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A combinatorial solution to intertwined recurrences

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Abstract

We provide combinatorial derivations of solutions to intertwined second order linear recurrences (such as $a_n = pb_{n-1} + qa_{n-2},\ b_n = ra_{n-1} + sb_{n-2}$) by counting tilings of length n strips with squares and dominoes of various colors and shades. A similar approach can be applied to intertwined third order recurrences with coefficients equal to one. Here we find that all solutions can be expressed in terms of tribonacci numbers. The method can also be easily extended to solve and combinatorially comprehend kth order Fibonacci recurrences.

1 Introduction

In the recent book [1], the authors asked for solutions to "intertwined" second and third order linear recurrences. For instance, beginning with arbitrary initial conditions a_0, a_1, b_0, b_1, the authors request a closed form for a_n and b_n defined for $n \geq 2$ by the recurrences

$$\begin{aligned} a_n &= pb_{n-1} + qa_{n-2} \\ b_n &= ra_{n-1} + sb_{n-2} \end{aligned}$$

In addition, they request closed forms for two other systems (where a_n depends on a_{n-1} and b_{n-2} or a_n depends on b_{n-1} and b_{n-2}). Even simply stated third order recurrences, such as

$$\begin{aligned} a_n &= b_{n-1} + b_{n-2} + a_{n-3} \\ b_n &= a_{n-1} + a_{n-2} + b_{n-3} \end{aligned}$$

(with arbitrary $a_0, a_1, a_2, b_0, b_1, b_2$) were presented without a closed form, along with six other intertwined recurrences. Hirschhorn [5, 6] used generating functions to derive closed forms for all of these problems. In this paper, we demonstrate how all of these recurrences can be derived by elementary combinatorial arguments, leading to alternative solutions. We conclude with a discussion of a general method for solving intertwined k-th order recurrences like the example above.

As is discussed extensively in [3], every k-th order linear recurrence with constant coefficients can be given a simple combinatorial interpretation. Suppose $c_1, \ldots, c_k$ are nonnegative integers, and consider the problem of counting the ways to tile a strip of length n with colored tiles, where for $1 \leq j \leq k$, a tile of length j can be assigned one of c_j different colors. If we let u_n denote the number of ways to tile such a strip, then u_n satisfies the recurrence: For $n \geq 1$,

$$u_n = c_1 u_{n-1} + \ldots c_k u_{n-k} \tag{1}$$

with initial conditions $u_j = 0$ for $j < 0$ and $u_0 = 1$. This is easily proved by induction on n and considering the length of the last tile, since the number of tilings that end with a colored tile of length j is $c_j u_{n-j}$.

Another way to think about u_n that allows c_j to be negative or real (or complex) is that each tile of length j is assigned a weight of c_j and the weight of a tiling is the product of the weights of its tiles. Thus by the same argument as before, u_n is the sum of the weights of all tilings of length n. Thus we have a combinatorial interpretation for any recurrence of type (1) subject to the *ideal* initial conditions $u_j = 0$ for $j < 0$ and $u_0 = 1$.

With arbitrary initial conditions, $a_0, a_1, \ldots, a_{k-1}$, and for $n \geq k$,

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} \tag{2}$$

a_n can also be given a combinatorial interpretation. As the next theorem indicates, changing the *initial conditions* from the ideal ones merely changes the weight of the *initial tile*.

Theorem 1 *For $n \geq 1$, if a_n satisfies recurrence (2), then a_n is the sum of the weights of all length n tilings, where the weight of a tiling is the product of the weights of its tiles. Except for the initial tile, the weight of a tile of length j is c_j; if the initial tile has length j, then the initial tile has weight has weight $w_j = a_j - \sum_{i=1}^{j-1} c_i a_{j-i}$.*

Proof. By induction on n. When $n = 1$, $w_1 = a_1$, as desired. For $1 \leq j \leq k$, w_j is chosen so that $a_1, a_2, \ldots, a_k$ are consistent with their combinatorial interpretation. Specifically, consider the last tile of a length j tiling, where $2 \leq j \leq k$. Either this tile has length j and thus weight w_j or it has length $i \leq j-1$ with weight c_i, preceded by a tiling of length $n - i$. Thus, by the induction hypothesis, $a_j = w_j + \sum_{i=1}^{j-1} c_i a_{j-i}$. For $n > k$, the theorem follows by induction and considering the length of the last tile. □

We note that when the initial tile has maximum length k, the recurrence $a_k = \sum_{i=1}^{k} c_i a_{k-i}$ implies that $w_k = c_k a_0$.

2 Intertwined second order recurrences

We now focus our attention on second order recurrences. Suppose that a_0, a_1, p, q are arbitrary real numbers and for $n \geq 2$,

$$a_n = pa_{n-1} + qa_{n-2} \tag{3}$$

It follows from Theorem 1 that for $n \geq 1$, a_n is the sum of the weights of all length n tilings with weighted squares and dominoes where, except for the first tile, all *squares* (length one) have weight p, and all *dominoes* (length two) have weight q. The initial tile has weight $w_1 = a_1$ or $w_2 = qa_0$, depending on whether it is a square or domino, respectively.

As is well known, the solution to (3) can be expressed as a linear combination of the powers of the roots of the characteristic polynomial, $\lambda^2 - p\lambda - q$. (Even this can be derived combinatorially, as done in [2, 3].) Alternatively, the solution to (3) can be expressed as a combinatorial sum.

Theorem 2 *Let a_n satisfy recurrence (3); then for $n \geq 1$,*

$$a_n = a_1 \sum_{t_1+2t_2} \sum_{=n-1} \binom{t_1+t_2}{t_1} p^{t_1} q^{t_2} + qa_0 \sum_{t_1+2t_2} \sum_{=n-2} \binom{t_1+t_2}{t_1} p^{t_1} q^{t_2}.$$

Proof. The first summand provides the sum of all weighted tilings that begin with a square tile (with weight a_1) followed by a tiling with t_1 squares and t_2 dominoes of length $t_1 + 2t_2 = n - 1$. These $t_1 + t_2$ tiles can be arranged $\binom{t_1+t_2}{t_2}$ ways and each of these tilings has weight $p^{t_1}q^{t_2}$. By the same reasoning, the second summand is the sum of the weights of those tilings whose initial tile is a domino with weight qa_0. □

More generally, the k-th order recurrence $a_n = c_1a_{n-1} + \ldots c_ka_{n-k}$ has solution

$$\sum_{j=1}^{t} w_j \sum_{t_1+2t_2} \sum_{+\ldots+\ kt_k} \cdots \sum_{=\ n-j} \binom{t_1+t_2+\ldots+t_k}{t_1, t_2, \ldots, t_k} c_1^{t_1} c_2^{t_2} \ldots c_k^{t_k}$$

where $w_j = a_j - \sum_{i=1}^{j-1} c_j a_{j-i}$ is the weight of the initial tile of length j, and the multinomial coefficient $\binom{t_1+t_2+\ldots+t_k}{t_1,t_2,\ldots,t_k} = \frac{(t_1+t_2+\ldots+t_k)!}{t_1!t_2!\ldots t_k!}$ counts the ways to arrange t_1 tiles of length one, t_2 tiles of length 2, ..., and t_k tiles of length k with total length $n-j$, each with weight $c_1^{t_1}c_2^{t_2}\ldots c_k^{t_k}$. Note that under the ideal initial conditions where $a_j = 0$ for $j < 0$ and $a_0 = 1$, then $w_j = c_j$ for all j, and we have

$$a_n = \sum_{t_1+2t_2+\cdots+kt_k} \sum \cdots \sum_{=\ n} \binom{t_1+t_2+\ldots+t_k}{t_1, t_2, \ldots, t_k} c_1^{t_1} c_2^{t_2} \ldots c_k^{t_k}.$$

For real numbers p, q, r, s, we now consider the intertwined recurrence: for $n \geq 2$,

$$a_n = pb_{n-1} + qb_{n-2} \qquad b_n = ra_{n-1} + sa_{n-2} \tag{4}$$

where a_0, a_1, b_0, b_1 are arbitrary real numbers. To view this recurrence combinatorially, consider the question of counting the ways to tile a strip of length n with squares and dominoes as before, but now the tiles come in two different *shades*, light or dark. The numbers p, q, r, s denote the respective weights of light squares, light dominoes, dark squares, and dark dominoes, with the exception of the initial tile, which is given a weight of a_1, qb_0, b_1 or sa_0, respectively. Furthermore we require that our tiling obey the *Predecessor Rule:* Every tile, except for the initial tile, is preceded by a tile of opposite shade.

Theorem 3 *For the tilings described by recurrence (4), for $n \geq 1$, a_n is the total weight of all length n shaded tilings that end with a light tile, and b_n is the total weight of all length n shaded tilings that end with a dark tile.*

Proof. By induction on n. When $n = 1$, a single light square has weight a_1 and a single dark square has weight b_1. When $n = 2$, there are two tilings that end in a light tile, namely a single domino with weight qb_0 or a dark square followed by a light square with weight pb_1; hence the total weight is $pb_1 + qb_0 = a_2$ by our recurrence. Likewise, the total weight of length two tilings that end in a dark tile is $ra_1 + sa_0 = b_2$. For $n \geq 3$, tilings that end in a light square (of weight p) are preceded by a length $n-1$ tiling that ends in a dark tile; tilings that end in a light domino (of weight q) are preceded by a length $n-2$ tiling that ends in a dark tile. Consequently, by induction, $a_n = pb_{n-1} + qb_{n-2}$, as desired. By the same argument, $b_n = ra_{n-1} + sa_{n-2}$, and the induction is complete. □

Theorem 3 was proved by focusing on the last tile. By turning things around, we can also count a_n by breaking it into four cases, depending on the first tile. Except for renaming our indices, we arrive at the same non-recursive formula for a_n as obtained in [5].

Theorem 4 *Suppose a_n and b_n are determined by recurrence (4). Then for $n \geq 1$,*

$$\begin{aligned} a_n \;=\;& a_1 \sum_{k\geq 0} \sum_{\substack{\ell_1+\ell_2=k,\ d_1+d_2=k \\ \ell_1+2\ell_2+d_1+2d_2=n-1}} \binom{k}{\ell_2}\binom{k}{d_2} p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2} \\ &+ qb_0 \sum_{k\geq 0} \sum_{\substack{\ell_1+\ell_2=k,\ d_1+d_2=k \\ \ell_1+2\ell_2+d_1+2d_2=n-2}} \binom{k}{\ell_2}\binom{k}{d_2} p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2} \\ &+ b_1 \sum_{k\geq 0} \sum_{\substack{\ell_1+\ell_2=k,\ d_1+d_2=k \\ \ell_1+2\ell_2+d_1+2d_2=n-1}} \binom{k+1}{\ell_2}\binom{k}{d_2} p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2} \\ &+ sa_0 \sum_{k\geq 0} \sum_{\substack{\ell_1+\ell_2=k,\ d_1+d_2=k \\ \ell_1+2\ell_2+d_1+2d_2=n-2}} \binom{k+1}{\ell_2}\binom{k}{d_2} p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2}. \end{aligned}$$

Proof. From Theorem 3, a_n is the total weight of all shaded length n tilings that end with a light tile. The first summand gives the total weight of all length n shaded tilings that end with a light tile and begin with a light square. To see this, note that all such tilings begin with a light tile of weight a_1, which, by the Predecessor Rule, will then be followed by an equal number of dark tiles and light tiles in an alternating sequence of length $n-1$. Let ℓ_1, ℓ_2, d_1, d_2, and k denote the number of light squares, light dominoes, dark squares, dark dominoes, and dark tiles, respectively, that occur after the initial tile. Any such tiling would have weight $a_1 p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2}$. By definition, $d_1 + d_2 = k$, and since there are an equal number of light and dark tiles, we also have $\ell_1 + \ell_2 = k$. The total length of these tiles satisfies $\ell_1 + 2\ell_2 + d_1 + 2d_2 = n-1$. The number of arrangements with these parameters is $\binom{k}{\ell_2}\binom{k}{d_2}$ since the k light tiles can be arranged $\binom{k}{\ell_2}$ ways (equal to zero when $\ell_2 < 0$ or $\ell_2 > k$) and the dark tiles can be independently arranged $\binom{k}{d_2}$ ways. The other three summands can be explained in exactly the same way, except these tilings begin with a light domino, dark square, or dark domino, respectively. □

We remark that by the linear relationship between ℓ_1, ℓ_2, d_1, d_2, and k, we could simplify the quintuple sums above to be double sums. For instance, knowing just two variables, say k and d_2 forces the values of the other three parameters and the first summand could be expressed as

$$a_1 \sum_{k \geq 0} \sum_{d_2 \geq 0} \binom{k}{n-1-2k-d_2} \binom{k}{d_2} p^{3k+d_2+1-n} q^{n-1-2k-d_2} r^{k-d_2} s^{d_2}$$

but, for the sake of clarity, we will refrain from writing our formulas this way. Also, using the same sort of argument as above, and interpreting the parameters in the exact same way, an analogous expression can also be given for b_n as the total weight of those length n tilings that end with a dark tile. We state that expression here, but will not state it for most of the remaining identities in this paper.

Corollary 5 *Let a_n and b_n be determined by recurrence (4). Then for $n \geq 1$,*

$$\begin{aligned} b_n \;=\; & a_1 \sum_{k\geq 0} \sum_{\substack{\ell_1+\ell_2=k,\ d_1+d_2=k \\ \ell_1+2\ell_2+d_1+2d_2=n-1}} \binom{k-1}{\ell_2}\binom{k}{d_2} p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2} \\ & + q b_0 \sum_{k\geq 0} \sum_{\substack{\ell_1+\ell_2=k,\ d_1+d_2=k \\ \ell_1+2\ell_2+d_1+2d_2=n-2}} \binom{k-1}{\ell_2}\binom{k}{d_2} p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2} \\ & + b_1 \sum_{k\geq 0} \sum_{\substack{\ell_1+\ell_2=k,\ d_1+d_2=k \\ \ell_1+2\ell_2+d_1+2d_2=n-1}} \binom{k}{\ell_2}\binom{k}{d_2} p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2} \\ & + s a_0 \sum_{k\geq 0} \sum_{\substack{\ell_1+\ell_2=k,\ d_1+d_2=k \\ \ell_1+2\ell_2+d_1+2d_2=n-2}} \binom{k}{\ell_2}\binom{k}{d_2} p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2}. \end{aligned}$$

The combinatorial interpretations and closed forms of the other intertwined recurrences are more interesting.

Theorem 6 *Given real numbers* $p, q, r, s, a_0, a_1, b_0, b_1$, *and for* $n \geq 2$,

$$a_n = pa_{n-1} + qb_{n-2} \qquad b_n = rb_{n-1} + sa_{n-2}. \tag{5}$$

Then for $n \geq 1$,

$$\begin{aligned} a_n &= a_1 \sum_{4k+\ell_1}\sum_{+d_1}\sum_{=n-1} \binom{k+\ell_1}{\ell_1}\binom{k+d_1-1}{d_1} p^{\ell_1} q^k r^{d_1} s^k \\ &+ qb_0 \sum_{4k+\ell_1}\sum_{+d_1}\sum_{=n-2} \binom{k+\ell_1}{\ell_1}\binom{k+d_1-1}{d_1} p^{\ell_1} q^k r^{d_1} s^k \\ &+ b_1 \sum_{4k+\ell_1}\sum_{+d_1}\sum_{=n-3} \binom{k+\ell_1}{\ell_1}\binom{k+d_1}{d_1} p^{\ell_1} q^{k+1} r^{d_1} s^k \\ &+ sa_0 \sum_{4k+\ell_1}\sum_{+d_1}\sum_{=n} \binom{k+\ell_1-1}{\ell_1}\binom{k+d_1-1}{d_1} p^{\ell_1} q^k r^{d_1} s^{k-1}. \end{aligned}$$

Proof. As in the proof of Theorem 4, one can show by induction on n that a_n is the total weight of tilings of a strip of length n, ending with a *light* tile, using light squares, light dominoes, dark squares, and dark dominoes where, except for the initial tile, each tile has respective weight p, q, r, or s, and obeys the following Predecessor Rule: Every square is preceded by a tile of the same shade, and every domino is preceded by a tile of opposite shade; the initial tile is given a respective weight of a_1, qb_0, b_1, sa_0. As before, the weight of a tiling is the product of its weights, and b_n is the total weight of the same objects, but where the last tile is constrained to be a *dark* tile.

Now consider the weight of a tiling beginning with a light square, followed by (in some order) ℓ_1 light squares, ℓ_2 light dominoes, d_1 light squares and d_2 dark dominoes. Such a tiling has weight $a_1 p^{\ell_1} q^{\ell_2} r^{d_1} s^{d_2}$ and satisfies $\ell_1 + 2\ell_2 + d_1 + 2d_2 = n - 1$. Furthermore, such a string is necessarily of the form

$$\text{(Initial light square)}(S)^{x_0}\mathbf{D_1}(\mathbf{S})^{x_1}D_2(S)^{x_2}\mathbf{D_3}(\mathbf{S})^{x_3}\cdots D_{2k}(S)^{x_{2k}}$$

for some nonnegative integers $k, x_0, \ldots, x_{2k}$, where the dominoes (denoted by $D_1, \ldots,$ D_{2k}) alternate from **D**ark, Light, **D**ark, ..., Light, and each $(S)^{x_{2i}}$ or $(\mathbf{S})^{x_{2i+1}}$ is a (possibly empty) string of light squares or a string of dark squares, respectively. Hence the number of light dominoes equals the number of black dominoes (say $\ell_2 = d_2 = k$) and therefore $4k + \ell_1 + d_1 = n - 1$. The number of light squares after the initial tile satisfies

$$x_0 + x_2 + \ldots + x_{2k} = \ell_1$$

which has $\binom{k+\ell_1}{\ell_1}$ nonnegative integer solutions. The number of dark squares satisfy $x_1 + x_3 + \ldots + x_{2k-1} = d_1$, which has $\binom{k+d_1-1}{d_1}$ independent solutions. Consequently, the total weight of these tilings is

$$a_1 \sum_{4k+\ell_1}\sum_{+d_1}\sum_{=n-1} \binom{k+\ell_1}{\ell_1}\binom{k+d_1-1}{d_1} p^{\ell_1} q^k r^{d_1} s^k$$

which is the first summand. The other three summands, which count the same kind of tilings, but with a different initial tile, can be derived in exactly the same way. □

The remaining intertwined second order recurrence can be solved with the same ideas.

Theorem 7 *Given real numbers* $p, q, r, s, a_0, a_1, b_0, b_1$, *and for* $n \geq 2$,

$$a_n = pb_{n-1} + qa_{n-2}, \qquad b_n = ra_{n-1} + sb_{n-2}.$$

Then for $n \geq 1$,

$$\begin{aligned} a_n \;=\;& a_1 \sum\sum\sum_{2(k+\ell_2+d_2)=n-1} \binom{k+\ell_2}{\ell_2}\binom{k+d_2-1}{d_2} p^k q^{\ell_2} r^k s^{d_2} \\ &+ qa_0 \sum\sum\sum_{2(k+\ell_2+d_2)=n-2} \binom{k+\ell_2}{\ell_2}\binom{k+d_2-1}{d_2} p^k q^{\ell_2} r^k s^{d_2} \\ &+ b_1 \sum\sum\sum_{2(k+\ell_2+d_2)=n-2} \binom{k+\ell_2}{\ell_2}\binom{k+d_2}{d_2} p^{k+1} q^{\ell_2} r^k s^{d_2} \\ &+ sb_0 \sum\sum\sum_{2(k+\ell_2+d_2)=n-3} \binom{k+\ell_2}{\ell_2}\binom{k+d_2}{d_2} p^{k+1} q^{\ell_2} r^k s^{d_2}. \end{aligned}$$

Proof. The proof is almost exactly the same as in the previous theorem. Here a_n and b_n count the total weight of the same tilings as before, but with a different Predecessor Rule: Except for the initial tile, every square tile is preceded by a tile of opposite shade, and every domino tile is preceded by a tile of the same shade; here, the initial tile has weight a_1, qa_0, b_1, or sb_0, depending on whether it is a light square, light domino, dark square, or dark domino, respectively. respectively. □

3 Intertwined Third Order Recurrences

We define the tribonacci numbers by the initial conditions $T_{-2} = 0$, $T_{-1} = 1$, $T_0 = 1$, and for all $n \geq 1$,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \tag{6}$$

T_n can be computed directly from the roots of $\lambda^3 - \lambda^2 - \lambda - 1$, but we shall focus on its combinatorial interpretation. By Theorem 1, T_n is the sum of the weights of all tilings of a strip of length n with squares, dominoes, and *trominoes* (tiles of length three). Here every tile, and thus every tiling, has weight one, and so T_n is also equal to the number of length n tilings.

For the remainder of this section we define, for $m \geq -2$, an m-tiling to be a tiling of length m using (unweighted, unshaded) squares, squares, dominoes, and trominoes. The number of m-tilings is T_m. (Note that there are zero tilings of length -1 or -2, and one empty tiling length zero, consistent with our initial conditions.) Later in this section, we will have more to say about T_k for negative values of k.

When the initial conditions are arbitrary real numbers a_0, a_1, a_2 and for $n \geq 3$,

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \tag{7}$$

then by applying Theorem 1, we have that for $n \geq 1$, a_n denotes the total weight of all length n tilings where all tiles have weight one, except for the initial tile. An initial square has weight a_1, an initial domino has weight $a_2 - a_1$, and an initial tromino has weight a_0. It immediately follows, by considering the first tile, that for all $n \geq 1$,

$$a_n = a_1 T_{n-1} + (a_2 - a_1) T_{n-2} + a_0 T_{n-3} \tag{8}$$

In this section, we consider, (as was done in [1] and [6]), the seven ways that the tribonacci recurrence can be coupled or intertwined with another tribonacci recurrence. Specifically, we consider sequences generated by arbitrary real numbers $a_0, a_1, a_2, b_0, b_1, b_2$, and for $n \geq 3$,

$$a_n = c_{n-1} + d_{n-2} + e_{n-3} \qquad b_n = \bar{c}_{n-1} + \bar{d}_{n-2} + \bar{e}_{n-3} \tag{9}$$

where c is equal to a or b, and $\bar{c}$ denotes the opposite choice of c; we define $d, \bar{d}, e, \bar{e}$ the same way. For example, the choice of $c = a, d = a, e = b$, results in $\bar{c} = b, \bar{d} = b, \bar{e} = a$ and the recurrence

$$a_n = a_{n-1} + a_{n-2} + b_{n-3} \qquad b_n = b_{n-1} + b_{n-2} + a_{n-3} \tag{10}$$

There are eight such recurrences, including the uncoupled recurrence where $c = a,\ d = a,\ e = a$. In the uncoupled case, we showed that a_n (and of course b_n) can be expressed entirely in terms of the tribonacci numbers, a result that is algebraically and combinatorially clear. But surprisingly, all seven other coupled recurrences can also be expressed in terms of tribonacci numbers (sometimes with negative indices) as shown in [6]. Here, we derive the same closed form solutions by elementary combinatorial arguments. We begin by giving a combinatorial interpretation for a_n and b_n.

Theorem 8 *Let $a_0, a_1, a_2, b_0, b_1, b_2$ be arbitrary real numbers, and for $n \geq 3$, let a_n and b_n be determined by recurrence (9). Then for $n \geq 1$, a_n is the sum of the weights of length n tilings with* shaded *tiles (light or dark) of length $1, 2,$ or 3, that end with a* light *tile, subject to the* Predecessor Rule: *A non-initial tile of length 1 (respectively, 2 or 3) is preceded by a tile of the same shade if and only if $c = a$ (respectively, $d = a$ or $e = a$). All tiles, except for the initial tile, have weight one. The weight of an initial tile of length j is w_j for light tiles and $\mathbf{w_j}$ for dark tiles, given by $w_1 = a_1$, $w_2 = a_2 - c_1$, $w_3 = a_0$, $\mathbf{w_1} = b_1$, $\mathbf{w_2} = \mathbf{b_2} - \bar{\mathbf{c}}_\mathbf{1}$, $\mathbf{w_3} = \mathbf{b_0}$. For $n \geq 1$, b_n is defined the same way, but with the restriction that the tiling ends with a* dark *tile.*

For example, from the coupled recurrence of equation (10), a_n is the total weight of all tilings of a length n strip with shaded squares, dominoes, and trominoes, ending in a light tile, with the restriction that, except for the initial tile, all squares and dominoes are preceded by a tile of the same shade, and all trominoes are preceded by a tile of opposite shade. All tiles have weight one, except for the initial tile, which has weight equal to one of $w_1 = a_1$, $w_2 = a_2 - a_1$, $w_3 = e_0$, $\mathbf{w_1} = \mathbf{b_1}$, $\mathbf{w_2} = \mathbf{b_2} - \mathbf{b_1}$, or $\mathbf{w_3} = \mathbf{\bar{e}_0}$, depending on its length and shade.

Proof. By induction on n. The weights of the initial tile are chosen so that the combinatorial interpretation is valid for $a_1, a_2, a_3, b_1, b_2, b_3$. (Details: For tilings ending in a light tile, by considering the length of the last light tile, the length one tiling has weight w_1; the length two tilings have total weight $w_2 + c_1$; the length three tilings have total weight $w_3 + c_2 + d_1$. Thus we set $w_1 = a_1, w_2 = a_2 - c_1$, and $w_3 = a_3 - c_2 - d_1 = e_0$, by recurrence (9). Likewise, $\mathbf{w}_1 = b_1, \mathbf{w}_2 = b_2 - \bar{c}_1$, and $\mathbf{w}_3 = b_3 - \bar{c}_2 - \bar{d}_1 = \bar{e}_0$.) For $n \geq 3$, the induction is completed by considering the length of the last tile. □

For the tilings described by Theorem 8, we say that a tile is *shade-changing* if it must be preceded by a tile of the opposite shade, otherwise it is called *shade-preserving*. For the tilings described by (10), trominoes are shade-changing; squares and dominoes are shade-preserving. Now consider an n-tiling that ends with a light tile, as enumerated by a_n. If that tiling begins with a light tile (respectively, a dark tile), then it contains an even number (respectively, an odd number) of shade-changing tiles after the initial tile. Conversely, given any *unshaded* tiling with an even number (odd number) of shade-changing tiles after the initial tile, then there is exactly one way to shade the tiles so that it begins with a light tile (dark tile) and ends with a light tile. For $m \geq 0$, we let E_m (respectively, O_m) denote the number of unshaded m-tilings with an even number (respectively, odd number) of shade-changing tiles. For combinatorial interpretation, we shall let $E_0 = 1$ (the empty tiling) and let $O_0 = E_{-1} = O_{-1} = E_{-2} = O_{-2} = 0$. Thus, if we can determine E_m and O_m, we can immediately compute a_n by the following nonrecursive formula.

Theorem 9 *For a_n and b_n defined by recurrence (9), we have, for $n \geq 1$,*

$$\begin{aligned} a_n &= a_1 E_{n-1} + (a_2 - c_1)E_{n-2} + e_0 E_{n-3} + b_1 O_{n-1} + (b_2 - \bar{c}_1)O_{n-2} + \bar{e}_0 O_{n-3} \\ b_n &= a_1 O_{n-1} + (a_2 - c_1)O_{n-2} + e_0 O_{n-3} + b_1 E_{n-1} + (b_2 - \bar{c}_1)E_{n-2} + \bar{e}_0 E_{n-3}. \end{aligned}$$

Proof. From the above discussion, by considering the initial tile of the corresponding tiling problem, we have

$$\begin{aligned} a_n &= w_1 E_{n-1} + w_2 E_{n-2} + w_3 E_{n-3} + \mathbf{w_1 O_{n-1}} + \mathbf{w_2 O_{n-2}} + \mathbf{w_3 O_{n-3}} \\ b_n &= w_1 O_{n-1} + w_2 O_{n-2} + w_3 O_{n-3} + \mathbf{w_1 E_{n-1}} + \mathbf{w_2 E_{n-2}} + \mathbf{w_3 E_{n-3}}. \end{aligned}$$

The theorem follows from Theorem 8, since $w_1 = a_1$, $w_2 = a_2 - c_1$, $w_3 = e_0$, $\mathbf{w}_1 = b_1$, $\mathbf{w_2} = \mathbf{b_2} - \mathbf{\bar{c}_1}$, $\mathbf{w_3} = \mathbf{\bar{e}_0}$. □

Since the number of unshaded m-tilings is T_m, we have for $m \geq 0$,

$$T_m = E_m + O_m. \tag{11}$$

We define

$$\Delta_m = E_m - O_m.$$

Once we determine a closed form for Δ_m, then we can solve for E_m and O_m:

$$E_m = \frac{1}{2}(T_m + \Delta_m) \qquad O_m = \frac{1}{2}(T_m - \Delta_m). \tag{12}$$

Hence, by Theorem 9, the problem of finding a closed for a_n and b_n reduces to solving for Δ_m.

We note that for the uncoupled system, defined by recurrence (7), there are no shade-changing tiles, and therefore all tilings have an even number (namely, zero) of shade-changing tiles. Thus, for this recurrence we have for $m \geq -2$, $E_m = T_m$ and $O_m = 0$, i.e., $\Delta_m = T_m$, and Theorem 9 reduces to equation (8).

In the seven coupled versions of recurrence (9), each has a different set of shade-changing tiles, yet for each of these recurrences, Δ_m (and hence E_m and O_m) can be easily determined by combinatorial considerations. The resulting closed forms given by are practically the same as those presented in [6], which were derived by much heavier algebraic machinery. In each problem, the challenge will be to find an almost one-to-one correspondence between those tilings with an even number of shade-changing tiles and those with an odd number of shade-changing tilings. We derive the seven solutions in order of increasing combinatorial complexity. The first four solutions all take advantage of the same correspondence. We shall sometimes represent our tilings as words from the alphabet $\{s, d, t\}$, where each s, d, or t represents a square, domino or tromino, respectively. For example, the tiling $d^2s^4dt^2s$ is a tiling of length 17, consisting of two dominoes, followed by four squares, a domino, two trominoes, and a square. We begin with the recurrence already described by recurrence (10).

Theorem 10 *For the coupled recurrence*

$$a_n = a_{n-1} + a_{n-2} + b_{n-3}, \qquad b_n = b_{n-1} + b_{n-2} + a_{n-3}$$

we have, for $m \geq -2$, $\Delta_m = \lfloor \frac{m+2}{2} \rfloor$.

Proof. For this recurrence, the shade-changing tiles are the trominoes, and therefore $\Delta_m = E_m - O_m$, where E_m (respectively, O_m) count the unshaded m-tilings with an even (respectively, odd number) of trominoes. The items counted by E_m and O_m can be paired up as follows. For any m-tiling X, find the last occurrence of a "threesome", defined to be any tromino or a square immediately followed by a domino. The m-tiling X is paired up with the m-tiling X' where the last threesome of X is transformed into the other kind of threesome. More formally, the m-tiling $X = YtZ$ (where, for some $k \geq 0$, Y is a k-tiling, t is a tromino, and Z is an $(m-k-3)$-tiling that contains no threesome) is paired up with the m-tiling $X' = YsdZ$. Clearly, for any tiling X, $(X')' = X$, and X' has one more or one fewer tromino than X and therefore the parity of the number of trominoes has changed. The only objects that are not paired up by this rule are those m-tilings with no trominoes and no square immediately followed by a domino. There are exactly $\lfloor \frac{m+2}{2} \rfloor$ of these, namely those tilings of the form $d^i s^{m-2i}$ for $0 \le i \le \lfloor \frac{m}{2} \rfloor$. These exceptional tilings all have zero trominoes, and therefore belong to the set of tilings enumerated by E_m. Therefore, $\Delta_m = \lfloor \frac{m+2}{2} \rfloor$, as asserted. □

Corollary 11 *For the coupled recurrence*

$$a_n = b_{n-1} + a_{n-2} + a_{n-3}, \qquad b_n = a_{n-1} + b_{n-2} + b_{n-3},$$

we have, for $m \ge -2$,

$$\Delta_m = (-1)^m \left\lfloor \frac{m+2}{2} \right\rfloor .$$

For the coupled recurrence

$$a_n = a_{n-1} + b_{n-2} + a_{n-3}, \qquad b_n = b_{n-1} + a_{n-2} + b_{n-3},$$

we have, for $m \ge -2$,

$$\Delta_m = \begin{cases} 1, & \text{if } m \equiv 0 \text{ or } 1 \pmod 4; \\ 0, & \text{if } m \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

For the coupled recurrence

$$a_n = b_{n-1} + b_{n-2} + b_{n-3}, \qquad b_n = a_{n-1} + a_{n-2} + a_{n-3},$$

we have, for $m \ge -2$,

$$\Delta_m = \begin{cases} (-1)^m, & \text{if } m \equiv 0 \text{ or } 1 \pmod 4; \\ 0, & \text{if } m \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

Proof. In these problems, E_m denotes the number of m-tilings with an even number of dominoes, an even number of squares, and an even number of tiles, respectively. Observe that the correspondence given in the proof of Theorem 10 not only changes the parity of the number of trominoes, but also changes the parity of the number of squares, dominoes, and tiles, as well. Hence, we need only compute $\Delta_m = E_m - O_m$ for the exceptional tilings of the form $d^i s^{m-2i}$ for $0 \le i \le \lfloor \frac{m}{2} \rfloor$. For recurrence (11), notice that the number of squares in each of these exceptional tilings has the same parity as m; hence, either E_m or O_m will be zero, depnding on whether m is odd or even. Thus, $\Delta_m = (-1)^m \lfloor \frac{m+2}{2} \rfloor$.

For recurrence (11), the number of dominoes in the exceptional tilings ranges from 0 to $\lfloor \frac{m}{2} \rfloor$. Thus $\Delta_m = 1$, if $\lfloor \frac{m}{2} \rfloor$ is even, i.e., when $m = 4k$ or $4k+1$, and otherwise, $\Delta_m = 0$.

For recurrence (11), the number of tiles in the exceptional tilings, ranges from m down to $\lceil \frac{m}{2} \rceil$. Thus, $\Delta_m = (-1)^m$ if $m - \lceil \frac{m}{2} \rceil$ is even, i.e., when $m = 4k$ or $4k+1$, and otherwise, $\Delta_m = 0$. □

Theorem 12 *For the coupled recurrence*

$$a_n = b_{n-1} + a_{n-2} + b_{n-3} \qquad b_n = a_{n-1} + b_{n-2} + a_{n-3} \tag{13}$$

we have, for $m \geq -2$,

$$\Delta_m = (-1)^m T_m.$$

Proof. Here, the shade-changing tiles are trominoes and squares, so E_m counts the m-tilings that have an even number of trominoes plus squares. But since dominoes have even length, such a tiling requires m to be even. Thus, if m is even, then $E_m = T_m$ and $O_m = 0$; if m is odd, then $E_m = 0$ and $O_m = T_m$. Thus $\Delta_m = E_m - O_m = (-1)^m T_m$. □

The solution of our remaining two recurrences make use of negatively indexed tribonacci numbers. Here are the tribonacci numbers, listed in both directions, with $T_{-3} = 1$, $T_{-2} = 0$, $T_{-1} = 0$, $T_0 = 1$.

$$\ldots, -47, 9, 18, -20, 7, 5, -8, 4, 1, -3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, 13, \ldots$$

The tribonacci recurrence, when written from "right to left" says for all n,

$$T_{n-3} = -T_{n-2} - T_{n-1} + T_n$$

If we let $u_n = T_{-(n+3)}$, then for all n,

$$u_n = -u_{n-1} - u_{n-2} + u_{n-3} \tag{14}$$

with ideal initial conditions $u_0 = 1$, $u_{-1} = u_{-2} = 0$. By our discussion of equation (1), we know that for $n \geq -2$, u_n is the sum of the weights of all weighted n-tilings, where every square and domino has weight -1, and every tromino has weight one. Here, each tiling will have weight $(-1)^k$, where k is the number of squares and dominoes in it. Thus, the negatively indexed tribonacci numbers have a simple combinatorial interpretation.

Theorem 13 *For all $n \geq 1$, the negatively indexed tribonacci number $T_{-n} = u_{n-3}$ is equal to the number of ways to tile a board of length $n-3$ (using squares, dominoes, and trominoes) with an even number of squares and dominoes minus the number of ways to tile such a board with an odd number of squares and dominoes.*

For the coupled recurrence

$$a_n = b_{n-1} + b_{n-2} + a_{n-3} \qquad b_n = a_{n-1} + a_{n-2} + b_{n-3} \tag{15}$$

the shade-changing tiles are squares and dominoes. As an immediate consequence of Theorem 13, we have the following.

Corollary 14 *For the coupled recurrence (15), we have, for $m \geq -2$,*

$$\Delta_m = T_{-(m+3)}$$

We have only one remaining coupled recurrence, and we give two proofs of its solution.

Corollary 15 *For the coupled recurrence recurrence*

$$a_n = a_{n-1} + b_{n-2} + b_{n-3} \qquad b_n = b_{n-1} + a_{n-2} + a_{n-3} \tag{16}$$

we have, for $m \geq -2$,

$$\Delta_m = (-1)^m T_{-(m+3)}$$

Proof. The shade-changing tiles of recurrence (16) are dominoes and trominoes. Consequently, $\Delta_m = v_m$ where v_n satisfies the recurrence for $n \geq 1$,

$$v_n = v_{n-1} - v_{n-2} - v_{n-3}$$

with ideal initial conditions $v_0 = 1$, $v_{-1} = v_{-2} = 0$. Behold, if we let $v_n = (-1)^n u_n = (-1)^n T_{-(n+3)}$, and multiply equation (14) by $(-1)^n$, that v_n satisfies the desired recurrence and initial conditions.

Alternatively, for a more bijective proof, let E'_m (respectively, O'_m) denote the number of m-tilings with an even number (respectively, odd number) of dominoes and *squares*, implicitly counted in the proof of Corollary 14. Likewise, define O'_m and $\Delta'_m = E'_m - O'_m$. But if m is even, then an m-tiling has an even number of dominoes and trominoes iff it has an even number of dominoes and squares; whence, $E_m = E'_m$, $O_m = O'_m$, and $\Delta_m = \Delta'_m$. By the same logic, if m is odd, then $E_m = O'_m$, $O_m = E'_m$ and thus $\Delta_m = -\Delta'_m$. Therefore, $\Delta_m = (-1)^m \Delta'_m = (-1)^m T_{-(m+3)}$. Either way, we have established our last intertwined recurrence. □

4 Intertwined k-th Order Fibonacci Recurrences

We remark that we could have used the first proof strategy of Corollary 15 to arrive at some of the other solutions in the last section. In fact, we can apply the same ideas to arrive at a general method for solving intertwined k-th order Fibonacci recurrences.

We define the k-th order Fibonacci number U_n as follows. $U_{-(k-1)} = \ldots = U_{-1} = 0$, $U_0 = 1$, and for $n \geq 1$,

$$U_n = U_{n-1} + U_{n-2} + \ldots + U_{n-k}$$

Thus, U_n can be interpreted as the number of ways to tile a strip of length n with tiles of any length up to length k, where all tiles, including the initial one, have weight one. Consider the k-th order intertwined recurrence, with arbitrary real numbers $a_0, a_1, \ldots, a_{k-1}$, $b_0, b_1, \ldots, b_{k-1}$, and for $n \geq k$,

$$a_n = c_{n-1} + d_{n-2} + e_{n-3} + \ldots + m_{n-k} \qquad b_n = \bar{c}_{n-1} + \bar{d}_{n-2} + \bar{e}_{n-3} + \ldots + \bar{m}_{n-k} \tag{17}$$

where c is equal to a or b, and $\bar{c}$ denotes the other choice. We define $d, \bar{d}, e, \bar{e}, \ldots, m, \bar{m}$ the same way. Associated with this recurrence, we have, for $1 \leq j \leq k$, a tile of length j is shade-changing if the recurrence for a_n in equation (17) has a term of the form b_{n-j}. Then by the same logic used to derive Theorems 8 and 9, we have that a_n is the sum of the weights of all shaded tilings of length n ending in a light tile, where all tiles except the initial tile, have weight one, and a non-initial tile of length j is preceded by a tile of oppposite shade if and only j is a shade-changing tile. The tilings counted by b_n end in a dark tile. For $1 \leq j \leq k$, the weights of the initial tiles of length j (w_j and $\mathbf{w}_j$) can be chosen as before to be consistent with the initial conditions. For $m \geq 0$, we let E_m and O_m denote the number of (unshaded, unweighted) m-tilings with an even and odd number of color-changing tiles. Then

$$\begin{aligned} a_n = & \textstyle\sum_{j=1}^{k} w_j E_{n-j} + \sum_{j=1}^{k} \mathbf{w}_j O_{n-j} \\ b_n = & \textstyle\sum_{j=1}^{k} w_j O_{n-j} + \sum_{j=1}^{k} \mathbf{w}_j E_{n-j} \end{aligned}$$

By definition, $E_m + O_m = U_m$, and if we define $\Delta_m = E_m - O_m$, then we determine E_m and O_m by solving Δ_m, which satisfies $\Delta_{-(k-1)} = \ldots \Delta_{-1} = 0$, $\Delta_0 = 1$, and the uncoupled recurrence: for $n \geq 1$,

$$\Delta_n = \sum_{j=1}^{k} \varepsilon_j \Delta_{n-j}$$

where $\varepsilon_j = 1$ if tiles of length j are shade-preserving and $\varepsilon_j = -1$ if tiles of length j are shade-changing. Hence, the solution to recurrence (17) can be expressed as a linear combination of U_m and Δ_m, or in terms of E_m and O_m, as desired.

We note that when $k = 2$, the solutions to recurrence (17) have an especially simple form. Here, we consider sequences generated by arbitrary real numbers a_0, a_1, b_0, b_1, and for $n \geq 2$,

$$a_n = c_{n-1} + d_{n-2} \qquad b_n = \bar{c}_{n-1} + \bar{d}_{n-2} \tag{18}$$

where c is equal to a or b, and $\bar{c}$ denotes the other choice; likewise, we define d and $\bar{d}$ the same way. Proceding exactly as we did in Theorems 8 and 9, we have,

Theorem 16 *For a_n and b_n defined by recurrence (18), we have, for $n \geq 1$,*

$$\begin{aligned} a_n = & a_1 E_{n-1} + d_0 E_{n-2} + b_1 O_{n-1} + \bar{d}_0 O_{n-2} \\ b_n = & a_1 O_{n-1} + d_0 O_{n-2} + b_1 E_{n-1} + \bar{d}_0 E_{n-2} \end{aligned}$$

Recall that the Fibonacci number f_m (with initial conditions $f_{-1} = 0$, $f_0 = 1$) counts m-tilings with squares and dominoes. We note that in the uncoupled situation, where $a_n = a_{n-1} + a_{n-2}$, no tiles are shade-changing, therefore $O_m = 0$ for all m, and Theorem 16 reduces to the well-known "Gibonacci" formula $a_n = a_1 f_{n-1} + a_0 f_{n-2}$.

We define for $m \geq -1$, $\Delta_m = E_m - O_m$. As before, the solution to recurrence (18) is equivalent to finding Δ_m, since $E_m + O_m = f_m$, we have $E_m = (f_m + \Delta_m)/2$ and $O_m = (f_m - \Delta_m)/2$. In the theorems that follow, we provide short bijective proofs for Δ_m, resulting in formulas previously obtained (through generating functions and other means) in [4] and [1]. The first identity and proof is very similar to Theorem 12.

Theorem 17 *For the coupled recurrence*

$$a_n = b_{n-1} + a_{n-2} \qquad b_n = a_{n-1} + b_{n-2} \tag{19}$$

we have, for $m \geq -1$,

$$\Delta_m = (-1)^m f_m$$

Proof. E_m counts m-tilings with an even number of squares, equal to f_m when m is even, and zero when m is odd. A similar conclusion for O_m immediately implies $\Delta_m = E_m - O_m = (-1)^m T_m$. □

Theorem 18 *For the coupled recurrence*

$$a_n = b_{n-1} + b_{n-2}, \qquad b_n = a_{n-1} + a_{n-2},$$

we have, for $m \geq -1$,

$$\Delta_m = \begin{cases} 1, & \text{if } m \equiv 0 \pmod 3; \\ -1, & \text{if } m \equiv 1 \pmod 3; \\ 0, & \text{if } m \equiv 2 \pmod 3; \end{cases}$$

For the coupled recurrence

$$a_n = a_{n-1} + b_{n-2}, \qquad b_n = b_{n-1} + a_{n-2},$$

we have, for $m \geq -1$,

$$\Delta_m = \begin{cases} 1, & \text{if } m \equiv 0 \text{ or } 1 \pmod 6; \\ -1, & \text{if } m \equiv 3 \text{ or } 4 \pmod 6; \\ 0, & \text{if } m \equiv 2 \text{ or } 5 \pmod 6; \end{cases}$$

Proof. In the first recurrence, all tiles are shade-changing. For any m-tiling that ends with a domino, then replace the final domino with two squares, and vice versa. If the tiling ends with square preceded by a domino, then ignore those two tiles and apply the same procedure to the $(m-3)$-tiling. In other words, we associate, for every m-tiling of the form $X = Yd(ds)^k$ the m-tiling $X' = Yss(ds)^k$. The only unpaired tilings are $(ds)^k$ (which has an even number of tiles and can only occur when $m = 3k$) and $s(ds)^k$ (which has an odd number of tilings and can only occur when $m = 3k+1$).

In the second recurrence, only the dominoes are shade-changing, but the bijection from the preceding recurrence also changes the number of dominoes, and therefore we have no unpaired tilings when m is of the form $3m+2$. Otherwise, we have exactly one unpaired m-tiling as before. That tiling has an even number of dominoes, when it is of the form $(ds)^{2k}$ or $s(ds)^{2k}$ (which can only occur when $m = 6k$ or $m = 6k+1$); that tiling has an odd number of dominoes when it is of the form $(ds)^{2k+1}$ or $s(ds)^{2k+1}$ (which can only occur when $m = 6k+3$ or $m = 6k+4$). □

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