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On the Ramsey number $R(3, 6)$

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Dedicated to Professor G.L. Cariolaro on the occasion of his 70th birthday.

Abstract

We give an easy proof for $R(3, 6) = 18$.

1 Introduction

The Ramsey number $R(k, l)$ is defined as the least positive integer n with the property that every graph on n vertices either contains k mutually adjacent vertices or l mutually nonadjacent vertices. A graph on $R(k, l)-1$ vertices which contains neither k mutually adjacent vertices nor l mutually nonadjacent vertices is called $R(k, l)$-*critical.*

Ramsey numbers are generally difficult to compute and only very few are known (see [7]). Most proofs of either exact or approximate estimation of Ramsey numbers involve computer computations. As far as we know there exist computer-free proofs of exact values of (nontrivial) Ramsey numbers only for $R(3, 3) = 6$, $R(3, 4) = 9$, $R(3, 5) = 14$, $R(3, 6) = 18$, $R(3, 7) = 23$ and $R(4, 4) = 18$.

We shall here only be concerned with the Ramsey number $R(3, 6)$. The proofs concerning $R(3, 6)$ known to the author are either not elementary [3], not immediately accessible [5] or in Hungarian [6].

We will provide an elementary proof for $R(3, 6) = 18$, which is shorter and simpler than all those mentioned above. Informally speaking, this says that, given 18 arbitrary people, there are either 3 who are mutually acquainted or there are 6 who are mutually strangers to each other, but the same fact does not necessarily hold if we replace the number 18 by 17.

2 The main result

We let $|G|$ denote the order (number of vertices) of graph G and, if $S \subset V(G)$, we let $N(S)$ denote the set of vertices which are adjacent to at least one vertex in S,

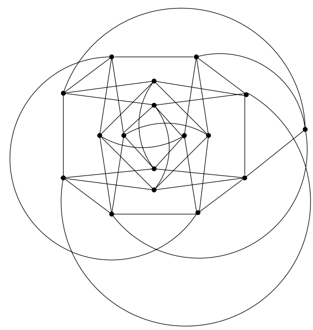

Figure 1: The existence of this graph proves that $R(3,6) \geq 18$.

and $N[S] = S \cup N(S)$.

An *IS* is an independent set of vertices and a k-*IS* is an IS of size k.

Theorem 1 $R(3,6) = 18$.

Proof. It may be checked (with a bit of patience) that the graph in Fig. 1 (taken from [3]) is a triangle-free graph of order 17 with no independent set of size 6, therefore proving the inequality $R(3,6) \geq 18$. Thus we are left only with the proof that $R(3,6) \leq 18$.

Let G be a triangle-free graph with 18 vertices. We shall prove that G contains a 6-IS. Arguing by contradiction, assume that G does not have a 6-IS.

Claim 1: G is 5-regular.

Since G is triangle-free, for any vertex v, $N(v)$ is an IS, and hence $|N(v)| \leq 5$, i.e. $\deg(v) \leq 5$. Suppose now that $\deg(v) < 5$. Let $H = G - N[v]$. Clearly $|H| \geq 13$. If $|H| \geq 14 = R(3,5)$, then H has a 5-IS, which together with v forms a 6-IS, giving a contradiction. Therefore $|H| = 13$, and hence $\deg(v) = 4$. Then H is the (unique) $R(3,5)$-critical graph and is in particular 4-regular. Let $t \in N(v)$. Then t has (by the first part of the proof) at least 3 neighbours t_1, t_2, t_3 in H, each of which is independent from $N(v) \setminus \{t\}$ (because t_1, t_2, t_3 have 4 neighbours in H and one more neighbour in $\{t\}$). Hence $(N(v) \setminus \{t\}) \cup \{t_1, t_2, t_3\}$ is a 6-IS, giving a contradiction.

Claim 2: For any vertex v there are exactly 4 non-neighbours p_i of v such that $|N(p_i) \cap N(v)| = 1$ and 8 non-neighbours q_i of v such that $|N(q_i) \cap N(v)| = 2$. Moreover the p_i's share 4 distinct neighbours with v and the q_i's share 8 distinct pairs of neighbours with v.

Let u, v be nonadjacent. We first prove that $1 \leq |N(u) \cap N(v)| \leq 2$. If $|N(u) \cap N(v)| = 0$ then, in particular, v is independent from $N(u)$, so that the set $\{v\} \cup N(u)$ is a 6-IS. Thus $|N(u) \cap N(v)| \geq 1$. Now suppose that $|N(u) \cap N(v)| \geq 3$. Let $H = G - N[u, v]$. Then $|H| \geq 9 = R(3,4)$ so that, since H is triangle-free, there is in H a 4-IS. This, together with u and v, gives a 6-IS. Thus $1 \leq |N(u) \cap N(v)| \leq 2$.

Let now $H = G - N[v]$. It is easy to see that there are exactly 20 edges between H and $N[v]$. Simply counting those vertices in H that send 2 edges to $N[v]$ and those that send only 1, we get the first part of the Claim.

For the second part, suppose that the vertices p_1, p_2 are adjacent to the same vertex $u \in N(v)$. Then in particular the set $\{p_1, p_2\} \cup (N(v) \setminus \{u\})$ is a 6-IS, contradicting the assumption. Thus each of p_1, p_2, p_3, p_4 is joined to a distinct vertex of $N(v)$. Finally, suppose that $q_1, q_2 \in V(H)$ are joined to the same pair $\{x, y\} \subset N(v)$. Then in particular the nonadjacent vertices x, y have the common neighbours $\{v, q_1, q_2\}$, contradicting the first part of Claim 2.

Claim 3: With the notations of Claim 2, $\{p_1, p_2, p_3, p_4\}$ induce a 4-cycle in G.

Label the vertices of G in such a way that $N(v) = \{t, s_1, s_2, s_3, s_4\}$, where, using Claim 2, we assume that $s_1p_1, s_2p_2, s_3p_3, s_4p_4$ are the only edges between the p_i's and $N(v)$. Notice that no p_i is a neighbour of t because the p_i's, by Claim 2, share only one neighbour with v. Rename the q_i's as follows: let $N(t) \setminus \{v\} = \{t_1, t_2, t_3, t_4\}$ and let the remaining q_i's be w_1, w_2, w_3, w_4. Thus $V(G) = \{v, t, s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4, p_1, p_2, p_3, p_4, w_1, w_2, w_3, w_4\}$.

Each of the s_i's sends exactly 1 edge to v, 1 edge to the p_i's, 1 edge to the t_i's and hence 2 edges to the w_i's. Moreover there cannot be two s_i's, say s_1, s_2, which are joined to the same pair, say $\{w_1, w_2\}$ of w_i's, otherwise s_1, s_2 would share the 3 neighbours $\{v, w_1, w_2\}$, contradicting Claim 2. Similarly no w_i is adjacent to more than two s_i's, since otherwise the pair $\{v, w_i\}$ would share too many neighbours.

Now suppose that two of the s_i's, say s_1, s_2, are adjacent to the same w_i, say w_1. None of the vertices p_1, p_2, s_1, s_2, w_1 is joined to any of the three independent vertices $\{s_3, s_4, t\}$, so that, to avoid a 6-IS, the subgraph induced by $\{p_1, p_2, s_1, s_2, w_1\}$ cannot contain a 3-IS, and hence (to avoid triangles) must be a 5-cycle. Thus, in particular, p_1 and p_2 are adjacent. A similar argument can be repeated for any pair of vertices in $\{s_1, s_2, s_3, s_4\}$ which have a w_i as common neighbour, and since there are exactly 4 such pairs there are exactly 4 edges in the subgraph induced by $\{p_1, p_2, p_3, p_4\}$, and hence (to avoid triangles) this subgraph is a 4-cycle, thus proving Claim 3.

Final step

Without loss of generality we assume that $p_1p_2p_3p_4p_1$ is the 4-cycle induced by $\{p_1, p_2, p_3, p_4\}$ in G. Each p_i shares at least one neighbour with t by Claim 2. Furthermore, by Claim 2 and the fact that G is triangle-free, the p_i's do not have common neighbours except in $\{p_1, p_2, p_3, p_4\}$. Thus each of the p_i's is joined to a single distinct t_i, and we shall assume (by possibly relabelling the t_i's) that $p_it_i \in E(G)$ for each $i = 1, 2, 3, 4$.

There are exactly 4 edges between the p_i's and the w_i's and (by possibly relabelling the w_i's) we can assume that they are the edges p_iw_i, $i = 1, 2, 3, 4$.

The vertices v and w_1 share exactly two neighbours and the only possible candidates are in $\{s_2, s_3, s_4\}$. Similarly t and w_1 share exactly two neighbours and the only possible candidates are in $\{t_2, t_3, t_4\}$.

Hence there is an $i \neq 1$ such that the vertex w_1 is joined to s_i and t_i. If $i = 2$ or $i = 4$, the vertices p_i and w_1 have 3 common neighbours, which contradicts Claim 2. Hence $i = 3$. By symmetry, we can further assume that $w_1 s_2 \in E(G)$. Hence, by the above remark, $w_1 t_4 \in E(G)$.

Consider now the vertex s_2. We proved above that each s_i is adjacent to exactly one t_i. It cannot be $s_2 t_2 \in E(G)$ to avoid the triangle $s_2 p_2 t_2$. Similarly it cannot be $s_2 t_3 \in E(G)$ to avoid the triangle $s_2 w_1 t_3$ and it cannot be $s_2 t_4 \in E(G)$ to avoid the triangle $s_2 w_1 t_4$. Thus the only possibility is that $s_2 t_1 \in E(G)$. But now s_2 and p_1 have the three common neighbours $\{p_2, w_1, t_1\}$, which contradicts Claim 2. This contradiction completes the proof.

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