interest for this paper.

Theorem 2.1 *[7] Every partite set of a strong n-partite tournament, $n \geq 3$, contains a vertex which lies on an m-cycle for each $m \in \{3, 4, \ldots, n\}$.*

Theorem 2.2 *[5] Every vertex in a strong n-partite tournament, $n \geq 3$, belongs to a cycle that contains vertices from exactly q partite sets for each $q \in \{3, 4, \ldots, n\}$.*

Theorem 2.3 *[11] Every strong n-partite tournament, $n \geq 3$, has at least $n-m+1$ cycles of length m for $3 \leq m \leq n$.*

3 Results related to Problem 1.1

The following theorem solves Problem 1.1 in the affirmative.

Proposition 3.1 *Let D be an n-partite tournament and let $4 \leq m \leq n$. Let $V_1, V_2, \ldots, V_n$ be partite sets of D and let $v_i \in V_i$, $i = 1, 2, \ldots, n$. If D satisfies the following conditions, then it has exactly $n - m + 1$ cycles of length m.*

1) $|V_i| = 1$ for every $i \neq n - m + 2$.

2) $C = v_1 v_2 \ldots v_n v_1$ is an n-cycle.

3) For every $s \in \{1, 2, \ldots, n-2\}$ and $r \in \{s+2, s+3, \ldots, n\}$, we have $v_r \to v_s$.

4) $v_n \to (V_{n-m+2} - \{v_{n-m+2}\}) \Rightarrow \{v_1, v_2, \ldots, v_{n-1}\}$.

Proof: By the conditions 2 and 3, the only path from vertex v_s to v_r, $r > s$ in $D\langle V(C)\rangle$ is $v_s v_{s+1} \ldots v_r$, which has $r + 1 - s$ vertices. Therefore, $D\langle V(C)\rangle$ has $n - m + 1$ cycles of length m. It is remain to show that there is no m-cycle C' that contains a vertex $x \in V_{n-m+2} - \{v_{n-m+2}\}$. Assume that $C' = x x_1 x_2 \ldots x_{m-1} x$ is an m-cycle through x. By the conditions 1 and 4 the only vertex that dominates a vertex in $V_{n-m+2} - \{v_{n-m+2}\}$ is v_n. Therefore all the vertices in $V(C') - \{x\}$ are in $V(C)$. Also $x_{m-1} = v_n$.

Let $x_1 = v_k$. By the conditions 2 and 3 the only path in $D\langle V(C)\rangle$ from v_k to v_n is $v_k v_{k+1} \ldots v_n$, which has $n + 1 - k$ vertices. So we have $n + 1 - k = m - 1$, i.e., $k = n - m + 2$. But we have $x \to x_1 = v_{n-m+2}$. This is a contradiction because v_{n-m+2} and x are in the same partite set. From the above we conclude that D has exactly $n - m + 1$ cycles of length m. □

It would be interesting to solve the following natural problem.

Problem 3.2 *Let $m \in \{3, 4, \ldots, n\}$. Find a characterization of strong n-partite tournaments having exactly $n - m + 1$ cycles of length m.*

This problem seems to be especially interesting for the case of Hamilton cycles, i.e., $m = n$. Tournaments with a unique Hamilton cycle were first characterized by Douglas [3]. Douglas's characterization is not simple even though the number of such tournaments on n vertices equals exactly the $(2n - 6)$th Fibonacci number [4, 10].

The following theorem might well be of interest for solving Problem 3.2.

Theorem 3.3 *Let $m \in \{3, 4, \ldots, n\}$ and let D be a strong n-partite tournament that has an m-cycle C containing vertices from less than m partite sets. Then D has more than $n - m + 1$ cycles of length m.*

Proof: If $m = n$, then by Theorem 2.1, there is another m-cycle that contains vertices from the partite set that does not have intersection with $V(C)$.

We prove the theorem by induction on $\ell = n - m + 1 \geq 1$. The above argument provides the basis of our induction ($\ell = 1$). Now assume that $\ell \geq 2$. Let V' be a maximal set such that $V(C) \subseteq V'$, V' does not contains vertices from all partite sets, and $D\langle V'\rangle$ is strong. If $D\langle V'\rangle$ contains vertices from $n - 1$ partite sets then by induction hypothesis $D\langle V'\rangle$ has more than $\ell - 1 = n - m$ cycles of length m. By Theorem 2.1 the remaining partite set has a vertex that is contained in an m-cycle. These imply that D has more than $n - m + 1$ cycles of length m. In particular, this argument extends the basis of our induction to $\ell = 2$.

Now we may assume that $\ell \geq 3$ and V' contains vertices from $q \leq n-2$ partite sets. Let t_1 be a vertex in $V(D) - V'$. Without loss of generality, assume that $V' \Rightarrow t_1$. Since D is strong there is a path from t_1 to a vertex $x \in V'$. Let $P = t_1 t_2 \ldots t_r x$ be such a path and assume that P is of minimum length. Therefore, we have $V' \Rightarrow \{t_2, t_3, \ldots, t_{r-1}\}$. If t_{r-1} and t_r are in partite sets that have intersection with V', then we can add t_{r-1} and t_r to V', a contradiction. Therefore one of them is in a partite set that does not have intersection with V'. If $q \leq n-3$ we can still add t_{r-1} and t_r to V', a contradiction.

Therefore the remaining case is $q = n-2$, and t_{r-1} and t_r are in two different partite sets that do not have intersection with V'. By our assumption we have $t_r \rightarrow V' \rightarrow t_{r-1} \rightarrow t_r$. Now consider C. We can find two distinct m-cycles that contain t_{r-1} and t_r, and some vertices from C. By induction hypothesis, $D\langle V' \rangle$ has more than $\ell - 2 = n-m-1$ distinct m-cycles. These imply that D has more than $n-m+1$ cycles of length m. □

Corollary 3.4 *Let D be a strong n-partite tournament and let D have exactly $n-m+1$ cycles of length m for some $m \in \{3, 4, \ldots, n\}$. Then every m-cycle of D has no pair of vertices from the same partite set.*

4 Results related to Problem 1.2

In this section we show that Problem 1.2 has a negative answer for $m \in \{n-1, n\}$. We denote, by $\mathcal{UC}_n$, the set of all strong n-partite tournaments, $n \geq 4$, which are not themselves tournaments, with exactly one cycle of length n.

Lemma 4.1 *If $D \in \mathcal{UC}_n$, $n \geq 4$, and C is its unique n-cycle, then there is a vertex $y \in D - V(C)$ such that $D\langle V(C) \cup \{y\} \rangle$ is strong.*

Proof: Let $D \in \mathcal{UC}_n$ and let C be its unique n-cycle. By Corollary 3.4, C contains a vertex from every partite set of D. Let $V_1, V_2, \ldots, V_n$ be partite sets of D and let $C = v_1 v_2 \ldots v_n v_1$, $v_i \in V_i$, $i = 1, 2, \ldots, n$.

Assume that there is no vertex $y \in D - V(C)$ for which $D\langle V(C) \cup \{y\} \rangle$ is strong. Then the following two sets S and T are non-empty: S (T) is the set of vertices in $D - V(C)$ that do not dominate (are not dominated by) any vertex in C. Since D is strong and $V(C) \cup S \cup T = V(D)$, there exist vertices $u \in S$ and $w \in T$ such that $u \rightarrow w$. Assume that $u \in V_i$, $w \in V_j$ $(i \neq j)$. If $i \neq j-2$, then $uwv_{j+1}v_{j+2} \ldots v_{j-2}u$ is an n-cycle of D distinct from C, which is impossible. If $i = j-2$, then $uwv_{j-1}v_j \ldots v_{j-4}u$ is an n-cycle of D distinct from C, which is impossible. □

Theorem 4.2 *There are no strong n-partite tournaments, $n \geq 4$, which are not themselves tournaments, with exactly one cycle of length n and two cycles of length $n-1$.*

Proof: Let $D \in \mathcal{UC}_n$. By Corollary 3.4, the unique n-cycle in D is $C = v_1v_2 \ldots v_nv_1$, where $v_i \in V_i,\ i = 1, 2, \ldots, n$. Let y be a vertex in $D - V(C)$ such that $D\langle V(C) \cup \{y\}\rangle$ is strong. By Theorem 2.2, y lies in a cycle C' of $D\langle V(C) \cup \{y\}\rangle$ that contains vertices from exactly $n-1$ partite sets. If C' contains v_i and v_i belongs to the same partite set as y, then the length of C' is n, a contradiction. Thus, C' is an $(n-1)$-cycle. It remains to observe that $D\langle V(C)\rangle$ has at least two $(n-1)$-cycles by Theorem 2.3. $\square$

Acknowledgment We would like to thank Anders Yeo and the referee for helpful comments.

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2000.

[2] J. Bang-Jensen and G. Gutin, Generalizations of tournaments: a survey, *J. Graph Theory* **28** (1998), 171–202.

[3] R.J. Douglas, Tournaments that admit exactly one hamiltonian circuit, *Proc. London Math. Soc.* **21** (1970), 716–730.

[4] M.R. Garey, On enumerating tournaments that admit exactly one hamiltonian circuit, *J. Combin. Theory Ser. B* **13** (1972), 266–269.

[5] W.D. Goddard and O.R. Oellermann, On the cycle structure of multipartite tournaments. In *Graph Theory Combin. Appl. 1*, Wiley, New York (1991), 525–533.

[6] Y. Guo, Semicomplete multipartite digraphs: a generalization of tournaments, Habilitation thesis, RWTH Aachen, Germany, 1998.

[7] Y. Guo and L. Volkmann, Cycles in multipartite tournaments, *J. Combin. Theory Ser. B* **62** (1994), 363–366.

[8] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey, *J. Graph Theory* **19** (1995), 481–505.

[9] J.W. Moon, On subtournaments of a tournament, *Canad. Math. Bull.* **9** (1966), 297–301.

[10] J.W. Moon, On the number of tournaments with a unique spanning cycle, *J. Graph Theory* **6** (1982), 303–308.

[11] L. Volkmann, Cycles in multipartite tournaments: results and problems, *Discrete Math.* **245** (2002), 19–53.

(Received 11 Mar 2004)