AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **39** (2007), Pages 163–174

Optimal double circulant self-dual codes over $\mathbb{F}_4$ II

T. Aaron Gulliver

Department of Electrical and Computer Engineering
University of Victoria
P.O. Box 3055, STN CSC
Victoria, B.C. V8W 3P6
Canada

Masaaki Harada   Hiroki Miyabayashi

Department of Mathematical Sciences
Yamagata University
Yamagata 990–8560
Japan

Abstract

The largest minimum weights among all double circulant Hermitian self-dual codes are known for lengths $n \le 40$. A classification of double circulant Hermitian self-dual codes with the largest minimum weights is known for lengths up to 26. We extend the classification of such double circulant Hermitian self-dual codes to lengths up to 40.

1 Introduction

For a Hermitian self-dual $[n, n/2, d]$ code over $\mathbb{F}_4$, the following upper bound on the minimum weight is known [7]:

$$d \le 2\left\lfloor \frac{n}{6} \right\rfloor + 2.$$

A Hermitian self-dual $[n, n/2, d]$ code with minimum weight $d = 2\lfloor n/6 \rfloor + 2$ is called *extremal*. For instance, extremal Hermitian self-dual codes are known to exist for even lengths $n \le 10$, $14 \le n \le 22$, and $n = 28, 30$, while there is no extremal code for lengths $n = 12, 24, 26$ (cf. [2], [6], [7], [9], [10]). The weight enumerator of an extremal Hermitian self-dual code of length n is uniquely determined (see Theorem 13 in [7]). For lengths 28 and 30, see W_{28} in [3] and Table I in [7], respectively. A Hermitian self-dual code with the largest minimum weight among all Hermitian self-dual codes of length n is called *optimal*. Of course, an extremal Hermitian self-dual code is optimal. All Hermitian self-dual codes have been classified for lengths $n \le 16$ [2], [7] and all extremal Hermitian self-dual codes are known for lengths 18 and 20 [4].

Binary double circulant self-dual codes provide many self-dual codes with large minimum weights (cf. [10]). The first author [3] determined the largest minimum weight $d_{DC}(n)$ among all double circulant Hermitian self-dual codes over $\mathbb{F}_4$ for lengths $n \leq 40$. For example, $d_{DC}(22) = d_{DC}(24) = d_{DC}(26) = 8$ and $d_{DC}(36) = 12$ (see Table VI in [3] (see also Table 7) for $d_{DC}(n)$ $(n \leq 40)$). In particular, a double circulant Hermitian self-dual $[36, 18, 12]$ code improved the lower bound on the largest minimum weight among all known $[36, 18]$ codes. Recently the third author [8] gave a classification of double circulant Hermitian self-dual codes with minimum weight 8 for lengths $n = 22, 24, 26$.

In this note, we extend the classification of double circulant Hermitian self-dual codes with minimum weight $d_{DC}(n)$ to lengths up to $n = 40$. We also give a remark on notions of equivalence for double circulant Hermitian self-dual codes.

2 Preliminaries

In this section, we give some basic definitions and properties. Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ be the Galois field with four elements, where $\bar{\omega} = \omega^2 = \omega + 1$. An $[n, k]$ code C over $\mathbb{F}_4$ is a k-dimensional subspace of $\mathbb{F}_4^n$. The value n is called the length of C. The weight $\text{wt}(x)$ of a vector $x \in \mathbb{F}_4^n$ is the number of non-zero components of x. The minimum non-zero weight of all codewords in C is called the minimum weight of C and an $[n, k]$ code with minimum weight d is called an $[n, k, d]$ code. The weight enumerator W of C is given by $W = \sum_{i=0}^{n} A_i y^i$ where A_i is the number of codewords of weight i in C.

For two vectors $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}_4^n$, the following inner product

$$x \cdot y = \sum_{i=1}^{n} x_i \, {y_i}^2,$$

is known as the Hermitian inner product. The *Hermitian dual code* $C^{\perp}$ of C is defined as

$$C^{\perp} = \{x \in \mathbb{F}_4^n | x \cdot c = 0 \text{ for all } c \in C\}.$$

A code C is called *Hermitian self-dual* if $C = C^{\perp}$. It is known that an $[n, k]$ code C is Hermitian self-dual if and only if C is even and $n = 2k$ [7].

Two codes C and C' are *equivalent* if there is some monomial matrix M over $\mathbb{F}_4$ such that $C' = CM = \{cM | c \in C\}$ [7]. A monomial matrix which maps C to itself is called an automorphism of C and the set of all automorphisms of C forms the automorphism group $\text{Aut}(C)$ of C.

Let D_p and D_b be codes with generator matrices of the form

$$\begin{pmatrix} I_m & R \end{pmatrix}, \tag{1}$$

and

$$\left(\begin{array}{c|cccc} & a & b & \cdots & b \\ & c & & & \\ I_{m+1} & \vdots & & R' & \\ & c & & & \end{array} \right), \tag{2}$$

which are $m \times 2m$ and $(m+1) \times 2(m+1)$ matrices, respectively, where I_m is the identity matrix of order m, R and R' are $m \times m$ circulant matrices and $a, b, c \in \mathbb{F}_4$. The codes D_p and D_b are called *pure double circulant* and *bordered double circulant*, respectively. The two families are called double circulant codes.

3 Results

In this section, we give a classification of double circulant Hermitian self-dual codes with minimum weight $d_{DC}(n)$ for lengths $n = 28, 30, \ldots, 40$.

3.1 Length 28

Using approaches similar to those given in [3] and [8], we have found all distinct extremal pure double circulant Hermitian self-dual $[28, 14, 10]$ codes which must be checked further for equivalence to complete the classification. This was done by considering all 14×14 circulant matrices R in the generator matrices (1), and produced 18 distinct codes. By MAGMA [1], we have verified that the codes are equivalent to the pure double circulant code P_{28} with the first row of R equal to

$$(\omega, \omega, 1, 1, \bar{\omega}, \bar{\omega}, \omega, \bar{\omega}, \omega, 1, 0, 1, 0, 0).$$

The automorphism group of this code is of order 42. In this work, the calculation of automorphism groups and determination of equivalence of codes were done using MAGMA.

Similarly, by considering all possible 13×13 circulant matrices R' and borders (a, b, c) in the generator matrices (2), we have verified that there is no extremal bordered double circulant Hermitian self-dual $[28, 14, 10]$ code. In this case, Lemma 8 in [8] can be used to reduce the number of possibilities substantially. Hence we have the following:

Proposition 1. *There is a unique extremal pure double circulant Hermitian self-dual* $[28, 14, 10]$ *code, up to equivalence. There is no extremal bordered double circulant Hermitian self-dual* $[28, 14, 10]$ *code.*

3.2 Length 30

By considering possible 15×15 circulant matrices in generator matrices (1) which must be checked further for equivalence to complete the classification, we have verified that all extremal pure double circulant Hermitian self-dual $[30, 15, 12]$ codes are equivalent to the pure double circulant code P_{30} with the first row of R equal to

$$(1, 1, 1, \omega, 0, \omega, \bar{\omega}, \omega, 0, \omega, 1, 1, 1, 0, 0).$$

The automorphism group of the code is of order 36540. Similarly, we have verified that all extremal bordered double circulant Hermitian self-dual $[30, 15, 12]$ codes are equivalent to the bordered double circulant code B_{30} with the first row of R' equal to

$$(\bar{\omega}, \bar{\omega}, 1, \bar{\omega}, 1, 0, \omega, 0, \omega, \omega, 1, 1, 0, 0),$$

and borders $(a, b, c) = (1, 1, 1)$. In addition, we have verified that P_{30} and B_{30} are equivalent. Hence we have the following:

Proposition 2. *There is a unique extremal pure double circulant Hermitian self-dual* $[30, 15, 12]$ *code, up to equivalence. There is a unique extremal bordered double circulant Hermitian self-dual* $[30, 15, 12]$ *code, up to equivalence. The two codes are equivalent.*

3.3 Length 32

By Theorem 13 in [7], the possible weight enumerators of Hermitian self-dual $[n, n/2, 2\lfloor n/6 \rfloor]$ codes are determined up to an integral parameter α. For example, the possible weight enumerators of Hermitian self-dual $[32, 16, 10]$ codes are as follows:

$$W_{32,\alpha} = 1 + (1968 + \alpha)y^{10} + (53928 - 7\alpha)y^{12} + (1056240 + 15\alpha)y^{14} + (12050514 + 15\alpha)y^{16} + (84996960 - 150\alpha)y^{18} + \cdots,$$

(see $W_{32}(x, y)$ in [5] for the full weight enumerator). Of course, a code with weight enumerator $W_{32,-1968}$ is extremal.

We have found all distinct pure double circulant Hermitian self-dual $[32, 16, 10]$ codes which must be checked further for equivalence. We complete the classification of the codes $P_{32,i}$ by listing the first rows r of R in (1) of the codes in Table 1 where the third column gives the integers α in the weight enumerators $W_{32,\alpha}$ and the last column gives the orders of the automorphism groups $\mathrm{Aut}(P_{32,i})$. In addition, we have verified that there is no bordered double circulant Hermitian self-dual $[32, 16, 10]$ code. Hence we have the following:

Proposition 3. *There are* 19 *inequivalent pure double circulant Hermitian self-dual* $[32, 16, 10]$ *codes. There is no bordered double circulant Hermitian self-dual* $[32, 16, 10]$ *code.*

3.4 Length 34

The possible weight enumerators of Hermitian self-dual $[34, 17, 10]$ codes are as follows:

$$W_{34,\alpha} = 1 + (2244 + \alpha)y^{10} + (28152 - 4\alpha)y^{12} + (767448 - 6\alpha)y^{14} + (11131362 + 60\alpha)y^{16} + (99245388 - 105\alpha)y^{18} + \cdots,$$

(see $W_{34}(x, y)$ in [5] for the full weight enumerator). Of course, a code with weight enumerator $W_{34,-2244}$ is extremal.

Table 1: Pure double circulant Hermitian self-dual $[32, 16, 10]$ codes

Codes	Rows r	α	$\lvert \mathrm{Aut} \rvert$
$P_{32,1}$	$(\bar{\omega},\bar{\omega},1,\bar{\omega},1,1,\bar{\omega},0,1,1,\omega,1,1,1,0,0)$	-768	48
$P_{32,2}$	$(\omega,1,\bar{\omega},\bar{\omega},\omega,\omega,\omega,\omega,1,1,\bar{\omega},1,\omega,1,1,0)$	-768	48
$P_{32,3}$	$(1,1,\omega,\bar{\omega},1,1,\bar{\omega},1,\bar{\omega},\bar{\omega},\omega,\omega,1,1,1,0)$	-624	48
$P_{32,4}$	$(1,\omega,1,0,\omega,\omega,\bar{\omega},0,\bar{\omega},0,\omega,1,1,0,1,0)$	-576	96
$P_{32,5}$	$(\omega,1,\bar{\omega},1,\bar{\omega},\omega,\omega,1,0,\omega,0,1,0,1,0,0)$	-336	96
$P_{32,6}$	$(1,1,\omega,\omega,0,1,\omega,1,0,\omega,\omega,1,1,0,0,0)$	-192	96
$P_{32,7}$	$(1,\omega,1,1,1,\omega,\bar{\omega},\omega,\bar{\omega},\omega,\omega,1,1,0,0,0)$	-192	48
$P_{32,8}$	$(1,1,1,\bar{\omega},\omega,\bar{\omega},1,\omega,\omega,\omega,1,0,1,1,0,0)$	-192	48
$P_{32,9}$	$(\bar{\omega},1,0,\bar{\omega},\bar{\omega},1,\omega,\omega,\omega,\omega,1,1,1,1,0,0)$	-192	48
$P_{32,10}$	$(\omega,1,\bar{\omega},\bar{\omega},1,0,\omega,\omega,1,\bar{\omega},1,\omega,1,1,0,0)$	-192	48
$P_{32,11}$	$(\omega,\bar{\omega},\bar{\omega},1,\omega,0,\omega,1,\bar{\omega},\bar{\omega},1,1,\omega,1,0,0)$	-192	48
$P_{32,12}$	$(\omega,1,\bar{\omega},\bar{\omega},\bar{\omega},1,\bar{\omega},\omega,\bar{\omega},\omega,\omega,1,1,1,1,0)$	-192	48
$P_{32,13}$	$(\omega,\bar{\omega},\bar{\omega},\omega,1,\bar{\omega},1,1,1,\omega,\omega,0,1,1,0,0)$	-144	48
$P_{32,14}$	$(\omega,1,0,\bar{\omega},\omega,\bar{\omega},\omega,0,1,1,0,1,1,1,0,0)$	-144	48
$P_{32,15}$	$(\omega,\omega,1,\bar{\omega},\bar{\omega},1,\omega,1,\bar{\omega},1,0,\omega,1,1,0,0)$	-144	48
$P_{32,16}$	$(1,\omega,\bar{\omega},1,1,1,1,\omega,\omega,0,\omega,0,1,0,0,0)$	-48	48
$P_{32,17}$	$(1,\bar{\omega},\omega,\bar{\omega},1,\bar{\omega},1,1,\omega,\bar{\omega},1,\omega,0,1,0,0)$	-48	48
$P_{32,18}$	$(1,\bar{\omega},\bar{\omega},1,\omega,\bar{\omega},\bar{\omega},\bar{\omega},0,\bar{\omega},\omega,\omega,1,1,0,0)$	-48	48
$P_{32,19}$	$(\omega,\bar{\omega},1,\omega,\bar{\omega},\bar{\omega},\bar{\omega},\omega,\omega,\omega,1,\omega,1,\omega,1,0)$	0	48

In Tables 2 and 3, we complete the classification of pure and bordered double circulant Hermitian self-dual $[34, 17, 10]$ codes $P_{34,i}$ and $B_{34,i}$ by listing the first rows r of R and R' in generator matrices (1) and (2), respectively, where the third column gives the integers α in the weight enumerators $W_{34,\alpha}$ and the last column gives the orders of the automorphism groups. Note that all bordered double circulant codes have borders $(a, b, c) = (\omega, 1, 1)$.

By comparing the automorphism groups, there is no pair of equivalent codes among $P_{34,i}$ $(i = 1, 2, \ldots, 58)$ and $B_{34,j}$ $(j = 1, 2, \ldots, 17)$. Hence we have the following:

Proposition 4. *There are* 58 *inequivalent pure double circulant Hermitian self-dual* $[34, 17, 10]$ *codes. There are* 17 *inequivalent bordered double circulant Hermitian self-dual* $[34, 17, 10]$ *codes. There are* 75 *inequivalent double circulant Hermitian self-dual* $[34, 17, 10]$ *codes.*

Table 2: Pure double circulant Hermitian self-dual $[34, 17, 10]$ codes

Codes	Rows r	α	$\lvert \mathrm{Aut} \rvert$
$P_{34,1}$	$(1,\omega,\omega,\omega,1,\bar{\omega},\omega,\omega,\omega,\bar{\omega},1,\omega,\omega,\omega,1,0,0)$	-2040	102
$P_{34,2}$	$(\omega,\omega,1,1,1,\omega,1,\omega,\omega,\bar{\omega},\omega,\omega,1,\omega,1,1,1)$	-2040	816
$P_{34,3}$	$(\omega,\omega,1,\bar{\omega},\bar{\omega},\omega,\bar{\omega},\bar{\omega},\omega,\bar{\omega},\bar{\omega},1,\omega,\omega,1,1,1)$	-2040	204
$P_{34,4}$	$(1,\omega,\omega,1,1,\omega,\omega,1,0,0,0,1,\omega,1,0,0,0)$	-1938	102
$P_{34,5}$	$(1,\omega,\bar{\omega},1,1,1,\omega,\omega,\omega,1,1,1,\bar{\omega},\omega,1,0,0)$	-1938	102
$P_{34,6}$	$(\omega,\bar{\omega},\bar{\omega},1,\omega,1,\omega,1,\bar{\omega},\bar{\omega},\omega,1,1,1,1,1,1)$	-1887	204
$P_{34,7}$	$(\bar{\omega},\omega,1,\omega,1,\bar{\omega},1,\bar{\omega},\bar{\omega},0,\omega,1,\omega,1,0,0,0)$	-1785	51
$P_{34,8}$	$(\omega,\bar{\omega},\bar{\omega},0,0,\bar{\omega},\omega,1,\omega,\omega,0,1,0,0,0,0,0)$	-1734	51
$P_{34,9}$	$(\bar{\omega},1,1,1,\bar{\omega},\omega,\bar{\omega},\bar{\omega},\omega,1,\omega,1,0,0,1,0,0)$	-1734	51
$P_{34,10}$	$(1,\omega,\bar{\omega},\bar{\omega},1,1,1,\bar{\omega},1,1,1,\bar{\omega},\bar{\omega},\omega,1,0,0)$	-1734	102
$P_{34,11}$	$(1,\omega,\bar{\omega},\bar{\omega},\omega,\bar{\omega},0,\omega,\bar{\omega},0,\omega,1,1,0,0,0,0)$	-1581	51
$P_{34,12}$	$(1,\omega,1,1,1,1,\omega,1,1,\omega,\omega,1,0,1,0,0,0)$	-1581	51
$P_{34,13}$	$(\bar{\omega},1,\bar{\omega},0,1,1,\bar{\omega},1,\omega,\omega,\omega,\omega,1,1,0,0,0)$	-1581	51
$P_{34,14}$	$(\bar{\omega},1,\omega,1,\bar{\omega},1,1,1,0,1,\bar{\omega},\omega,1,1,0,0,0)$	-1581	51
$P_{34,15}$	$(\omega,\bar{\omega},1,0,\bar{\omega},\bar{\omega},\omega,\bar{\omega},\omega,\omega,1,1,0,0,0,0,0)$	-1530	51
$P_{34,16}$	$(\bar{\omega},\omega,\omega,\bar{\omega},\omega,\bar{\omega},1,\bar{\omega},\omega,1,1,0,0,1,1,0,0)$	-1530	51
$P_{34,17}$	$(\bar{\omega},1,\omega,1,1,1,\omega,\bar{\omega},1,\omega,1,\omega,1,\omega,1,0,0)$	-1530	51
$P_{34,18}$	$(1,\bar{\omega},\omega,1,\omega,\bar{\omega},\omega,\bar{\omega},\bar{\omega},1,\omega,1,\omega,\omega,1,0,0)$	-1479	51
$P_{34,19}$	$(\bar{\omega},\omega,\bar{\omega},1,0,\omega,\bar{\omega},\omega,\bar{\omega},1,\omega,0,0,1,0,0,0)$	-1428	51
$P_{34,20}$	$(1,\bar{\omega},\bar{\omega},\bar{\omega},\omega,\bar{\omega},\bar{\omega},\omega,\omega,1,0,\omega,1,1,0,0,0)$	-1428	51
$P_{34,21}$	$(\bar{\omega},1,1,\bar{\omega},0,\omega,\bar{\omega},1,\omega,1,\omega,\omega,1,1,0,0,0)$	-1428	51
$P_{34,22}$	$(1,0,\omega,\omega,\omega,1,\omega,\omega,\omega,1,\omega,\omega,\omega,0,1,0,0)$	-1428	102
$P_{34,23}$	$(\bar{\omega},\omega,1,\bar{\omega},\bar{\omega},\bar{\omega},1,\omega,1,1,\omega,\bar{\omega},\omega,1,1,0,0)$	-1428	51
$P_{34,24}$	$(1,\omega,\omega,1,1,\bar{\omega},1,1,1,\bar{\omega},1,1,\omega,\omega,1,0,0)$	-1428	102
$P_{34,25}$	$(1,\bar{\omega},0,\omega,\omega,0,1,\bar{\omega},\bar{\omega},\bar{\omega},\omega,1,1,0,0,0,0)$	-1377	51
$P_{34,26}$	$(1,\bar{\omega},\omega,\omega,\omega,\bar{\omega},1,1,\omega,1,1,\omega,1,0,0,0,0)$	-1377	51
$P_{34,27}$	$(1,1,1,\omega,\bar{\omega},1,1,\bar{\omega},\omega,1,1,1,0,0,1,0,0)$	-1377	102
$P_{34,28}$	$(\omega,1,\bar{\omega},0,\bar{\omega},1,1,\bar{\omega},\omega,\bar{\omega},\omega,1,1,0,1,0,0)$	-1377	51
$P_{34,29}$	$(1,\bar{\omega},\omega,1,1,\bar{\omega},\bar{\omega},\omega,1,\omega,1,\omega,1,1,1,0,0)$	-1377	51
$P_{34,30}$	$(\bar{\omega},1,0,1,1,\bar{\omega},\omega,1,\omega,1,1,0,1,0,0,0,0)$	-1326	51
$P_{34,31}$	$(1,0,\omega,\bar{\omega},1,0,1,0,1,\bar{\omega},\omega,0,1,0,0,0,0)$	-1326	102
$P_{34,32}$	$(\omega,\bar{\omega},0,\omega,\omega,1,1,\omega,1,\bar{\omega},\omega,0,0,1,0,0,0)$	-1326	51
$P_{34,33}$	$(\omega,\bar{\omega},1,1,\omega,1,1,\bar{\omega},\omega,1,\bar{\omega},1,\bar{\omega},\omega,1,0,0)$	-1326	51
$P_{34,34}$	$(1,\bar{\omega},1,1,\bar{\omega},\omega,\bar{\omega},\bar{\omega},0,0,\omega,1,0,1,0,0,0)$	-1275	51
$P_{34,35}$	$(\omega,1,1,1,0,0,\omega,\omega,1,1,\bar{\omega},\omega,0,1,0,0,0)$	-1275	51
$P_{34,36}$	$(1,\omega,\omega,\omega,\bar{\omega},0,\omega,\bar{\omega},1,\omega,1,1,1,1,0,0,0)$	-1275	51
$P_{34,37}$	$(1,\bar{\omega},\bar{\omega},\omega,\omega,\bar{\omega},\bar{\omega},\omega,\omega,0,1,\omega,1,0,1,0,0)$	-1275	51
$P_{34,38}$	$(\omega,0,\bar{\omega},\omega,1,\bar{\omega},1,\omega,\bar{\omega},1,\bar{\omega},1,\omega,0,1,0,0)$	-1275	51
$P_{34,39}$	$(\omega,1,1,\omega,\bar{\omega},\bar{\omega},1,\bar{\omega},1,\bar{\omega},1,1,\omega,1,1,0,0)$	-1275	51
$P_{34,40}$	$(\omega,1,1,1,\omega,\omega,1,\omega,1,1,\bar{\omega},\omega,\omega,1,1,0,0)$	-1275	51
$P_{34,41}$	$(\omega,1,\bar{\omega},\bar{\omega},1,\omega,1,1,\omega,\bar{\omega},1,1,1,\omega,1,0,0)$	-1275	51
$P_{34,42}$	$(\omega,0,\bar{\omega},\omega,\bar{\omega},1,1,\omega,\bar{\omega},\bar{\omega},1,\omega,1,0,1,0,0)$	-1224	51
$P_{34,43}$	$(1,\bar{\omega},1,\omega,\bar{\omega},\omega,\bar{\omega},\omega,1,\bar{\omega},\bar{\omega},1,\omega,1,1,0,0)$	-1224	51
$P_{34,44}$	$(\omega,1,\bar{\omega},\omega,1,1,\bar{\omega},1,\omega,\bar{\omega},\bar{\omega},\omega,1,\omega,1,0,0)$	-1224	51
$P_{34,45}$	$(1,\bar{\omega},1,0,\omega,\omega,\bar{\omega},0,1,\omega,0,1,0,0,0,0,0)$	-1122	51
$P_{34,46}$	$(\omega,\bar{\omega},1,1,\bar{\omega},0,\omega,\bar{\omega},\bar{\omega},\bar{\omega},\bar{\omega},\omega,\omega,1,0,0,0)$	-1122	51
$P_{34,47}$	$(1,\bar{\omega},\omega,\omega,\omega,1,\omega,\bar{\omega},1,0,\bar{\omega},\bar{\omega},\omega,1,0,0,0)$	-1122	51
$P_{34,48}$	$(\bar{\omega},\omega,\omega,\bar{\omega},0,\bar{\omega},1,\bar{\omega},\omega,\omega,1,1,1,0,1,0,0)$	-1122	51
$P_{34,49}$	$(1,1,\omega,\omega,\bar{\omega},\omega,\omega,1,1,\bar{\omega},1,1,\omega,1,1,0,0)$	-1122	51
$P_{34,50}$	$(\omega,\bar{\omega},1,\omega,\bar{\omega},1,\bar{\omega},\bar{\omega},\omega,\omega,\omega,1,\omega,\omega,1,0,0)$	-1071	51
$P_{34,51}$	$(1,\bar{\omega},\bar{\omega},1,\omega,\bar{\omega},\bar{\omega},1,\bar{\omega},0,\omega,0,1,0,0,0,0)$	-1020	51
$P_{34,52}$	$(\bar{\omega},1,1,\omega,\omega,\omega,\bar{\omega},1,1,\omega,1,1,0,0,1,0,0)$	-1020	51
$P_{34,53}$	$(\omega,\omega,\bar{\omega},\bar{\omega},\omega,1,\bar{\omega},\omega,1,\omega,\omega,1,1,0,0,0,0)$	-969	51
$P_{34,54}$	$(1,\bar{\omega},0,\bar{\omega},\bar{\omega},\bar{\omega},1,\bar{\omega},\bar{\omega},1,1,\omega,1,1,0,0,0)$	-969	51
$P_{34,55}$	$(\omega,\bar{\omega},1,\omega,1,\bar{\omega},1,1,\omega,1,\omega,1,1,0,0,0,0)$	-918	51
$P_{34,56}$	$(\bar{\omega},1,\omega,0,1,1,\bar{\omega},\bar{\omega},1,\omega,1,0,0,1,0,0,0)$	-918	51
$P_{34,57}$	$(1,1,0,0,0,1,0,1,1,\omega,1,1,0,1,0,0,0)$	-816	408
$P_{34,58}$	$(1,\omega,1,\bar{\omega},\omega,1,\bar{\omega},\omega,\omega,1,1,\omega,\omega,1,1,0,0)$	-765	51

Table 3: Bordered double circulant Hermitian self-dual $[34, 17, 10]$ codes

Codes	Rows r	α	$\lvert \mathrm{Aut} \rvert$
$B_{34,1}$	$(\omega,\omega,\bar\omega,0,\omega,1,\omega,1,\bar\omega,1,\omega,1,0,1,0,0)$	-2136	96
$B_{34,2}$	$(1,\omega,\omega,1,\omega,1,\omega,\omega,1,0,0,0,1,0,0,0)$	-1800	96
$B_{34,3}$	$(1,\bar\omega,\omega,\bar\omega,1,0,0,\bar\omega,\omega,\omega,1,\bar\omega,1,1,0,0)$	-1800	48
$B_{34,4}$	$(1,\bar\omega,0,1,\omega,0,\bar\omega,\bar\omega,\bar\omega,0,0,1,0,0,0,0)$	-1704	48
$B_{34,5}$	$(\omega,\bar\omega,\omega,\bar\omega,1,1,0,0,1,\omega,1,0,1,0,0,0)$	-1608	48
$B_{34,6}$	$(1,\bar\omega,0,\omega,\omega,\bar\omega,\omega,\bar\omega,1,\bar\omega,\omega,\omega,1,0,0,0)$	-1512	48
$B_{34,7}$	$(\bar\omega,\bar\omega,\bar\omega,\bar\omega,0,\omega,\omega,1,0,\omega,1,1,0,0,0,0)$	-1464	48
$B_{34,8}$	$(\bar\omega,\bar\omega,1,\bar\omega,0,\bar\omega,\bar\omega,\omega,0,\bar\omega,1,1,0,0,0,0)$	-1416	48
$B_{34,9}$	$(1,\omega,\omega,1,\omega,1,0,1,1,1,1,0,1,1,0,0)$	-1368	48
$B_{34,10}$	$(\bar\omega,\bar\omega,\omega,\bar\omega,\omega,0,\omega,\omega,\omega,0,\bar\omega,1,0,0,0,0)$	-1320	48
$B_{34,11}$	$(1,1,\omega,0,1,\bar\omega,\omega,\bar\omega,\omega,1,0,0,1,0,0,0)$	-1320	48
$B_{34,12}$	$(1,\omega,\bar\omega,\bar\omega,0,1,\omega,0,\bar\omega,\bar\omega,\omega,0,1,0,0,0)$	-1320	48
$B_{34,13}$	$(1,\bar\omega,0,1,\omega,\bar\omega,1,\bar\omega,1,\bar\omega,1,1,1,0,0,0)$	-1272	48
$B_{34,14}$	$(1,\bar\omega,0,\bar\omega,1,1,1,\omega,\omega,1,1,\omega,1,0,0,0)$	-1176	48
$B_{34,15}$	$(\omega,1,\bar\omega,0,\bar\omega,\omega,\bar\omega,\omega,0,0,\bar\omega,1,1,0,0,0)$	-1128	48
$B_{34,16}$	$(1,1,\bar\omega,\bar\omega,\omega,1,0,1,\bar\omega,\bar\omega,0,1,0,0,0,0)$	-1080	48
$B_{34,17}$	$(\bar\omega,1,1,0,1,1,1,1,1,\omega,\bar\omega,1,0,1,0,0)$	-840	96

3.5 Length 36

The possible weight enumerators of Hermitian self-dual $[36, 18, 12]$ codes are as follows:

$$\begin{aligned} W_{36,\alpha} =& 1 + (19548 + \alpha)y^{12} + (536544 - 12\alpha)y^{14} + (9136314 + 66\alpha)y^{16} \\ &+ (102310560 - 220\alpha)y^{18} + (741700476 + 495\alpha)y^{20} \\ &+ (3467116224 - 792\alpha)y^{22} + (10288224072 + 924\alpha)y^{24} \\ &+ (18805089600 - 792\alpha)y^{26} + (20146300020 + 495\alpha)y^{28} \\ &+ (11672461152 - 220\alpha)y^{30} + (3176328573 + 66\alpha)y^{32} \\ &+ (305900064 - 12\alpha)y^{34} + (4353588 + \alpha)y^{36}. \end{aligned}$$

Of course, a code with weight enumerator $W_{36,-19548}$ is extremal.

We complete the classification of pure double circulant Hermitian self-dual $[36, 18, 12]$ codes by giving the first rows of R in (1) of the two inequivalent codes $P_{36,1}$ and $P_{36,2}$:

$$(\omega,0,0,\bar\omega,1,1,\omega,1,1,\omega,1,\omega,0,0,1,0,0,0), \qquad (\omega,1,\omega,\omega,1,\omega,1,\omega,\omega,1,\omega,0,0,1,0,1,0,0),$$

respectively. We have verified that any bordered double circulant Hermitian self-dual $[36, 18, 12]$ code is equivalent to the bordered double circulant code B_{36} with the first row of R' equal to

$$(1,1,\omega,1,\omega,\omega,\omega,1,1,\omega,\omega,\omega,1,\omega,1,1,0),$$

and borders $(a,b,c)=(0,1,1)$. In addition, we have verified that $P_{36,2}$ and B_{36} are equivalent. Hence we have the following:

Proposition 5. *There are two inequivalent pure double circulant Hermitian self-dual* $[36,18,12]$ *codes. There is a unique bordered double circulant Hermitian self-dual* $[36,18,12]$ *code, up to equivalence. There are two inequivalent double circulant Hermitian self-dual* $[36,18,12]$ *codes.*

The codes $P_{36,1}$ and $P_{36,2}$ have weight enumerators with $W_{36,1296}$ and $W_{36,9216}$, and they have automorphism groups of orders 108 and 14688, respectively.

3.6 Length 38

The possible weight enumerators of Hermitian self-dual $[38,19,12]$ codes are as follows:

$$\begin{aligned}W_{38,\alpha}=&1+(9348+\alpha)y^{12}+(346104-9\alpha)y^{14}+(6949098+30\alpha)y^{16}\\&+(94631856-22\alpha)y^{18}+(851995188-165\alpha)y^{20}\\&+(5077669896+693\alpha)y^{22}+(19871074152-1452\alpha)y^{24}\\&+(50072165280+1980\alpha)y^{26}+(78688642956-1881\alpha)y^{28}\\&+(73258213512+1265\alpha)y^{30}+(37222230573-594\alpha)y^{32}\\&+(8956265904+186\alpha)y^{34}+(767905596-35\alpha)y^{36}+(9807480+3\alpha)y^{38}.\end{aligned}$$

Of course, a code with weight enumerator $W_{38,-9348}$ is extremal.

We complete the classification of pure double circulant Hermitian self-dual $[38,19,12]$ codes by giving the first rows of R in (1) of the four inequivalent codes $P_{38,1}, P_{38,2}, P_{38,3}$ and $P_{38,4}$:

$$\begin{aligned}&(1,0,0,1,1,1,\omega,1,\omega,1,1,1,0,0,1,0,0,0,0),\\&(1,\omega,\bar\omega,\omega,\bar\omega,\omega,\bar\omega,\omega,\bar\omega,\omega,1,0,0,0,1,1,0,0,0),\\&(1,\omega,\bar\omega,\bar\omega,\omega,1,\bar\omega,\bar\omega,1,\omega,\bar\omega,\bar\omega,\omega,1,0,0,1,0,0),\\&(1,0,1,\omega,\bar\omega,1,1,\omega,\omega,\omega,1,1,\bar\omega,\omega,1,0,1,0,0),\end{aligned}$$

respectively. The codes $P_{38,1}, P_{38,2}, P_{38,3}$ and $P_{38,4}$ have weight enumerators $W_{38,\alpha}$ with $\alpha=399,2052,2052,9633$ and automorphism groups of orders $114,114,114$, 75924, respectively.

We complete the classification of bordered double circulant Hermitian self-dual $[38,19,12]$ codes $B_{38,i}$ by listing the first rows r of R' in Table 4, where the third column gives the integers α in the weight enumerators $W_{38,\alpha}$ and the last column gives the orders of the automorphism groups. Note that all bordered double circulant codes have borders $(a,b,c)=(\omega,1,1)$.

In addition, we have verified that $P_{38,4}$ and $B_{38,13}$ are equivalent. By comparing the automorphism groups, there is no other pair of equivalent codes among $P_{38,i}$ $(i=1,2,3,4)$ and $B_{38,j}$ $(j=1,2,\dots,13)$. Hence we have the following:

Table 4: Bordered double circulant Hermitian self-dual $[38,19,12]$ codes

Codes	Rows r	α	$\lvert \mathrm{Aut} \rvert$
$B_{38,1}$	$(\bar\omega,\bar\omega,\omega,1,\omega,\bar\omega,\omega,\bar\omega,\omega,0,0,\omega,1,1,0,0,0,0)$	489	54
$B_{38,2}$	$(1,\omega,\bar\omega,\bar\omega,\bar\omega,\bar\omega,\bar\omega,\omega,\bar\omega,1,\omega,0,1,0,0,0,0,0)$	714	108
$B_{38,3}$	$(\bar\omega,\bar\omega,\bar\omega,\omega,1,\omega,\omega,\bar\omega,1,\omega,0,1,1,\omega,1,0,0,0)$	1101	108
$B_{38,4}$	$(\bar\omega,0,1,\omega,1,\omega,1,1,\bar\omega,0,0,\omega,\omega,\omega,1,0,0,0)$	1254	54
$B_{38,5}$	$(\bar\omega,\omega,\bar\omega,\omega,\bar\omega,1,\bar\omega,\omega,\omega,1,1,0,\omega,0,1,0,1,0)$	1794	108
$B_{38,6}$	$(\bar\omega,\bar\omega,\bar\omega,\omega,\bar\omega,1,1,\bar\omega,\bar\omega,0,0,\omega,\omega,0,1,0,0,0)$	2037	108
$B_{38,7}$	$(1,\omega,0,\bar\omega,1,\omega,\bar\omega,\bar\omega,\omega,\omega,1,0,\bar\omega,\omega,1,1,0,0)$	2037	108
$B_{38,8}$	$(\bar\omega,\bar\omega,0,1,\omega,\bar\omega,0,0,\bar\omega,\bar\omega,\omega,\bar\omega,\omega,1,0,1,0,0)$	2145	108
$B_{38,9}$	$(1,\bar\omega,0,0,\omega,0,\bar\omega,\omega,\omega,\omega,\omega,\bar\omega,\bar\omega,1,1,0,0,0)$	2343	54
$B_{38,10}$	$(1,1,\bar\omega,\omega,\omega,\omega,\omega,\bar\omega,0,\omega,0,\omega,1,\omega,1,1,0,0)$	2433	54
$B_{38,11}$	$(\bar\omega,\omega,\bar\omega,\omega,\omega,\omega,1,\omega,\omega,1,1,\omega,1,1,1,0,1,0)$	2613	108
$B_{38,12}$	$(\omega,1,\bar\omega,\bar\omega,\bar\omega,\omega,\bar\omega,\omega,\bar\omega,\bar\omega,0,0,1,0,1,0,0,0)$	3801	108
$B_{38,13}$	$(\omega,1,\bar\omega,\bar\omega,\bar\omega,\omega,1,\bar\omega,\bar\omega,0,\bar\omega,0,0,\omega,1,0,0,0)$	9633	75924

Proposition 6. *There are four inequivalent pure double circulant Hermitian self-dual* $[38,19,12]$ *codes. There are* 13 *inequivalent bordered double circulant Hermitian self-dual* $[38,19,12]$ *codes. There are* 16 *inequivalent double circulant Hermitian self-dual* $[38,19,12]$ *codes.*

3.7 Length 40

The possible weight enumerators of Hermitian self-dual $[40,20,12]$ codes are as follows:

$$\begin{aligned} W_{40,\alpha} =& 1 + (1560+\alpha)y^{12} + (223200-6\alpha)y^{14} + (4917510+3\alpha)y^{16} \\ &+ (79659840+68\alpha)y^{18} + (875026152-231\alpha)y^{20} \\ &+ (6471372960+198\alpha)y^{22} + (32287341060+627\alpha)y^{24} \\ &+ (107307849600-2376\alpha)y^{26} + (232470362280+4059\alpha)y^{28} \\ &+ (317475663648-4378\alpha)y^{30} + (259204959585+3201\alpha)y^{32} \\ &+ (116446075200-1596\alpha)y^{34} + (24948981720+523\alpha)y^{36} \\ &+ (1917133920-102\alpha)y^{38} + (22059540+9\alpha)y^{40}. \end{aligned}$$

Of course, a code with weight enumerator $W_{40,-1560}$ is extremal.

We complete the classification of pure and bordered double circulant Hermitian self-dual $[40,20,12]$ codes $P_{40,i}$ and $B_{40,i}$ by listing the first rows r of R and R' in Tables 5 and 6, respectively, where the third column gives the integers α in the weight enumerators $W_{40,\alpha}$ and the last column gives the orders of the automorphism groups. Note that all bordered double circulant codes have borders $(a,b,c)=(0,1,1)$. By comparing the automorphism groups, there is no pair of equivalent codes among $P_{40,i}$ $(i=1,2,\dots,26)$ and $B_{40,j}$ $(j=1,2,\dots,13)$. Hence we have the following:

Proposition 7. *There are* 26 *inequivalent pure double circulant Hermitian self-dual* $[40,20,12]$ *codes. There are* 13 *inequivalent bordered double circulant Hermitian self-*

dual $[40, 20, 12]$ *codes. There are* 39 *inequivalent double circulant Hermitian self-dual* $[40, 20, 12]$ *codes.*

Table 5: Pure double circulant Hermitian self-dual $[40, 20, 12]$ codes

Codes	Rows r	α	$\lvert\mathrm{Aut}\rvert$
$P_{40,1}$	$(1,\omega,\bar\omega,1,\bar\omega,0,\bar\omega,0,\bar\omega,1,\bar\omega,\omega,1,0,0,0,0,0,0,0)$	7260	120
$P_{40,2}$	$(\omega,\bar\omega,\omega,1,\omega,\bar\omega,0,1,\omega,\omega,\bar\omega,1,1,1,\omega,1,0,0,0,0)$	5370	60
$P_{40,3}$	$(\omega,1,1,\bar\omega,\omega,1,1,\bar\omega,1,\bar\omega,0,\bar\omega,1,1,\omega,1,0,0,0,0)$	2970	60
$P_{40,4}$	$(\omega,\bar\omega,\bar\omega,\bar\omega,0,0,\bar\omega,1,\bar\omega,0,\omega,1,\bar\omega,1,\omega,1,0,0,0,0)$	3690	60
$P_{40,5}$	$(\omega,\omega,0,1,1,0,\omega,1,\omega,\omega,1,\omega,\omega,0,1,0,1,0,0,0)$	5550	60
$P_{40,6}$	$(\bar\omega,\bar\omega,\omega,\omega,\bar\omega,\bar\omega,\omega,\bar\omega,\bar\omega,0,\omega,\omega,\omega,\omega,0,1,1,0,0,0)$	2850	60
$P_{40,7}$	$(\omega,1,\bar\omega,\omega,\bar\omega,0,1,\bar\omega,0,0,\bar\omega,\omega,1,1,0,\omega,1,0,0,0)$	4650	60
$P_{40,8}$	$(\omega,\bar\omega,\omega,1,1,\bar\omega,\bar\omega,\omega,\bar\omega,1,\omega,\bar\omega,1,\bar\omega,1,\omega,1,0,0,0)$	3210	60
$P_{40,9}$	$(\omega,\bar\omega,1,0,\omega,1,1,\omega,\bar\omega,\omega,\omega,1,\bar\omega,\omega,1,0,0,1,0,0)$	3870	60
$P_{40,10}$	$(\omega,\bar\omega,1,\omega,\omega,1,\omega,\omega,\omega,1,\omega,0,1,1,\omega,0,0,1,0,0)$	3180	60
$P_{40,11}$	$(1,\omega,\omega,1,1,\bar\omega,1,\omega,1,\omega,1,1,\omega,\omega,1,1,0,1,0,0)$	3600	60
$P_{40,12}$	$(\bar\omega,0,\omega,1,\omega,1,1,\bar\omega,\bar\omega,\bar\omega,\omega,0,1,\bar\omega,\omega,1,0,1,0,0)$	3510	60
$P_{40,13}$	$(1,\omega,1,1,\bar\omega,1,1,\bar\omega,\omega,1,0,1,1,\omega,0,\omega,0,1,0,0)$	4770	60
$P_{40,14}$	$(\bar\omega,1,\omega,1,0,1,\bar\omega,\bar\omega,\bar\omega,1,1,\bar\omega,\bar\omega,\bar\omega,0,\omega,0,1,0,0)$	4470	60
$P_{40,15}$	$(\omega,\bar\omega,\bar\omega,\bar\omega,\bar\omega,\bar\omega,\bar\omega,\bar\omega,\bar\omega,\bar\omega,\omega,1,\omega,1,1,\omega,0,1,0,0)$	5370	60
$P_{40,16}$	$(\bar\omega,1,\omega,0,\bar\omega,\bar\omega,\bar\omega,1,0,1,\bar\omega,\bar\omega,\bar\omega,1,1,\omega,0,1,0,0)$	3150	60
$P_{40,17}$	$(\bar\omega,\omega,\bar\omega,\bar\omega,\omega,\omega,1,0,\bar\omega,1,\omega,1,1,\omega,0,0,1,1,0,0)$	4410	60
$P_{40,18}$	$(\bar\omega,\bar\omega,\bar\omega,\omega,1,\omega,\omega,\bar\omega,\omega,\omega,\bar\omega,\omega,\omega,\omega,1,0,1,1,0,0)$	4050	60
$P_{40,19}$	$(\omega,\bar\omega,1,\bar\omega,1,1,\omega,\bar\omega,\omega,1,\bar\omega,1,\omega,0,\omega,1,1,1,0,0)$	3390	60
$P_{40,20}$	$(\bar\omega,\bar\omega,1,\omega,0,1,1,\omega,\omega,1,\bar\omega,\omega,1,\bar\omega,\omega,\omega,1,1,0,0)$	3660	60
$P_{40,21}$	$(\omega,\bar\omega,\omega,1,\bar\omega,\omega,\bar\omega,\bar\omega,1,\omega,\omega,\bar\omega,1,1,0,1,\omega,1,0,0)$	3990	60
$P_{40,22}$	$(1,\omega,\omega,0,\omega,1,\omega,\omega,1,\omega,\bar\omega,1,\omega,\omega,\bar\omega,1,\omega,1,0,0)$	5070	60
$P_{40,23}$	$(\omega,\bar\omega,\omega,\bar\omega,1,\omega,1,1,\omega,\omega,1,\omega,\omega,1,1,0,1,0,1,0)$	3840	60
$P_{40,24}$	$(\omega,1,1,\omega,\omega,0,1,1,\omega,\omega,\omega,\bar\omega,\omega,\bar\omega,\omega,1,1,0,1,0)$	4020	60
$P_{40,25}$	$(1,\omega,\omega,\bar\omega,\omega,\bar\omega,1,1,\omega,\omega,1,0,\omega,1,1,\omega,1,0,1,0)$	4320	60
$P_{40,26}$	$(1,\bar\omega,\bar\omega,\omega,\bar\omega,1,\omega,\omega,\omega,\omega,\omega,\bar\omega,\bar\omega,\bar\omega,\omega,1,1,1,1,0)$	3630	60

3.8 Summary

In Table 7, we summarize the results from this note and [8] which classify all double circulant Hermitian self-dual codes of lengths n and minimum weights $d_{DC}(n)$ for $n = 22, 24, \ldots, 40$. In the table, N_{pDCC}, N_{bDCC} and N_{DCC} denote the numbers of inequivalent pure double circulant Hermitian self-dual codes, bordered double circulant Hermitian self-dual codes, and double circulant Hermitian self-dual codes with minimum weight $d_{DC}(n)$, respectively.

4 Remark on Notions of Equivalence

We employ a notion of equivalence given in [7]. Often this equivalence is called a monomial equivalence. Here we say that two codes C_1 and C_2 are conjugation-equivalent if $C_1 = C_2M$ or $C_1 = \tau(C_2M)$ for some monomial matrix M where τ is the

Table 6: Bordered double circulant Hermitian self-dual $[40, 20, 12]$ codes

Codes	Rows r	α	$\lvert \mathrm{Aut} \rvert$
$B_{40,1}$	$(\bar{\omega}, 1, \omega, 1, \bar{\omega}, 0, 1, \omega, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0)$	2601	57
$B_{40,2}$	$(1, \bar{\omega}, \omega, \omega, \bar{\omega}, 0, \omega, 0, \bar{\omega}, \bar{\omega}, 0, \omega, 1, 0, 0, 0, 0, 0, 0)$	3570	57
$B_{40,3}$	$(1, 1, 0, \bar{\omega}, \omega, \omega, \bar{\omega}, \omega, \bar{\omega}, \bar{\omega}, 1, \omega, 0, 0, 1, 0, 0, 0, 0)$	4254	57
$B_{40,4}$	$(1, \omega, \omega, \bar{\omega}, \omega, 1, \bar{\omega}, \omega, \omega, 0, \omega, 0, 0, 1, 1, 0, 0, 0, 0)$	4197	57
$B_{40,5}$	$(1, 1, \omega, 1, 1, \omega, \omega, 0, \omega, \omega, 1, 1, \omega, 1, 1, 0, 0, 0, 0)$	5223	114
$B_{40,6}$	$(1, \omega, \bar{\omega}, \bar{\omega}, 1, 1, 1, \bar{\omega}, \omega, 0, \bar{\omega}, 1, \omega, \omega, 1, 0, 0, 0, 0)$	4197	57
$B_{40,7}$	$(1, \bar{\omega}, \omega, 0, \bar{\omega}, \omega, \bar{\omega}, \bar{\omega}, \omega, 1, \omega, 1, 1, 1, 0, 1, 0, 0, 0)$	4539	57
$B_{40,8}$	$(1, 1, \omega, 0, 1, \bar{\omega}, 1, \bar{\omega}, \bar{\omega}, 1, \bar{\omega}, 1, 0, \omega, 1, 1, 0, 0, 0)$	6192	114
$B_{40,9}$	$(\omega, \bar{\omega}, \bar{\omega}, \bar{\omega}, 1, \bar{\omega}, \omega, \omega, \bar{\omega}, \omega, \omega, \bar{\omega}, 1, \omega, 1, 1, 0, 0, 0)$	2715	57
$B_{40,10}$	$(1, 1, \bar{\omega}, \omega, 1, \omega, 1, 1, \omega, 1, \bar{\omega}, \omega, \omega, \omega, 1, 1, 0, 0, 0)$	4254	57
$B_{40,11}$	$(\omega, \omega, \omega, 1, 1, \omega, \omega, \omega, \bar{\omega}, 1, \bar{\omega}, \bar{\omega}, \bar{\omega}, \omega, \omega, 0, 1, 0, 0)$	3912	57
$B_{40,12}$	$(\bar{\omega}, \omega, \bar{\omega}, \omega, \bar{\omega}, \omega, 1, \omega, 1, 1, \bar{\omega}, \omega, \omega, 0, 1, 1, 1, 0, 0)$	3912	57
$B_{40,13}$	$(1, \bar{\omega}, 1, \bar{\omega}, \bar{\omega}, 1, \bar{\omega}, \omega, \omega, \omega, \omega, \bar{\omega}, 1, \omega, \bar{\omega}, \omega, 1, 1, 0)$	4083	171

Table 7: Summary

Length n	$d_{DC}(n)$	N_{pDCC}	N_{bDCC}	N_{DCC}
22	8	3	0	3
24	8	3	2	4
26	8	11	8	19
28	10	1	0	1
30	12	1	1	1
32	10	19	0	19
34	10	58	17	75
36	12	2	1	2
38	12	4	13	16
40	12	26	13	39

conjugation of $\mathbb{F}_4$ sending $x \in \mathbb{F}_4$ to x^2 and $\tau(C) = \{(\tau(x_1), \ldots, \tau(x_n)) | (x_1, \ldots, x_n) \in C\}$. Some authors use the last notion of equivalence (cf. [4]). We denote two monomial equivalent (resp. conjugation-equivalent) codes C_1, C_2 by $C_1 \cong_M C_2$ (resp. $C_1 \cong_\tau C_2$). In this section, we show that there is no difference between these two notions of equivalence for double circulant Hermitian self-dual codes.

Lemma 8. *Let C be a double circulant Hermitian self-dual code. Then C and $\tau(C)$ are (monomial) equivalent.*

Proof. Suppose that C is pure, that is, its generator matrix is of the form (1). Since C is Hermitian self-dual, the codes with generator matrices $\begin{pmatrix} I & R \end{pmatrix}$ and $\begin{pmatrix} \tau(R)^T & I \end{pmatrix}$ are equivalent, where A^T denotes the transpose of A and $\tau(A) = (b_{ij})$ with $b_{ij} = \tau(a_{ij})$ for $A = (a_{ij})$. Since R is a circulant matrix, the codes with generator matrices $\begin{pmatrix} I & R \end{pmatrix}$ and $\begin{pmatrix} I & R^T \end{pmatrix}$ are equivalent. Hence the codes with generator matrices $\begin{pmatrix} I & R \end{pmatrix}$ and $\begin{pmatrix} I & \tau(R) \end{pmatrix}$ are equivalent, that is, $C \cong_M \tau(C)$.

Now consider the case that C is bordered. Any bordered double circulant Hermitian self-dual code is equivalent to a bordered double circulant code with borders $(1,0,0),(0,1,1)$, or $(1,1,1)$ (see Remark 9 in [8]). Thus we may assume that the borders of C are one of $(1,0,0),(0,1,1),(1,1,1)$. Hence the borders are invariant under the conjugation τ. This means that the argument for a pure double circulant code C can be applied to the bordered case. □

Proposition 9. *Let C_1, C_2 be double circulant Hermitian self-dual codes. Then C_1 and C_2 are conjugation-equivalent if and only if C_1 and C_2 are (monomial) equivalent.*

Proof. It is sufficient to show that if $C_1 \cong_\tau C_2$ then $C_1 \cong_M C_2$. Suppose that $C_1 \cong_\tau C_2$. Thus $C_2 \cong_M C_1$ or $C_2 \cong_M \tau(C_1)$. For the later case, by Lemma 8, $C_1 \cong_M \tau(C_1)$. Hence $C_1 \cong_M C_2$. □

References

[1] W. Bosma and J. Cannon, Handbook of Magma Functions, Department of Mathematics, University of Sydney, Available online at
`http://magma.maths.usyd.edu.au/magma/`

[2] J.H. Conway, V. Pless and N.J.A. Sloane, Self-dual codes over $GF(3)$ and $GF(4)$ of length not exceeding 16, *IEEE Trans. Inform. Theory* **25** (1979), 312–322.

[3] T.A. Gulliver, Optimal double circulant self-dual codes over $\mathbb{F}_4$, *IEEE Trans. Inform. Theory* **46** (2000), 271–274.

[4] W.C. Huffman, Characterization of quaternary extremal codes of lengths 18 and 20, *IEEE Trans. Inform. Theory* **43** (1997), 1613–1616.

[5] J.-L. Kim, New self-dual codes over GF(4) with the highest known minimum weights, *IEEE Trans. Inform. Theory* **47** (2001), 1575–1580.

[6] C.W.H. Lam and V. Pless, There is no $(24, 12, 10)$ self-dual quaternary code, *IEEE Trans. Inform. Theory* **36** (1990), 1153–1156.

[7] F.J. MacWilliams, A.M. Odlyzko, N.J.A. Sloane and H.N. Ward, Self-dual codes over $GF(4)$, *J. Combin. Theory Ser. A* **25** (1978), 288–318.

[8] H. Miyabayashi, A classification of double circulant Hermitian self-dual codes over $\mathbb{F}_4$ of lengths up to 26, *Bull. Yamagata Univ. (Natural Sci.)*, **16** (2007), 81–91.

[9] P.R.J. Östergård, There exists no Hermitian self-dual quaternary $[26, 13, 10]_4$ code, *IEEE Trans. Inform. Theory* **50** (2004), 3316–3317.

[10] E.M. Rains and N.J.A. Sloane, Self-dual codes, in *Handbook of Coding Theory*, V.S. Pless and W.C. Huffman, eds., Elsevier, Amsterdam, (1998), 177–294.

(Received 11 Sep 2006; revised 9 Jan 2007)