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# On the super edge-magic deficiencies of graphs

A.A.G. Ngurah

*Department of Civil Engineering*
*Universitas Merdeka Malang*
*Jalan Taman Agung 1 Malang*
*Indonesia*

E.T. Baskoro    R. Simanjuntak

*Combinatorial Mathematics Research Group*
*Faculty of Mathematics and Natural Sciences*
*Institut Teknologi Bandung*
*Jalan Ganesa 10 Bandung*
*Indonesia*
{s304agung, ebaskoro, rino}@dns.math.itb.ac.id

**Abstract**

A graph $G$ is called *edge-magic* if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ such that $f(x) + f(xy) + f(y)$ is a constant for every edge $xy \in E(G)$. A graph $G$ is said to be *super edge-magic* if $f(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$. Furthermore, the *edge-magic deficiency* of a graph $G$, $\mu(G)$, is defined as the minimum nonnegative integer $n$ such that $G \cup nK_1$ is edge-magic. Similarly, the *super edge-magic deficiency* of a graph $G$, $\mu_s(G)$, is either the minimum nonnegative integer $n$ such that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer $n$.

In this paper, we present the super edge-magic deficiencies of some classes of graphs.

## 1 Introduction

We consider finite and simple graphs; we denote the vertex and edge sets of a graph $G$ by $V(G)$ and $E(G)$, respectively, and $p = |V(G)|$ and $q = |E(G)|$. For most graph theory notions, we refer the reader to Chartrand and Lesniak [3].

An *edge-magic total labeling* of $G$ is a bijective function $f$ from $V(G) \cup E(G)$ to $\{1, 2, 3, \ldots, p + q\}$ such that $f(x) + f(xy) + f(y)$ is a constant $k$, which is called the *magic constant* of $f$ for any edge $xy$ of $G$. An edge-magic total labeling $f$ is called *super edge-magic total* if $f(V(G)) = \{1, 2, 3, \ldots, p\}$. A graph $G$ is called *edge-magic* (*super edge-magic*) if there exists an edge-magic (super edge-magic, respectively) total labeling of $G$. The concept of edge-magic total labeling was first introduced and studied by Kotzig and Rosa [12, 13], using a different name "magic valuations". Meanwhile, the super edge-magic total labeling was introduced by Enomoto, Lladó, Nakamigawa and Ringel [4]. Recently several papers of (super) edge-magic total labeling have been published by several authors; for instance see [8, 9, 10, 11, 15].

In [12], Kotzig and Rosa proved that for every graph $G$ there exists an edge-magic graph $H$ such that $H \cong G \cup nK_1$ for some nonnegative integer $n$. This fact motivates the emergence of the concept of the edge-magic deficiency of a graph. The *edge-magic deficiency* of a graph $G$, $\mu(G)$, is the minimum nonnegative integer $n$ such that $G \cup nK_1$ has an edge-magic total labeling. Kotzig and Rosa [12] give an upper bound of the edge-magic deficiency for a graph $G$ with $n$ vertices, that is, $\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1)$, where $F_n$ is the $n$-th Fibonacci number.

Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno *et al.* [6] defined a similar concept for the super edge-magic total labeling. The *super edge-magic deficiency* of a graph $G$, $\mu_s(G)$, is the minimum nonnegative integer $n$ such that $G \cup nK_1$ has a super edge-magic total labeling or $+\infty$ if there exists no such $n$. Unlike the edge-magic deficiency, not all graphs have finite super edge-magic deficiency. Examples of such graphs can be found in [6]. As a consequence of the above two definitions, we have that for every graph $G$, $\mu(G) \leq \mu_s(G)$.

Figueroa-Centeno *et al.* in two separate papers [6, 7] provided the exact values of (super) edge-magic deficiencies of several classes of graphs, such as cycles, complete graphs, some classes of forests, 2-regular graphs, and complete bipartite graphs $K_{2,m}$. They also provided some upper bounds of the super edge-magic deficiency of complete bipartite graphs $K_{m,n}$.

In this paper, we present the exact value of the super edge-magic deficiency of a particular type of chain graphs, and of fans $F_n$, double fans $F_{n,2}$, and wheels $W_n$, for small values of $n$. We also describe some upper and lower bounds of the super edge-magic deficiency for general chain graphs, fans, double fans, wheels, bipartite and tripartite graphs.

The following three lemmas will be used frequently. The first lemma characterizes super edge-magic graphs. The second lemma gives a sufficient condition for nonexistence of a super edge-magic total labeling of a graph. The last lemma gives a sufficient condition for graphs with infinite super edge-magic deficiencies.

**Lemma 1** [5] *A graph $G$ with $p$ vertices and $q$ edges is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the set $S = \{f(x) + f(y) : xy \in E(G)\}$ consists of $q$ consecutive integers. In such a case, $f$ extends to a super edge-magic total labeling of $G$ with the magic constant $k = p+q+s$, where $s = min(S)$.*

**Lemma 2** [4] *If a graph $G$ with $p$ vertices and $q$ edges is super edge-magic, then $q \leq 2p-3$.*

**Lemma 3** [6] *If $G$ is a graph of size $q$ such that the degrees of all vertices are even and $q \equiv 2$ (mod 4), then $\mu_s(G) = +\infty$.*

In addition to these three lemmas, the notion of dual labeling will also appear frequently in the next sections. A *dual* labeling of a super edge-magic total labeling $f$ is defined as

$$f'(x) = p + 1 - f(x), \text{ for all } x \in V(G),$$

and

$$f'(xy) = 2p + q + 1 - f(xy), \text{ for all } xy \in E(G).$$

It has been proved in [2] that the dual of a super edge-magic total labeling is also a super edge-magic total labeling.

## 2 Super edge-magic deficiencies of chain graphs

By a block of a graph we mean a maximal subgraph with no cut-vertex. Following [3], we define a *block-cut-vertex graph* of a graph $G$ as a graph $H$ where vertices of $H$ are blocks and cut-vertices in $G$ and two vertices are adjacent in $H$ if and only if one vertex is a block in $G$ and the other is a cut-vertex in $G$ belonging to the block.

Barrientos [1] defines a *chain graph* as a graph with blocks $B_1, B_2, B_3, \ldots, B_k$ such that for every $i$, $B_i$ and $B_{i+1}$ have a common vertex in such a way that the block-cut-vertex graph is a path. We denote by *$kK_n$-path* a chain graph with $k$ blocks where each block is identical and isomorphic to the complete graph $K_n$.

In this section, we consider super edge-magic deficiencies of $kK_n$-paths for $n = 2, 3,$ and 4. If $n = 2$ then $kK_2$-path $\cong P_{k+1}$. It is well known that $P_n$ is super edge-magic. Consequently, $\mu_s(kK_2\text{-path}) = 0$.

If $n = 3$ then $kK_3$-path is a triangular snake which is considered in [14], where Lee and Wang showed that $kK_3$-path is super edge-magic if and only if $k \equiv 0, 1$ (mod 4). Hence, $\mu_s(kK_3\text{-path}) = 0$ for $k \equiv 0, 1$ (mod 4). Additionally, as a consequence of Lemma 3, we have $\mu_s(kK_3\text{-path}) = +\infty$ for $k \equiv 2$ (mod 4). The next theorem gives an upper bound for $\mu_s$ ($kK_3$-path), $k \equiv 3$ (mod 4).

**Theorem 1** *If $G$ is a $kK_3$-path where $k \equiv 3$ (mod 4) then $\mu_s(G) \leq k-1$.*

**Proof** As explained before, $\mu_s(G) \geq 1$.

Next, define $H \cong G \cup (k-1)K_1$ as a graph with

$$V(H) = \{u_i : 1 \leq i \leq k+1\} \cup \{v_i : 1 \leq i \leq k\} \cup \{w_i : 1 \leq i \leq k-1\}$$

and

$$E(H) = \{u_i u_{i+1} : 1 \le i \le k\} \cup \{u_i v_i, u_{i+1} v_i : 1 \le i \le k\},$$

where $k \equiv 3 \pmod 4$.

To show $\mu_s(G) \le k - 1$, we define a vertex labeling $f$ as follows.

$$f(x) = \begin{cases} \frac{1}{2}(1+i), & \text{if } x = u_i \text{ for ood } i, \\ \frac{1}{2}(k+i+1), & \text{if } x = u_i \text{ for even } i, \\ 2k+2-i, & \text{if } x = v_i \text{ for ood } i, \\ 3k+2-i, & \text{if } x = v_i \text{ for even } i, \end{cases}$$

where the remaining labels are placed in the isolated vertices $w_i$, $1 \le i \le k-1$. It can be checked that $f$ extends to a super edge-magic total labeling of $G \cup (k-1)K_1$ with the magic constant $\frac{1}{2}(13k+5)$. Hence, $\mu_s(G) \le k-1$. □

**Open problem 1** *Find a better upper bound of the super edge-magic deficiencies of $kK_3$-paths.*

The last chain graphs to be considered in this paper are $kK_4$-paths. First, we define the $kK_4$-path as a graph having

$$V(kK_4\text{-path}) = \{u_i : 1 \le i \le k+1\} \cup \{v_i, w_i : 1 \le i \le k\},$$

and

$$E(kK_4\text{-path}) = \{u_i u_{i+1}, u_i v_i, u_i w_i, v_i w_i, v_i u_{i+1}, w_i u_{i+1} : 1 \le i \le k\}.$$

Observe that a $kK_4$-path has $3k+1$ vertices and $6k$ edges. By Lemma 2, it is not a super edge-magic graph. Consequently, $\mu_s(kK_4\text{-path}) \ge 1$. In fact, the super edge-magic deficiency of $kK_4$-paths is 1 as we state in the following theorem.

**Theorem 2** *For every integer $k$, the super edge-magic deficiency of a $kK_4$-path is* 1.

**Proof** Applying Lemma 1, the following vertex labeling of $kK_4$-path $\cup K_1$ extends to a super edge-magic total labeling with the magic constant $9k+6$.

$$f(x) = \begin{cases} 3i-2, & \text{if } x = u_i \text{ for } i \le i \le k+1, \\ 3i, & \text{if } x = v_i \text{ for } i \le i \le k, \\ 3i+2, & \text{if } x = w_i \text{ for } i \le i \le k, \\ 2, & \text{if } x = K_1. \end{cases}$$

Therefore, $\mu_s(kK_4\text{-path}) = 1$. □

**Open problem 2** *Find the super edge-magic deficiencies of $kK_n$-paths, $n \ge 5$.*

In the next section, we consider three types of graphs which are often considered to be closely connected to each other. They are fans, double fans, and wheels.

# 3 Super edge-magic deficiencies of fans, double fans and wheels

The fan $F_n \cong P_n + K_1$ is a graph resulting from connecting each vertex in a path to a single vertex; this gives $q = 2p - 3$. In [5], Figueroa-Centeno *et al.* proved that $F_n$ is super edge-magic if and only if $1 \leq n \leq 6$. Thus the super edge-magic deficiency of $F_n$ is 0 for $1 \leq n \leq 6$ and at least 1 for $n \geq 7$.

For the case $n = 7, 8, 9, 10$ and 11, label $K_1$ with 4 and the vertices of the path $P_n$ with $7-5-9-6-2-1-3$, $5-8-7-10-6-2-1-3$, $5-11-8-10-7-6-2-1-3$, $10-8-11-9-12-5-6-2-1-3$, and $8-11-9-12-10-13-5-6-2-1-3$, respectively. Each of these labelings extends to a super edge-magic total labeling of $F_n \cup K_1$. Hence, for $7 \leq n \leq 11$, $\mu_s(F_n) = 1$.

For the rest of the cases, we could only find upper bounds for the super edge-magic deficiency of $F_n$. For $12 \leq n \leq 16$, we have that $\mu_s(F_n) \leq 2$, as can be shown through the following labelings:

- For $n = 12$, label $K_1$ and $P_{12}$ with 13 and $15 - 11 - 14 - 9 - 4 - 8 - 3 - 7 - 2 - 6 - 1 - 5$, respectively.
- For $n = 13$, label $K_1$ and $P_{13}$ with 15 and $13 - 14 - 16 - 10 - 5 - 9 - 4 - 8 - 3 - 7 - 2 - 6 - 1$, respectively.
- For $n = 14, 15$ and 16, label $K_1$ with 16 and $P_n$ with $14-15-17-11-5-10-4-9-3-8-2-7-1-6$, $15-14-18-17-11-5-10-4-9-3-8-2-7-1-6$ and $15 - 14 - 18 - 19 - 17 - 11 - 5 - 10 - 4 - 9 - 3 - 8 - 2 - 7 - 1 - 6$, respectively.

**Open problem 3** *Find the exact values of* $\mu_s(F_n)$, $12 \leq n \leq 16$.

For other values of $n$, the upper bound of the super edge-magic deficiency of $F_n$ is stated in the following theorem.

**Theorem 3** *For every* $n \geq 17$, $\mu_s(F_n) \leq \lfloor \frac{1}{2}(n-2) \rfloor$.

**Proof** Let $H$ be isomorphic to $F_n \cup \lfloor \frac{1}{2}(n-2) \rfloor K_1$ with

$$V(H) = \{v_i : 1 \leq i \leq n\} \cup \{c\} \cup \{u_i : 1 \leq i \leq \lfloor \frac{1}{2}(n-2) \rfloor\}$$

and

$$E(H) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{cv_i : 1 \leq i \leq n\}.$$

We found that the labelings for $H$ are different for even and odd $n$ as described in the following two cases.

**Case 1** : $n$ odd.

Consider the vertex labeling $h : V(H) \to \{1, 2, 3, \ldots, \frac{3n-1}{2}\}$ such that

$$h(x) = \begin{cases} \frac{1}{2}i, & \text{if } x = v_i \text{ for even } i, \\ \frac{1}{2}(n+i), & \text{if } x = v_i \text{ for odd } i, \\ \frac{1}{2}(3n-1), & \text{if } x = c, \end{cases}$$

and the remaining labels are placed arbitrarily on the isolated vertices. By applying Lemma 1, $h$ extends to a super edge-magic total labeling of $H$ with the magic constant $k = 4n$.

**Case 2** : $n$ even.

Consider the vertex labeling $g : V(H) \to \{1, 2, 3, \ldots, \frac{3}{2}n\}$ such that

$$g(x) = \begin{cases} \frac{1}{2}i, & \text{if } x = v_i \text{ for even } i, \\ \frac{1}{2}(n+1+i), & \text{if } x = v_i \text{ for odd } i, \\ \frac{3}{2}n, & \text{if } x = c, \end{cases}$$

and the remaining labels are placed arbitrarily on the isolated vertices. This vertex labeling extends to a super edge magic total labeling of $H$ with the magic constant $k = 4n + 1$. This completes the proof. □

**Open problem 4** *Find a better upper bound of the super edge-magic deficiency of* $F_n$*,* $n \geq 17$*.*

To generalize the fan, we can always multiply the number of single vertices to be connected to the vertices on the path. The simplest graph resulting from this construction is the double fan $F_{n,2} \cong P_n + 2K_1$ which is a graph with $n+2$ vertices and $3n-1$ edges.

Let

$$V(F_{n,2}) = \{x, y\} \cup \{z_i | 1 \leq i \leq n\}$$

and

$$E(F_{n,2}) = \{xz_i | 1 \leq i \leq n\} \cup \{yz_i | 1 \leq i \leq n\} \cup \{z_i z_{i+1} | 1 \leq i \leq n-1\}.$$

The next theorem gives the only two super edge-magic double fans.

**Theorem 4** *The graph* $F_{n,2}$ *is super edge-magic if and only if* $n \leq 2$*.*

**Proof** First, we show that $F_{n,2}$ is super edge-magic for $n = 1, 2$. The graph $F_{1,2} \cong P_3$ is trivially super edge-magic. For $n = 2$, label $2K_1$ with $\{1, 4\}$ and $P_2$ with $\{2, 3\}$. Then by Lemma 1, $F_{2,2}$ is super edge-magic.

Conversely, let $F_{n,2}$ be a super edge-magic graph. Then, by Lemma 2 we have $n \leq 2$. □

Since there is no super edge-magic total labeling of double fans for most values of $n$, we thus try to find its super edge-magic deficiency. The following theorem gives the upper and lower bounds of the deficiency.

**Theorem 5** *The super edge-magic deficiency of* $F_{n,2}$ *satisfies* $\lfloor \frac{n-1}{2} \rfloor \leq \mu_s(F_{n,2}) \leq n-2$ *for all* $n \geq 2$.

**Proof** Define a labeling $f$ as follows:

$$f(x) = 1, \quad f(y) = 2n.$$

The rest of the vertices are labeled in the following way.

If $n$ is even,

$$f(z_i) = \begin{cases} \frac{1}{2}(n+1+i), & \text{for } i = 1, 3, 5, \ldots, n-1, \\ \frac{1}{2}(2n+i), & \text{for } i = 2, 4, 6, \ldots, n; \end{cases}$$

and if $n$ is odd,

$$f(z_i) = \begin{cases} \frac{1}{2}(n+i), & \text{for } i = 1, 3, 5, \ldots, n, \\ \frac{1}{2}(2n+i), & \text{for } i = 2, 4, 6, \ldots, n-1. \end{cases}$$

We can see that these labels of vertices constitute a set $\{f(x)+f(y) \mid xy \in E(F_{n,2})\}$ of $3n-1$ consecutive integers. We can also see that no labels are repeated. However, the largest vertex label used is $2n$ and there exist $2n-(n+2) = n-2$ labels that are not utilized. So, for each of the numbers between 1 and $2n$ that has not been used as a label, we introduce a new vertex with that number as its label, which gives $n-2$ new isolated vertices. By Lemma 1, this yields a super edge-magic total labeling of a double fan $F_{n,2} \cup (n-2)K_1$ with the magic constant $\lfloor \frac{1}{2}(11n+2) \rfloor$. Hence,

$$\mu_s(F_{n,2}) \leq n-2.$$

For a lower bound, by Lemma 2, it is easy to check that

$$\mu_s(F_{n,2}) \geq \lfloor \tfrac{n-1}{2} \rfloor. \qquad \square$$

For small values of $n$, we have found the exact values of the super edge-magic deficiency of double fans $F_{n,2}$ such as $\mu_s(F_{3,2}) = \mu_s(F_{4,2}) = 1$, $\mu_s(F_{5,2}) = \mu_s(F_{6,2}) = 2$, and $\mu_s(F_{7,2}) = 3$. The label of vertices $\{x, y; z_1, z_2, \ldots, z_n\}$ of $F_{n,2}$ for $3 \leq n \leq 7$ are $\{1, 6; 2, 4, 3\}$, $\{3, 5; 1, 2, 6, 7\}$, $\{2, 9; 3, 1, 5, 6, 7\}$, $\{4, 7; 2, 1, 3, 8, 10, 9\}$, and $\{3, 7; 4, 2, 1, 9, 11, 10, 12\}$, respectively.

**Open problem 5** *For* $n \geq 8$, *find the exact values of the super edge-magic deficiencies of* $F_{n,2}$.

Let us now determine the super edge-magic deficiency of the wheel $W_n \cong C_n + \{c\}$, $n \geq 3$. Let

$$V(W_n) = \{x_i \mid 1 \leq i \leq n\} \cup \{c\},$$

and

$$E(W_n) = \{cx_i \mid 1 \leq i \leq n\} \cup \{x_i x_{i+1} \mid 1 \leq i \leq n-1\} \cup \{x_n x_1\}.$$

By Lemma 2, it is easy to see that $W_n$ is not super edge-magic. Consequently, $\mu_s(W_n) \geq 1$. In the next theorem we give exact values of super edge-magic deficiencies of $W_n$ for small values of $n$, and the lower bound for other values of $n$.

**Theorem 6** *Let $W_n$ be a wheel with $n+1$ vertices. Then,*

$$\mu_s(W_n) = \begin{cases} 1, & \text{for } n = 3, 4, 6, 7, \\ 2, & \text{for } n = 5, 9, 10, 11, 12, 13, \end{cases}$$

*and for other values of $n$, $\mu_s(W_n) \geq 2$.*

**Proof** For $W_3, W_4, W_6$ and $W_7$, label $(c; x_1, x_2, \ldots, x_n)$ as follows: $(5; 1, 2, 3)$, $(3; 1, 4, 6, 5)$, $(3; 1, 2, 4, 8, 6, 7)$ and $(4; 1, 2, 8, 7, 9, 5, 3)$, respectively. Thus, $\mu_s(W_n) = 1$ for $n = 3, 4, 6, 7$.

Next, we will show that $\mu_s(W_5) = 2$. Suppose that $W_5 \cup K_1$ is a super edge-magic graph. Then, there exists a vertex labeling $f : V(W_5 \cup K_1) \longrightarrow \{1, 2, \ldots, 7\}$ that extends to a super edge-magic total labeling of $W_5 \cup K_1$. Let $S = \{f(x) + f(y) \mid xy \in E(W_5 \cup K_1)\}$. Then, we have two possible $S$, namely $S_1 = \{3, 4, 5, \ldots, 12\}$ or $S_2 = \{4, 5, 6, \ldots, 13\}$. Note that $S_1$ and $S_2$ are dual to each other under the labeling $f'(x) = 8 - f(x)$, for all $x \in V(W_5 \cup K_1)$. So, it suffices to consider only one of them. The sum of all elements of $S_1$ is 75. This sum contains each label of $x_i$, $1 \leq i \leq 5$, three times, and each label of $c$ five times. Thus, we have

$$75 = 2\sum_{i=1}^{5} x_i + 4f(c) + \sum_{i=1}^{7} i - f(K_1).$$

Clearly, $f(K_1)$ must be odd. On the other hand, labels 1, 3, 5 and 7 must be assigned to the vertices of $W_5$, since otherwise we cannot obtain 4 and 12 in $S_1$. So, it is a contradiction to the fact that $f(K_1)$ is odd. Hence $W_5 \cup K_1$ cannot be a super edge-magic graph. Thus $\mu_s(W_5) \geq 2$. Furthermore, if we label $(c; x_1, x_2, x_3, x_4, x_5)$ by $(8; 1, 4, 2, 5, 3)$, this labeling extends to a super edge-magic total labeling of $W_5 \cup 2K_1$. Hence, we have $\mu_s(W_5) = 2$.

For $n \geq 8$, we will show that $W_n \cup K_1$ is not super edge-magic by contradiction. Assume that $W_n \cup K_1$ is super edge-magic with a labeling $g$. Then, there are two possible cases of $S = \{g(x) + g(y) \mid xy \in E(W_n \cup K_1)\}$, namely $S_1 = \{3, 4, 5, \ldots, 2n + 2\}$ and $S_2 = \{4, 5, 6, \ldots, 2n + 3\}$. Since they are dual to each other, it suffices to consider only one of them. Let us consider $S = \{3, 4, 5, \ldots, 2n + 2\}$. The only possibility to get the sums 3, 4 and 5 in $S$ is $3 = 1 + 2$, $4 = 1 + 3$ and $5 = 2 + 3 = 1$

$+\ 4$. Then, the vertices of labels 1, 2 and 3 must form a triangle or the vertex of label 1 is adjacent to the vertices of labels 2, 3 and 4. So, $2n+2$, $2n+1$, $2n$, $\ldots$, $n+g(c)+3$ should be the sum of labels of two vertices in the cycle $C_n$ of $W_n$. To obtain $2n+2$, $2n+1$, $2n$, $2n-1$ we only have two possibilities: $(n-3)-(n+2)-(n)-(n+1)-(n-1)$ or $(n-2)-(n+1)-(n-1)-(n+2)-(n)$. However for $n = 8$ and $g(c) = 4$ this construction fails, since either edge-weight 8 or 9 or 10 cannot be obtained in this case. Furthermore, if $g(c) = 1, 2$ or 3 and $n \geq 8$ the edge-weight $2n-2$ is impossible to be obtained. Hence, $W_n \cup K_1$ is not super edge-magic for $n \geq 8$. In other word, $\mu_s(W_n) \geq 2$ for $n \geq 8$.

Further, we show that for $9 \leq n \leq 13$, $\mu_s(W_n) = 2$. For $W_n$, $9 \leq n \leq 13$ label $(c; x_1, x_2, \ldots, x_n)$ as follows: $(7; 1, 4, 2, 5, 9, 12, 8, 10, 3)$, $(13; 1, 5, 2, 6, 3, 7, 4, 9, 12, 11)$, $(3; 1, 2, 4, 8, 10, 12, 11, 13, 6, 14, 7)$, $(4; 1, 2, 8, 15, 7, 14, 12, 13, 11, 9, 5, 3)$ and $(13; 1, 5, 2, 6, 3, 7, 4, 8, 14, 16, 15, 11, 12)$, respectively. This completes the proof. □

An upper bound of the super edge-magic deficiency of $W_n$ can be found in the following theorem. For $n \equiv 3 \pmod 4$, this bound can be used as upper bound for edge-magic deficiency which is better than Kotzig and Rosa's one in [12].

**Theorem 7** *For odd $n$ and $n \geq 3$, $\mu_s(W_n) \leq \frac{1}{2}(n-1)$.*

**Proof** Let $G \cong W_n \cup \frac{1}{2}(n-1)K_1$ be a graph with

$$V(G) = V(W_n) \cup \{u_i : 1 \leq i \leq \frac{1}{2}(n-1)\} \text{ and } E(G) = E(W_n).$$

It suffices to show that $G$ admits a super edge magic total labeling.

Now, consider the vertex labeling $f : V(G) \to \{1, 2, 3, \ldots, \frac{1}{2}(3n+1)\}$ such that

$$f(v) = \begin{cases} \frac{1}{2}(i+1), & \text{if } v = x_i \text{ for odd } i, \\ \frac{1}{2}(n+1+i), & \text{if } v = x_i \text{ for even } i, \\ \frac{1}{2}(3n+1), & \text{if } x = c, \end{cases}$$

and $f(U) = \{n+1, n+2, n+3, \ldots, \frac{1}{2}(3n-1)\}$, where $U = \{u_i : 1 \leq i \leq \frac{1}{2}(n-1)\}$. Then by Lemma 1, $f$ extends to a super edge magic total labeling of $G$ with the magic constant $k = 4n+2$. Consequently, for odd $n$, $n \geq 3$, $\mu_s(W_n) \leq \frac{1}{2}(n-1)$. □

Some open problems related the super edge-magic deficiency of $W_n$ are presented below.

**Open problem 6** *Find an upper bound of $\mu_s(W_n)$ for even $n$ and $n \geq 14$. Further, find a better lower bound of $\mu_s(W_n)$ for every $n \geq 14$ and a better upper bound of $\mu_s(W_n)$ for odd $n$.*

## 4 Super edge-magic deficiencies of complete multipartite graphs

In [4], it is proved that the only super edge-magic complete multipartite graphs are $K_{1,n}$ and $K_{1,1,n}$ where $n \geq 1$. In the next theorems, we give some upper bounds of the super edge-magic deficiencies of particular complete 3-partite and 4-partite graphs.

**Theorem 8** *$\mu_s(K_{1,m,n}) \leq mn - n - 1$ for all positive integers $m \geq 2$ and $n \geq 2$.*

**Proof** Let $V_1$, $V_2$ and $V_3$ be the partite sets of $K_{1,m,n}$, and let $G \cong K_{1,m,n} \cup (mn-n-1)K_1$. We will show that $G$ is a super edge-magic graph. Without loss of generality, assume that $m \leq n$.

Consider the vertex labeling $f : V(G) \to \{1, 2, 3, \ldots, m(n+1)\}$ such that $f(V_1) = \{n+2\}$, $f(V_2) = \{1, 2(n+1), 3(n+1), \ldots, m(n+1)\}$, and $f(V_3) = \{2, 3, 4, \ldots, n+1\}$ and the isolated vertices are labelled by the remaining labels. It is easy to verify that $S = \{f(a) + f(b) \mid ab \in E(G)\} = \{3, 4, 5, \ldots, (n+1)(m+1)+1\}$ consists of $mn + m + n$ consecutive integers.

Thus, $f$ extends to a super edge-magic total labeling of $G$ with the magic constant $k = (n+1)(2m+1) + 2$. Consequently, $G$ is a super edge-magic graph. □

**Theorem 9** *For $m = 1, 2$ and $n \geq 1$, $\mu_s(K_{1,1,m,n}) \leq (m+1)(n-1) + m$.*

**Proof** Let $V_1$, $V_2$, $V_3$ and $V_4$ be the partite sets of $K_{1,1,m,n}$, and let $H \cong K_{1,1,m,n} \cup ((m+1)(n-1)+m)K_1$. We will show that $H$ is a super edge-magic graph.

Now, consider the following function

$$f : V(H) \to \{1, 2, 3, \ldots, m(n+1) + 2n + 1\}$$

defined as follows.

**Case 1** : For $m = 1$

$$f(V_i) = \{i\} \text{ for } i = 1, 2, 3 \text{ and } f(V_4) = \{5, 8, 11, \ldots, 3n+2\}.$$

**Case 2** : For $m = 2$

$$f(V_i) = \{i+1\} \text{ for } i = 1, 2,\ f(V_3) = \{1, 4\}, \text{ and}$$
$$f(V_4) = \{7, 11, 15, \ldots, 4n+3\}.$$

The remaining labels from 1 to $m(n+1) + 2n + 1$ are used to label isolated vertices of $H$. It can be verified that $S = \{f(x) + f(y) \mid xy \in E(H)\} = \{3, 4, 5, \ldots, 3n+5\}$ or $S = \{3, 4, 5, \ldots, 4n+7\}$ if $m = 1$ or 2, respectively. By Lemma 1, $f$ extends to a super edge-magic total labeling of $H$ with the magic constant $k = 6n + 8$ (for $m = 1$), or $k = 8n + 11$ (for $m = 2$). □

By using Lemma 2, we have the following lower bounds for general complete 3-partite and 4-partite graphs.

**Theorem 10** *For $n_1 \geq 1$, and $n_2, n_3 \geq 2$*

$$\mu_s(K_{n_1,n_2,n_3}) \geq \lfloor \frac{1}{2}(n_2n_3 + (n_1-2)(n_2+n_3) - 2n_1 + 4) \rfloor.$$

**Theorem 11** *For all $n_1, n_2, n_3$, and $n_4$*

$$\mu_s(K_{n_1,n_2,n_3,n_4}) \geq \lfloor \frac{1}{2}(n_1(n_2+n_3+n_4-2) + n_2(n_3+n_4-2) - n_3(n_4-2) - 2n_4 + 4) \rfloor.$$

**Open problem 7** *Find an upper bound of the super edge-magic deficiency of $K_{n_1,n_2,n_3}$ and $K_{n_1,n_2,n_3,n_4}$ for all positive integers $n_1, n_2, n_3, n_4$.*

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