AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **40** (2008), Pages 41–56

Irregular edge-colorings of sums of cycles of even lengths

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Abstract

P. Wittmann showed that for the irregular coloring number $c(G)$ of a simple 2-regular graph of order n the inequality $c(G) \leqslant \sqrt{2n} + O(1)$ holds. We determine the exact value of this number in the case when the 2-regular graph consists of cycles of even lengths. For this purpose we consider decompositions of several classes of graphs.

1 Introduction

Consider a simple (without loops and multiple edges) nondirected graph G. Let C be a color set, $w\colon E(G) \to C$ an edge coloring and let $S(v)$ denote a **multiset** of colors of all edges incident with a vertex v in G. A coloring w is said to be *irregular* if for any two distinct vertices u, v the corresponding multisets satisfy $S(u) \neq S(v)$. We ask for the minimal number of necessary colors to obtain an irregular edge coloring and we call it an *irregular coloring number*. Moreover, we denote by $c(G)$ the irregular coloring number of a given graph G.

Such a number also has another interesting interpretation, as a variant of irregular weighting of edges of a graph G with positive integers. Namely, $c(G)$ can be considered as the minimal cardinality of such a subset of $\mathbb{N}$ that allows us to distinguish all vertices of G by the sums of labels of edges adjacent to them; see [11] for details.

The irregular coloring number is, for instance, determined to be equal to three for graphs K_n and $K_{n,n}$. Other results for connected graphs, such as multipartite graphs, are also known, see [11]. However, since we are interested in global distinguishing of vertices (not only neighboring ones), these results do not yield an instant generalization for nonconnected graphs, even if these graphs are disjoint unions of the connected graphs mentioned above. Already 2-regular graphs turned out to be

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problematic. A representative of such a family is a disjoint union of cycles and can be denoted as $G = C_{a_1} \cup \cdots \cup C_{a_p}$, where C_i is a cycle of length i. The following upper bound was first established by M. Aigner et al.

Theorem 1 ([1]) *Let* $G = C_{a_1} \cup \cdots \cup C_{a_p}$ *be a simple* 2*-regular graph of order* $n = \sum_{i=1}^{p} a_i$. *Then*

$$c(G) \leqslant \sqrt{8n} + O(1).$$

It was then improved by P. Wittmann.

Theorem 2 ([11]) *Let* $G = C_{a_1} \cup \cdots \cup C_{a_p}$ *be a simple* 2*-regular graph of order* $n = \sum_{i=1}^{p} a_i$. *Then*

$$c(G) \leqslant \sqrt{2n} + O(1).$$

This result is best possible except for an additive constant term. In this paper, which is a continuation of our reasonings from [7], we determine the exact value of the irregular coloring number for 2-regular graphs consisting of cycles of even lengths.

2 Coloring and decomposition

Similarly to the authors mentioned above, we use the following correspondence between an irregular edge coloring w of a 2-regular graph $G = C_{a_1} \cup \cdots \cup C_{a_p}$ with r colors and an (edge-disjoint) packing of (connected) Eulerian subgraphs into the graph M_r, where M_r is a complete graph K_r with a loop at each vertex added. (Although M_r contains loops, we shall call it a graph.)

First identify the vertices of M_r with the colors of w. Now choose an arbitrary C_{a_i} and for any two colors appearing in some $S(u)$ of C_{a_i} draw an edge or a loop between the corresponding vertices of M_r. (Notice that each multiset $S(u)$ consists of just two colors and that we draw a loop in M_r only if these colors are the same.) Since $S(u) \neq S(v)$ for any two distinct vertices of C_{a_i}, we never draw an edge of M_r twice. Moreover, as in the following example (see Figure 1), traversing C_{a_i} yields a corresponding Eulerian subgraph G_{a_i} of size a_i in M_r. Since w is an irregular edge

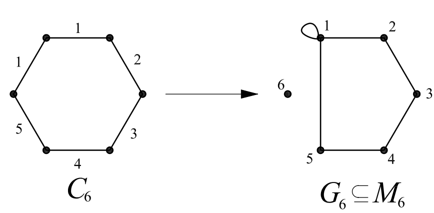

Figure 1: C_6 producing a closed trail G_6 in M_6.

coloring of the graph $G = C_{a_1} \cup \cdots \cup C_{a_p}$, we obtain edge-disjoint Eulerian subgraphs of sizes $a_1, \dots, a_p$ in M_r. Clearly, this procedure works the other way around as well, hence we have reduced the problem of irregular edge coloring to the following packing problem:

Let $G = C_{a_1}, \dots, C_{a_p}$; then $c(G)$ is the smallest number r such that we can (edge-disjointly) pack Eulerian subgraphs of sizes $a_1, \dots, a_p$ into M_r.

We shall call a (connected) Eulerian graph (or subgraph) of size n, a *closed trail* of length n. Notice that such a closed trail can be identified with any sequence $(v_1, v_2, \dots, v_{n+1})$ of its, not necessarily distinct, vertices such that $v_i v_{i+1}$ are its distinct edges for $i = 1, \dots, n$, and hence $v_1 = v_{n+1}$. Moreover, given two edge-disjoint closed trails A_1, A_2 which are *not* disjoint on vertices, we shall write $A_1.A_2$ for their union, which is a closed trail as well. Notice also that if $A_1, \dots, A_p$ are edge-disjoint closed trails in M_n, their union forms an Eulerian subgraph (or a union of Eulerian subgraphs) of M_n. Therefore, instead of the problem of packing, we consider a problem of decomposing into closed trails of even lengths of L_n, which is defined to be a maximal, in terms of size, Eulerian subgraph of M_n with an even number of edges.

Therefore, we introduce the following definitions. A sequence $\tau = (a_1, \dots, a_p)$ of integers is called *admissible for* G if its elements add up to $\|G\|$, the size of the graph G, and $a_i \geqslant 3$ for $i = 1, \dots, p$. Moreover, if G can be (edge-disjointly) decomposed into closed trails $A_1, A_2, \dots, A_p$ of lengths $a_1, a_2, \dots, a_p$, respectively, then τ is called *realizable in* G and the sequence $(A_1, A_2, \dots, A_p)$ is said to be *a* G*-realization of* τ or *a realization of* τ *in* G. Since we are interested in unions of cycles of *even* lengths, we exclusively investigate sequences consisting of numbers divisible by two and we call them *even sequences*. Moreover, we say that a graph G is *arbitrarily decomposable* into closed trails (of even lengths) if each (even) admissible for G sequence is also realizable in this graph.

Our main aim will be to show that L_n (and L_n with a pair of loops removed) is arbitrarily decomposable into closed trails of even lengths for (almost) every n. In other words, we show that we can (edge-disjointly) pack as many closed trails of even lengths into M_n as is possible, taking into account the necessary conditions, i.e. the size of the graph and the fact that these closed trails are its Eulerian subgraphs; see Theorem 14. Such an optimal solution of this problem results in determining $c(G)$ for 2−regular graphs consisting of cycles of even lengths; see Theorem 15.

It is also worth mentioning here that similar problems concerning decompositions, but in the case of complete graphs, were investigated by P.N. Balister, whose best known result is as follows.

Theorem 3 ([2]) *Let the sum* $\sum_{i=1}^{p} a_i$, $a_i \geqslant 3$, *be equal* $\binom{n}{2}$ *when* n *is odd and* $\binom{n}{2} - \frac{n}{2} - 2 \leqslant \sum_{i=1}^{p} a_i \leqslant \binom{n}{2} - \frac{n}{2}$ *when* n *is even. Then we can decompose some subgraph of* K_n *into closed trails of lengths* $a_1, \dots, a_p$.

This theorem enabled the value of a variant of irregular coloring number for 2-regular graphs to be established in the case when we assume this coloring to be proper. Also directed graphs were discussed by the same author, see [3]. Other related problems appear in [5] and [10]. In our proof, the most useful will be the following result of M. Horňák and M. Woźniak.

Theorem 4 ([10]) *If a, b are positive even integers, then if $\sum_{i=1}^{p} a_i = a \cdot b$ and there is a closed trail of length a_i in $K_{a,b}$ (for all $i \in \{1, \ldots, p\}$), then $K_{a,b}$ can be (edge-disjointly) decomposed into closed trails $A_1, A_2, \ldots, A_p$ of lengths $a_1, a_2, \ldots, a_p$, respectively.*

Let us observe that $K_{2,b}$ contains only closed trails of lengths $4i$, where $i = 1, 2, \ldots, \frac{b}{2}$, whereas $K_{a,b}$, for $a, b \geqslant 4$, contains closed trails of lengths $2j$, where $j = 2, 3, \ldots, \frac{ab-4}{2}$, $\frac{ab}{2}$; see [10].

3 Decomposition of L_n^m

Notice first that L_n for *even* n is a complete graph K_n with a single loop at each vertex added and minus a 1-factor. Let L_n' (n even) denote a graph L_n with two loops at non-adjacent vertices removed. In [7] we proved the following theorem.

Theorem 5 *Graphs L_n and L_n' with even n are arbitrarily decomposable into closed trails of even lengths, unless $n = 4$ (and $\tau = (4, 4)$).*

Observe in turn, that if we take an *odd* n into account, then L_n is equal to M_n if $n \equiv 3 (\mathrm{mod}\ 4)$ and is equal to M_n with one loop removed otherwise. This time let L_n' denote a graph L_n with an arbitrary pair of loops removed. For technical reasons, to show that graphs L_n and L_n' (n odd) are arbitrarily decomposable into closed trails of even lengths, we first have to show that such a statement remains true for another family of graphs.

So for even integers m, n, with $0 \leqslant m \leqslant n$, let $\mathcal{L}_n^m$ stand for a family of all graphs we may receive by adding single loops at m vertices of K_n minus a 1-factor. Notice that $L_n \in \mathcal{L}_n^n$ and is actually (up to isomorphism) the only representative of this family and as well K_n minus a 1-factor is a single representative of $\mathcal{L}_n^0$. This is however not the case if $2 \leqslant m \leqslant n - 2$; therefore, from now on, by L_n^m we will usually mean an arbitrary representative of the family $\mathcal{L}_n^m$ if nothing else is stated. This section is dedicated to proving the following theorem.

Theorem 6 *Let m, n be even numbers, where $0 \leqslant m \leqslant n$. A graph $L_n^m \in \mathcal{L}_n^m$ is arbitrarily decomposable into closed trails of even lengths, unless $n = 4$ and $m = 4$.*

Let x be a vertex of L_n^m. The only nonadjacent to x vertex of L_n^m we shall denote by x', hence $(x')' = x$. We say that x is of type one if neither x nor x' has a loop, whereas it is of type three if there is a loop at both of them. Analogously, x is of type two or four if only x' or only x, respectively, has a loop. We shall write $t(x) = i$, where $i = 1, 3, 2, 4$, respectively, see Figure 2. Notice that the type of x' is the consequence of the type of x, since either $t(x) = 1 = t(x')$ or $t(x) = 3 = t(x')$ or $\{t(x), t(x')\} = \{2, 4\}$.

Observe that if $m = 0$, then Theorem 6 is true by Theorem 3, whereas for $m = n$ we are done by Theorem 5. Therefore, we can restrict our reasonings to the case

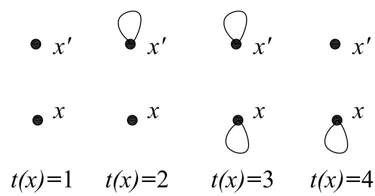

Figure 2: Types of a vertex x.

$2 \leqslant m \leqslant n-2$. But then, a graph L_n^m contains an induced subgraph H_1 or H_2 from Figure 3. It is because then there exist vertices $x, x', y, y' \in V(L_n^m)$, such that either $t(x) = t(y) = 2$ or $t(x) = 1$ and $t(y) = 3$. Notice that the graphs H_1 and

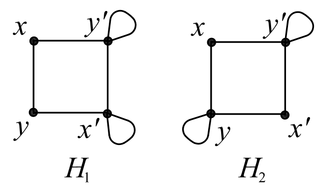

Figure 3: H_1 and H_2.

H_2 are actually the only nonisomorphic representatives of the family $\mathcal{L}_4^2$. Therefore, there exist $L_4^2 \in \mathcal{L}_4^2$ and $L_{n-4}^{m-2} \in \mathcal{L}_{n-4}^{m-2}$ such that $L_n^m = (L_4^2.K_{4,n-4}).L_{n-4}^{m-2}$. Let us then denote by $\mathcal{R}_n$ a family of graphs of the form $L_2^4.K_{4,n-4}$, where a vertex set of $L_4^2 \in \mathcal{L}_4^2$ coincide with a partition set of size 4 of $K_{4,n-4}$. An arbitrary representative of this family we shall denote by R_n, see Figure 4. The basic idea of our proof is to

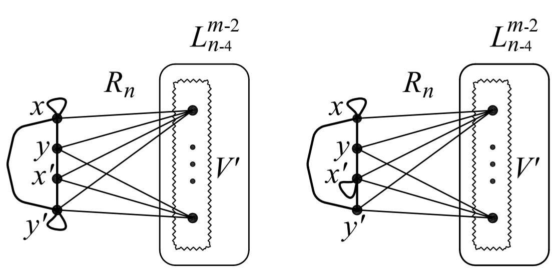

Figure 4: L_n^m as $R_n.L_{n-4}^{m-2}$.

consider a graph $G = L_n^m$ as a union $G'.G''$ of the two graphs and given an admissible for G sequence $\tau = (a_1, \ldots, a_p)$, divide it into two sequences $\tau' = (a_1, \ldots, a_i)$, $\tau'' = (a_{i+1}, \ldots, a_p)$ admissible for G', G'', respectively, and decompose these two graphs separately. Therefore, if $2 \leqslant m \leqslant n-2$, we can take $G' = R_n$ and $G'' = L_{n-4}^{m-2}$. If additionally $m \geqslant 4$, then we can assume $G' = R_n.L_2$ and $G'' = L_{n-4}^{m-4}$, where by $R_n.L_2$ we mean one of the members of the family $\mathcal{R}_n$ with two loops at two vertices from the partition set of size $n-4$ of $K_{4,n-4}$ added. Then we decompose G'' by induction and G' by one of the two lemmas below. It is however obvious, we can not always simply divide τ into τ' and τ'' as described above. Therefore, we split $a_i = a_i' + a_i''$ at times and search for realizations of $\tau_1' = (a_1, \dots, a_{i-1}, a_i')$ and $\tau_1'' = (a_i'', a_{i+1}, \dots, a_p)$ in G' and G'', respectively, and finally glue together closed trails of lengths a_i' and a_i'' to form the one of length a_i. This is, in turn, possible only if the closed trail of length a_i' meets at least one vertex from the partition set of size $n-4$ of $K_{4,n-4}$, see Figure 4.

From now on, we shall write $a_1^{r_1} \cdot a_2^{r_2} \cdot \dots \cdot a_s^{r_s}$ instead of the sequence $(\underbrace{a_1, \dots, a_1}_{r_1}, \underbrace{a_2, \dots, a_2}_{r_2}, \dots, \underbrace{a_s, \dots, a_s}_{r_s})$ for short. Moreover, if $r_i = 1$ for some i, we shall omit writing r_i in this shortened notation.

Lemma 7 *A graph $R_n \in \mathcal{R}_n$, with $n \geqslant 8$, is arbitrarily decomposable into closed trails of even lengths.*

Proof. Let $\tau = (a_1, \dots, a_p)$ be an even admissible sequence for R_n. Since $\|R_n\| \equiv 2 \pmod 4$, we may assume $a_1 \equiv 2 \pmod 4$ and find a realization of $\tau_1 = (a_1 - 6, a_2, \dots, a_p)$ in $K_{4,n-4}$ by Theorem 4 (in particular we may receive $a_1 - 6 = 0$). Then, by gluing together a closed trail of length $a_1 - 6$ with L_4^2, we receive a realization of τ in G. ■

Notice that if we want to be certain we can choose this realization of τ in such a way, that a closed trail of a given length, say a_1, is a subset of $K_{4,n-4}$, it is enough to assume either $a_j \notin \{4, 8\}$ or $a_{j_1}, a_{j_2} = 8$ for some $j, j_1, j_2 > 1$. It is obvious by the proof above when $a_j = 6$ or $a_j \geqslant 10$. It is enough to exchange a_1 with a_j. In the second case, if for instance $a_2 = a_3 = 8$, we find a realization of $\tau_2 = (a_1, a_2 - 2, a_3 - 4, a_4, \dots, a_p)$ in $K_{4,n-4}$ and glue together (possibly after a permutation of vertices of $K_{4,n-4}$) a closed trail of length $a_2 - 2$ with the two loops from L_4^2 and the one of length $a_3 - 4$ with the rest of L_4^2.

It is also obvious that in these two cases a closed trail of length a_1 has at least one vertex in the partition set of size $n-4$ of $K_{4,n-4}$. However, since $\|L_4^2\| = 6$, it is also the case if $a_1 \geqslant 8$.

Lemma 8 *If $\tau = (a_1, \dots, a_p)$ is an even admissible sequence for $G = R_n.L_2$, where $R_n \in \mathcal{R}_n$ and $n \geqslant 8$, then it is also G-realizable, unless $\tau = 4^r$ for some r.*

Proof. If one of the elements of τ, say a_1, is not smaller than 12, then we find a realization of $\tau_1 = (a_1 - 8, a_2, \dots, a_p)$ in $K_{4,n-4}$ by Theorem 4 and then glue together a closed trail of length $a_1 - 8$ with L_4^2 and L_2. Analogously, if there are at least two elements of τ, say a_1, a_2, not divisible by four, then we find a realization of $\tau_2 = (a_1 - 6, a_2 - 2, a_3, \dots, a_p)$ in $K_{4,n-4}$ and glue together a closed trail of length

$a_1 - 2$ with the two loops L_2 and a closed trail of length $a_2 - 6$ with L_4^2. In both cases some permutations of vertices of $K_{4,n-4}$ may be necessary. O Therefore, since $\|G\| \equiv 0 \pmod 4$, we can assume $a_j \in \{4, 8\}$ for all j and $a_1 = 8$ (because $\tau \neq 4^r$). Then we find a realization of $\tau_3 = (a_1 - 4, a_2 - 4, a_3, \ldots, a_p)$ in $K_{4,n-4}$ by Theorem 4 (in particular we may receive $a_2 - 4 = 0$) and we glue together a closed trail of length $a_1 - 4$ with L_2 and the two loops from L_4^2, and the one of length $a_2 - 4$ with the rest of L_4^2. Again, some permutations of vertices of $K_{4,n-4}$ may be necessary. ■

In addition to the lemmas above, we also make use of the following result by Chou, Fu and Huang to prove Theorem 6.

Theorem 9 ([9]) *Let $K_{a,b}$ be the complete bipartite graph and C_n be an elementary cycle of length n. Graph G can be decomposed into p copies of C_4, q copies of C_6 and r copies of C_8 for each triple p, q, r of nonnegative integers such that $4p+6q+8r = \|G\|$ in the following two cases:*

1. *$G = K_{a,b}$, if $a \geqslant 4$, $b \geqslant 6$ and a, b are even.*
2. *$G = K_{a,a}$ minus 1-factor if a is odd.*

Proof of Theorem 6. The cases for $n \leqslant 10$ have been analyzed by a computer programme we created. Assume then $n \geqslant 12$ and let us argue by induction on n.

Let $G = L_n^m$ be an arbitrary representative of a family $\mathcal{L}_n^m$ and let $\tau = (a_1, \ldots, a_p)$ be an even admissible sequence for G. Since the cases $m = 0$ and $m = n$ are the consequence of Theorems 3 and 5, we may assume $2 \leqslant m \leqslant n - 2$. Let $s_i = a_1 + a_2 + \cdots + a_i$ for each i.

Let first $\tau = 4^p$. If there exists a vertex $x \in V(G)$ such that $t(x) = 1$, then $G = K_{2,n-2}.L_{n-2}^m$ for some $L_{n-2}^m \in \mathcal{L}_{n-2}^m$, where one of the partition sets of $K_{2,n-2}$ equals $\{x, x'\}$. Hence, we can separately decompose $K_{2,n-2}$ and L_{n-2}^m into closed trails of length four by Theorem 4 and induction, respectively. Assume therefore that there does not exist a vertex of type one in G, hence $m \geqslant \frac{n}{2}$. Then, since m is an even number not greater than $n - 2$, either there exist $x, y, z \in V(G)$ such that $t(x) = 3$ and $t(y) = 2 = t(z)$ or $t(u) \in \{2, 4\}$ for each $u \in V(G)$. In the first case, we have $G = (L_6^4.K_{6,n-6}).L_{n-6}^{m-4}$, where $V(L_6^4) = \{x, x', y, y', z, z'\}$. Then we can decompose $K_{6,n-6}$ into closed trails of length four by Theorem 4 and the remaining two graphs by induction. In the last case, when $t(u) \in \{2, 4\}$ for each $u \in V(G)$, we have $m = \frac{n}{2}$ and $\|G\| = \frac{n(n-1)}{2}$. On the other hand, since $\tau = 4^p$ is admissible for G, $\|G\| \equiv 0 \pmod 4$. Therefore $n \equiv 0 \pmod 8$ and $G = (L_8^4.K_{8,n-8}).L_{n-8}^{m-4}$, hence we can, as above, decompose these three graphs into closed trails of length four separately.

We assume from now on, that the sequence τ is nonincreasing, i.e. $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_p$, and is not of the form 4^p. Let also $G = R_n.L_{n-4}^{m-2}$, where $R_n \in \mathcal{R}_n$, $L_{n-4}^{m-2} \in \mathcal{L}_{n-4}^{m-2}$ and $s = \|R_n\|$.

Case 1: For some i, $s_i = s$. Then we can find a realization of $\tau_1 = (a_1, \ldots, a_i)$ in R_n by Lemma 7 and then decompose L_{n-4}^{m-2} into closed trails of lengths $a_{i+1}, \ldots, a_p$ by induction.

Case 2: For some i, $s_{i-1} \leqslant s - 4$ and $s_i \geqslant s + 4$. Let $\tau_2 = (a_1, \ldots, a_{i-1}, a_i')$ and $\tau_3 = (a_i'', a_{i+1}, \ldots, a_p)$, where $a_i' = s - s_{i-1} \geqslant 4$ and $a_i'' = a_i - a_i' \geqslant 4$. Since $a_i \geqslant 8$ and τ is nonincreasing, then $a_j \geqslant 8$ for all $j < i$, hence we can find its realization in $R_n = L_4^2.K_{4,n-4}$ by Lemma 7. Moreover, by the comment below the proof of this lemma, we can find this realization in such a way that a closed trail of length a_i' contains at least one vertex from the partition set of size $n-4$ of $K_{4,n-4}$. Then, by induction, we find a realization of τ_3 in L_{n-4}^{m-2} and permute its vertices in such a way that the trails of lengths a_i' and a_i'' meet at some vertex forming a trail of length a_i.

Case 3: For some i, $s_i = s + 2$. If $m \geqslant 4$, we have $G = G_1.L_{n-4}^{m-4}$, where $G_1 = R_n.L_2$. Then $\|G_1\| = s + 2$ and the sequence $\tau_1 = (a_1, \ldots, a_i)$ is not of the form 4^r (because nonincreasing τ is not). Hence we can find a realization of τ_1 in G_1 by Lemma 8 and a realization of the remaining elements of τ in L_{n-4}^{m-4} by induction. Assume therefore $m = 2$. Then $G = G_2.L_{n-4}^2$, where $G_2 = L_4^0.K_{4,n-4}$ and $\|G_2\| = s_i - 4$. If $a_1 \geqslant 12$, then we find realizations of sequences $\tau_2 = (a_1 - 8, a_2, \ldots, a_i)$ and $\tau_3 = (4, a_{i+1}, \ldots, a_p)$ in $K_{4,n-4}$ and L_{n-4}^2 by Theorem 4 and induction, respectively. Then it is enough to glue together the closed trail of length $a_1 - 8$ with L_4^0 and a closed trail of length four from L_{n-4}^2 to form the one of length a_1. Analogously, if there exist $i_1, i_2 \leqslant i$ such that $a_{i_1}, a_{i_2} \neq 6$, then we find a $K_{4,n-4}$-realization of τ_1 with elements a_1, a_2 exchanged (or removed) by $a_{i_1} - 4$, $a_{i_2} - 4$ and an L_{n-4}^2-realization of τ_3. Now it is enough to glue together the closed trails of length $a_{i_1} - 4$, $a_{i_2} - 4$ with L_4^0 and a closed trail of length four from L_{n-4}^2, respectively. Consequently, we may assume $\tau_1 = 6^i$ or $\tau_1 = 8 \cdot 6^{i-1}$ or $\tau_1 = 10 \cdot 6^{i-1}$ or $\tau_1 = 6^{i-1} \cdot 4$, hence $a_2 = a_3 = 6$. If $t_p = 4$, then we find a realization of $\tau_4 = (a_1, a_p, a_4, \ldots, a_i)$ in $K_{4,n-4}$ by Theorem 4 and we are done, since $G = (L_4^2.K_{4,n-4}).L_n^0$, where L_4^2 is a closed trail of length a_3 and we can find a realization of the remaining elements of τ in L_n^0 by induction. Therefore, since τ is nonincreasing, we may assume that almost all of its elements (except possibly one) are equal to six. In such a case we have $G = (L_6^0.K_{6,n-6}).L_{n-6}^2$, where L_6, $K_{6,n-6}$ are decomposable into closed trails of length six and we can find an L_{n-6}^2-realization of the remaining elements of τ by induction.

Case 4: For some i, $s_i = s - 2$. In such a situation, if $a_p = 4$, we have $s_i + a_p = s + 2$ and we continue the proof the same way as in case 3. Therefore, we may assume $a_j \geqslant 6$ for each j and $s_i + a_p \geqslant s + 4$. If additionally $a_1 > a_p$, we have $s_i - a_1 + a_p \leqslant s - 4$ and we proceed the same way as in case 2. Hence, we are left with the case $\tau = t^r$ with $t \geqslant 6$.

Case 4.1: $\tau = 6^p$. Then we first find by induction a realization of the sequences $6^{p_1} \cdot 4$ in L_{n-4}^{m-2}, where a closed trail of length four we denote as $A_1 = (v_1, \ldots, v_4, v_1)$. Then we find a realization of the sequence $8 \cdot 6^{p_2}$ in $R_n = L_4^2.K_{4,n-4}$ by taking L_4^2 as one closed trail of length 6 and then decomposing $K_{4,n-4}$ into $p_2 - 1$ copies of C_6 and one C_8 by Theorem 9. We may assume $C_8 = (w_1, \ldots, w_8, w_1)$, where $w_1 = v_1$

and $w_5 = v_3$ (it is enough to permute the vertices of $K_{4,n-4}$). Then the union of such A_1 and C_8 can be easily decomposed into two cycles of length 6, namely of the form $(w_1, w_2, w_3, w_4, w_5, v_4, w_1)$ and $(w_1, v_2, w_5, w_6, w_7, w_8, w_1)$, see Figure 5. Since

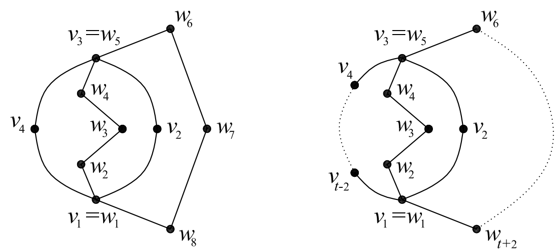

Figure 5: Intersecting closed trails.

obviously $p = p_1 + p_2 + 2$, we receive a realization of 6^p in G.

Case 4.2: $\tau = t^p$ and $t \geqslant 8$. As above, we first find a realization of the sequence $t^{p_1} \cdot (t-2)$ in L^{m-2}_{n-4}, with $A_1 = (v_1, \ldots, v_{t-2}, v_1)$ being a closed trail of length $t-2$. Then, by Lemma 7, we find a realization of the sequence $(t+2) \cdot t^{p_2}$ in $R_n = L^2_4.K_{4,n-4}$ in such a way that a closed trail $A_2 = (w_1, \ldots, w_{t+2}, w_1)$ of length $t+2$ is a subset of $K_{4,n-4}$ (or is equal to the entire R_n). It is possible by the comment below the proof of that lemma. Therefore, since $t+2 \geqslant 10$, A_2 contains at least three vertices from the partition set of size $n-4$ of $K_{4,n-4}$, and we may assume w_1, w_5 are distinct such vertices. Moreover, since we can permute the vertices of L^{m-2}_{n-4}, we may assume $v_1 = w_1$ and $v_3 = w_5$. Then the union of such A_1 and A_2 can be, analogously as in the previous subcase, decomposed into two closed trails of length t, see Figure 5. ∎

4 Decomposition of L_n with odd n

We can now prove a theorem which, together with Theorem 5, closes the subject of packing closed trails of even lengths into M_n for an arbitrary n, see Theorem 14.

Theorem 10 *Graphs L_n and L'_n with odd n are arbitrarily decomposable into closed trails of even lengths.*

Let n be an odd number and $G = L_n$ or $G = L'_n$. Let us fix a loopless vertex of G in the case $n \equiv 1 (\text{mod } 4)$ or any vertex with a loop otherwise and label it as x. Then take a subgraph of G of the form $L^m_{n-1} \in \mathcal{L}^m_{n-1}$, m even, containing all the vertices of G except x. Then $G = L^m_{n-1}.G_x$, where G_x has even size and is one of the forms presented below, see Figure 6. Observe that x is the only vertex that joins subsequent cells of the graph G_x, where by cells we mean triangles with possible loops at their bottom (as in Figure 6) and sometimes upper vertices. Since we expect to find a decomposition of G_x into closed trails, we are particularly interested in subgraphs of G_x consisting of sets of these cells glued at x. We introduce the following notion. Let

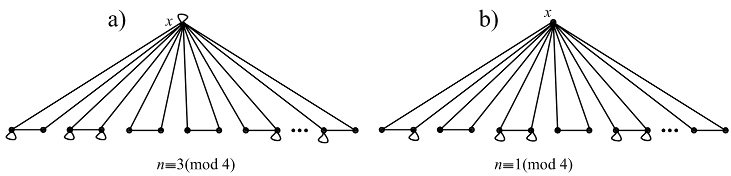

Figure 6: A graph G_x.

T_0 stand for a loopless triangle, T_1 for a triangle with one loop at a bottom vertex (no matter which one), T_2 for a triangle with two loops at the bottom vertices and T_{i+3}, where $i \in \{0, 1, 2\}$, for one of the above T_i's with a loop at its upper vertex added (e.g. T_4 will stand for a triangle with one loop at a bottom vertex and one at x), see Figure 7. Note that, by the choice of x, such a cell (T_3, T_4 or T_5) can only appear if

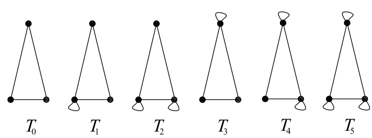

Figure 7: Cells appearing in G_x.

$n \equiv 3 (\mathrm{mod}\ 4)$. Furthermore, we denote a subgraph of G_x formed by subsequent cells $T_{i_1}, T_{i_2}, \dots, T_{i_j}$ glued at x by $T_{i_1}T_{i_2}\dots T_{i_j}$. Such a subgraph is obviously a closed trail. We will also write T_i^r for short instead of $\underbrace{T_i \dots T_i}_{r}$. For example, first five cells in Figure 6 (a) with a loop at x added to the second cell, form a closed trail $T_1T_5T_0^2T_1$ of length 20. For simplicity, we take the order of the cells into account and usually decompose G_x by cutting it into subsequent groups of cells.

We may however do everything the other way around and in our proof, we actually start by creating G_x by gluing together closed trails of the described form (and of proper lengths) and this way receive L_{n-1}^m by deleting all the edges of G_x (together with the vertex x) from G. Clearly, we have $G = L_{n-1}^m.G_x$ and we call such a graph L_{n-1}^m a completion of G_x in G. Analogously, we call G_x a completion of L_{n-1}^m in G. Notice that we have to be careful while creating G_x, since m has to be even if we want to use Theorem 6 in our proof. Moreover, we cannot use too many loops to construct G_x if $G = L'_n$.

To simplify the notation of the main proof, we formulate three simple observations. Let $G = L_n$ or $G = L'_n$ with odd $n \geqslant 7$ and let $\tau = (a_1, \dots, a_p)$ be an even admissible sequence for G. Assume $G = L_{n-1}^m.G_x$, where $L_{n-1}^m \in \mathcal{L}_{n-1}^m$, with even $m < n$ ($m < n-2$ if $G = L'_n$), and G_x are some subgraphs of G created as described above. Then the following statements hold true.

Observation 11 *If there exists a permutation $\tau_1 = (b_1, \dots, b_j, \dots, b_p)$ of τ such that the sequence $\tau_1' = (b_1, \dots, b_j)$ is realizable in G_x, then τ is realizable in G.*

Proof. It is true by Theorem 6, since we can find a realization of the remaining elements of τ in L^m_{n-1}. ■

Observation 12 *If there exists a permutation $\tau_1 = (b_1, \dots, b_j, \dots, b_p)$ of τ such that a sequence $\tau_1' = (b_1, \dots, b_{j-1}, b_j')$, where $b_j'' = b_j - b_j' \geqslant 4$, is admissible and realizable in G_x in such a way that B_j' (the respective realization of b_j') contains $T_2T_1T_0$ or $T_5T_1T_0$ as an induced subgraph, then τ is G-realizable.*

Proof. By Theorem 6, we find a realization of $\tau_1'' = (b_j'', b_{j+1}, \dots, b_p)$ in L^m_{n-1} with B_j'' being the respective realization of b_j''. Since B_j' contains all possible kinds of cells of G_x (not taking a possible loop at x into account), i.e. T_0, T_1 and T_2, then we can easily permute $n-1$ vertices of G_x (all except x) in such a way that B_j' meets at least one vertex of B_j''. Then $B_j = B_j'.B_j''$ is a closed trail of length b_j and we receive a realization of τ in G. ■

Observation 13 *If $a_p > \frac{3}{2}n - \frac{5}{2}$, then τ is realizable in G.*

Proof. We have $a_p \geqslant 3\frac{n-1}{2}$. Assume first $n \equiv 1 (\mathrm{mod}\ 4)$. Then if $a_p = 3\frac{n-1}{2}$, we take such a subgraph $L^m_{n-1} \in \mathcal{L}^m_{n-1}$ of G, that G_x being its completion in G is of the form T_0^r. Then $||G_x|| = a_p$ and we are done by Observation 11. Analogously, if $a_p = 3\frac{n-1}{2} + 2$, it is enough if $G_x = T_2T_0^r$. If finally $a_p = a_p' + a_p''$, where $a_p' = 3\frac{n-1}{2}$ and $a_p'' \geqslant 4$, we again take $G_x = T_0^r$ as a realization of (a_p'). Then by Theorem 6 we find a realization of the remaining elements of τ (and a_p'') in L^m_{n-1} and since G_x contains all the vertices of G, we easily glue together the closed trails of lengths a_p' and a_p''.

If now $n \equiv 3 (\mathrm{mod}\ 4)$, then $a_p \geqslant 3\frac{n-1}{2} + 1$, since a_p is an even number. Notice that if we take such a subgraph $L^m_{n-1} \in \mathcal{L}^m_{n-1}$ of G, that G_x being its completion in G is of the form $T_3T_0^s$, then $||G_x|| = 3\frac{n-1}{2} + 1$ and we prove that τ is G-realizable the same way as in the previous paragraph. ■

Proof of Theorem 10. We verified the cases for $n \leqslant 11$ using a computer program we created, hence we assume $n \geqslant 13$ is an odd number. Let $G = L_n$ or $G = L_n'$ and let $\tau = (a_1, \dots, a_p)$ be a nondecreasing even admissible sequence for G, hence $a_1 \leqslant a_2 \leqslant \dots \leqslant a_p$. In the first part of the proof we show that if τ consists of sufficiently many small numbers, it is easy to construct a proper G_x by gluing together short closed trails and then use Observation 11 to finish the proof (notice that in all the subsequent cases, whenever $G = L_n'$, we use at most $n-3$ loops, not taking the possible one at vertex x into account, to create G_x). In the second part we deal with the case when there exists a sufficiently big element in τ. Note first that for $G = L^m_{n-1}.G_x$, a graph G_x consists of odd number of cells T_i ($i \in \{0, \dots, 5\}$) if $n \equiv 3 (\mathrm{mod}\ 4)$ or of even number of T_i's if $n \equiv 1 (\mathrm{mod}\ 4)$.

Case 1: $a_{i_1}, \ldots, a_{i_l} = 6$ and $6l > \frac{3}{2}n - \frac{9}{2}$. Then we take such a subgraph $L^m_{n-1} \in \mathcal{L}^m_{n-1}$ of G, that its completion G_x in G is of the form $T_5T_0^r$ if $n \equiv 3(\text{mod } 4)$ or T_0^s otherwise. Notice that r and s are even numbers. Therefore, we can easily decompose G_x into closed trails of length 6, namely of the form T_0^2 (or T_5) and since $||G_x|| \leqslant 3 \cdot \frac{n-1}{2} + 3 = \frac{3}{2}n + \frac{3}{2}$, then there is enough a_{i_j}'s of length six to sum up to the size of G_x, hence we are done by Observation 11.

Case 2: $a_{i_1}, \ldots, a_{i_l} = 10$ and $10l > \frac{5}{2}n - \frac{35}{2}$. Then we take $G_x = T_3T_0^2T_2^r$ if $n \equiv 3(\text{mod } 4)$ or $G_x = T_1T_0^2T_1T_0^2T_2^s$ otherwise, where G_x is a graph of order n. Again r and s are even and we can easily decompose G_x into closed trails of length 10, namely of the form T_2^2, $T_3T_0^2$ or $T_1T_0^2$. Since $||G_x|| \leqslant \frac{5}{2}n - \frac{15}{2}$, we are, analogously as above, done by Observation 11.

Case 3: $a_{i_1}, \ldots, a_{i_l} = 14$ and $14l > \frac{7}{4}n - \frac{21}{4}$. We conduct the same reasoning as in the two above cases to find G_x of order n being easily decomposable into closed trails of length 14. Since we need to use exactly $\frac{n-1}{2}$ T_i's to construct G_x, therefore, if $\frac{n-1}{2} \equiv 0(\text{mod } 4)$ (hence $n \equiv 1(\text{mod } 4)$), then we construct G_x by gluing together trails $T_2T_0^3$. If $\frac{n-1}{2} \equiv 1(\text{mod } 4)$ (hence $n \equiv 3(\text{mod } 4)$), we take one trail $T_3T_2^2$, two trails $T_2^2T_1$ and trails $T_2T_0^3$. If $\frac{n-1}{2} \equiv 2(\text{mod } 4)$ (hence $n \equiv 1(\text{mod } 4)$), we take two trails $T_2^2T_1$ and trails $T_2T_0^3$. Finally, if $\frac{n-1}{2} \equiv 3(\text{mod } 4)$ (hence $n \equiv 3(\text{mod } 4)$), we construct G_x from one trail $T_3T_2^2$ and trails $T_2T_0^3$. It is easy to check that in all the cases the completion of G_x in G is of the form $L^m_{n-1} \in \mathcal{L}^m_{n-1}$ for some even m and by our construction $||G_x|| \leqslant \frac{7}{4}n + \frac{35}{4}$, hence the assumption made in this case is sufficient to guarantee existing enough number of 14's in τ to add up to the size of G_x. Therefore we are done by Observation 11.

Case 4: $a_{i_j} \equiv 0(\text{mod } 4)$ for $i = 1, \ldots, l$ and $a_{i_1} + \cdots + a_{i_l} > 2n - 6$. Then it is enough to take $G_x = T_3T_1^r$ if $n \equiv 3(\text{mod } 4)$ or $G_x = T_1^s$ otherwise. Since G_x consists of cells of size 4, we can easily cut it into closed trails of lengths divisible by 4 and by our assumption there is enough $a_{i_j} \equiv 0(\text{mod } 4)$ to cover G_x. Therefore, if there exists $k \leqslant l$ such that $a_{i_1} + \cdots + a_{i_k} = 2n - 2$ $(= ||G_x||)$, then we are done by Observation 11. If not, then there exists $k < l$ and $c, d > 0$ such that $a_{i_{k+1}} = 4c + 4d$ and $a_{i_1} + \cdots + a_{i_k} + 4c = 2n - 2$. In this case, we find a realization of $\tau_1' = (a_{i_1}, \ldots, a_{i_k}, 4c)$ in G_x. Moreover, similarly as in Observation 12, we can choose this realization in such a way that a closed trail of length $4c$ contains any of the cells of G_x, i.e. either T_1 or T_3. In other words, this closed trail may contain any of the vertices of G. Therefore, if using Theorem 6 we find a realization of the remaining elements of τ (and $4d$) in L^m_{n-1} being a completion of G_x in G, then we can glue together the proper closed trails of lengths $4c$ and $4d$ forming a one of length $a_{i_{k+1}}$. This way we receive a realization of τ in G.

Case 5: None of the above occurs. Let $\tau' = (b_1, \ldots, b_q)$ be a nondecreasing subsequence of τ consisting of all its elements b_j such that $b_j \equiv 2(\text{mod } 4)$ and $b_j \geqslant 10$. Since the inequality

$$\frac{n(n+1)}{2} - 3 > (2n-6) + (\frac{3}{2}n - \frac{9}{2}) + (\frac{5}{2}n - \frac{35}{2}) + (\frac{7}{4}n - \frac{21}{4}),$$

where $||G|| \geqslant \frac{n(n+1)}{2} - 3$, holds for each n, then, in view of the previous cases, at least one of the elements of the nondecreasing sequence τ' is not smaller than 18, hence $b_q \geqslant 18$. Moreover, since the inequality

$$\frac{n(n+1)}{2} - 3 > (2n-6) + (\frac{3}{2}n - \frac{9}{2}) + \frac{5(n-1)}{2}$$

holds for $n \geqslant 9$, then $b_1 + \cdots + b_q > \frac{5(n-1)}{2}$.

Notice that since $b_q \geqslant 18$, then τ is G-realizable by Observation 13 if $n = 13$. Therefore, we assume from now on, that $n \geqslant 15$ and $b_q \leqslant \frac{3}{2}n - \frac{5}{2}$.

Now, we will again construct a proper subgraph G_x of G whose completion in G will be of the form $L^m_{n-1} \in \mathcal{L}^m_{n-1}$ for some even m and use Observation 11 or 12 to finish the proof. Let for each $j < q$, $l(j)$ denote the greatest number of cells T_i ($i = 0, 1, 2$) which we have to use to construct a closed trail of length b_j, but under the condition that this trail consists of at least one loop. For instance, if $b_j = 10$, then we can construct a closed trail of this length in two ways, as T_2^2 and $T_1T_0^2$, hence $l(j) = 3$, but we remember that there is also a closed trail of length 10 using one less triangle, namely $l(j) - 1 = 2$ triangles. Analogously, $T_2^2T_1$ and $T_1^2T_0^2$ are both of length 14, but $l(j) = 4$ for $b_j = 14$. For $b_j = 18$ we have three representations $T_2^3T_0$, $T_2T_1T_0^3$, T_0^6 on different number of triangles, but only two first of them consist of at least one loop, hence $l(j) = 5$ in this case. However, whenever $b_j \geqslant 22$, we have at least three representations consisting of at least one loop, in particular on $l(j)$, $l(j) - 1$ and $l(j) - 2$ triangles. For example $l(j) = 7$ for $b_j = 22$, because $T_1T_0^6$ is of length 22, but $T_2^2T_0^4$ and $T_2^3T_1T_0$ are also closed trails of length 22. We will use "the widest" (consisting of maximal number of cells) of such closed trails to construct G_x so as to be able, in a way, squeeze them later if necessary. The requirement of one loop appearing in these closed trails is important in the case $n \equiv 3 (\mathrm{mod}\ 4)$, when there has to be a loop at vertex x in G_x. Moreover, we define $l(q)$ to be the number of T_i's in a closed trail of length b_q of the form $T_2T_1T_0T_{i_1}T_0^{r_1}$, where $i_1 \equiv b_q (\mathrm{mod}\ 3)$, $i_1 \in \{0, 1, 2\}$ and $r_1 \geqslant 1$, hence $l(q) = r_1 + 4$. We use this representation because it is the widest one containing $T_2T_1T_0$ as an induced subgraph, see Observation 12. Note also that $l(q) \geqslant 5$ and, since $b_q \geqslant b_j$ for all $j < q$, then $l(q) \geqslant l(j) - 1$ for $j < q$. Let now $k < q$ be the smallest number for which

$$\sum_{j=1}^{k} l(j) + l(q) \geqslant \frac{n-1}{2},$$

where $\frac{n-1}{2}$ is the number of triangles that we have to use to construct G_x (there exists such k, since $b_1 + \cdots + b_q > \frac{5(n-1)}{2}$).

Case 5.1: $l(1) + \cdots + l(k) = \frac{n-1}{2}$ (it is possible, since $l(q)$ may be equal to $l(k) - 1$). Then we construct G_x by joining together closed trails of lengths b_j, $j = 1, \ldots, k$, on $l(j)$ triangles described above (in the case when $n \equiv 3 (\mathrm{mod}\ 4)$, we use a slightly different closed trail of length b_1, namely we exchange either one T_1 to T_3 or one T_2 to T_4 in the described above representation). Since all b_j are even, it is also obvious that the completion of G_x in G will have an even number of loops, hence we are done by Observation 11.

Case 5.2: $l(1) + \cdots + l(k) + l(q) = \frac{n-1}{2}$. Then we are analogously done by Observation 11.

Case 5.3 $l(1) + \cdots + l(k) + l(q) = \frac{n-1}{2} + 1$. Then to construct G_x we may use instead of the closed trail $T_2T_1T_0T_{i_1}T_0^{r_1}$ of length b_q, another one, namely $T_2^3T_{i_1}T_0^{r_1-1}$, also of length b_q, but consisting of one less triangle. Hence, we are again done by Observation 11.

Case 5.4: $l(1) + \cdots + l(k) + l(q) \geqslant \frac{n-1}{2} + 2$ and $l(1) + \cdots + l(k) \leqslant \frac{n-1}{2} - 3$. Let R be such a number that $l(1) + \cdots + l(k) + R = \frac{n-1}{2}$, $R \geqslant 3$. If $R = 3$, then we take $b'_q = 12$, $b''_q = b_q - b'_q \geqslant 6$ and we construct G_x by joining described closed trails of lengths b_j, $j = 1, \ldots, k$, on $l(j)$ triangles and a closed trail $T_2T_1T_0$ of length 12. This way a sequence $\tau'' = (b_1, \ldots, b_k, b'_q)$ is realizable in G_x and we are done by Observation 12. If now $R = 4 + r_2$ for some $r_2 \geqslant 0$, then to construct G_x we take the proper closed trails of lengths b_j, $j = 1, \ldots, k$, and a closed trail $T_2T_1T_0T_{i_2}T_0^{r_2}$, where $i_2 = i_1$ if $r_1 - r_2$ is an even number or $i_2 \in \{0, 1\}$ and $|i_2 - i_1| = 1$ otherwise. Observe that since $l(1) + \cdots + l(k) + l(q) \geqslant \frac{n-1}{2} + 2$, we have $r_1 - r_2 \geqslant 2$. Therefore, if we take $b'_q = ||T_2T_1T_0T_{i_2}T_0^{r_2}||$ and $b''_q = b_q - b'_q$, then b''_q is an even number greater than four and a sequence $\tau'' = (b_1, \ldots, b_k, b'_q)$ is realizable in G_x, hence we are again done by Observation 12.

Case 5.5: $l(1) + \cdots + l(k) + l(q) \geqslant \frac{n-1}{2} + 2$ and $l(1) + \cdots + l(k) = \frac{n-1}{2} - 2$. Then instead of using a closed trail of length b_1 on $l(1)$ triangles to construct G_x, we use another one, on $l(1) - 1$ triangles, also described above. This way we have $(l(1) - 1) + l(2) + \cdots + l(k) = \frac{n-1}{2} - 3$ and (since $l(q) \geqslant 5$) $(l(1) - 1) + l(2) + \cdots + l(k) + l(q) \geqslant \frac{n-1}{2} + 2$, hence we get back to the case 5.4.

Case 5.6: $l(1) + \cdots + l(k) + l(q) \geqslant \frac{n-1}{2} + 2$ and $l(1) + \cdots + l(k) = \frac{n-1}{2} - 1$. If $k \geqslant 2$, then to construct G_x we can use instead of closed trails of lengths b_1, b_2 on $l(1)$, $l(2)$ triangles, another ones, on $l(1) - 1$ and $l(2) - 1$ triangles. This way we have $(l(1) - 1) + (l(2) - 1) + l(3) + \cdots + l(k) = \frac{n-1}{2} - 3$ and $(l(1) - 1) + (l(2) - 1) + l(3) + \cdots + l(k) + l(q) \geqslant \frac{n-1}{2} + 2$, hence we get back to the case 5.4. If on the other hand $k = 1$ and $b_1 \geqslant 22$, we can use instead of a closed trail of length b_1 on $l(1)$ triangles, another one, on $l(1) - 2$ triangles to construct G_x. This way we have $(l(1) - 2) = \frac{n-1}{2} - 3$ and $(l(1) - 2) + l(q) \geqslant \frac{n-1}{2} + 2$, hence we again get back to the case 5.4. To finish the proof, notice that it is not possible that $k = 1$ and $b_1 \leqslant 18$, because then we would have $l(1) \leqslant 5$ and consequently $n \leqslant 13$. ■

5 Final results and remarks

Here we sum up our results in two theorems second of which is a direct consequence of the first one.

Theorem 14 *Let $\tau = (a_1, \ldots, a_p)$ be a sequence of positive even integers greater than two. Then we can edge-disjointly pack closed trails of lengths $a_1, \ldots, a_p$ into M_r, whenever $\sum_{i=1}^{p} a_i \leqslant ||L_r||$, except for the case $\tau = (4,4)$ and $r = 4$.*

Proof. Let $n = \sum_{i=1}^{p} a_i$. Since $n \leqslant ||L_r||$, then by Theorem 5 or 10 we may find a realization of τ in L_r or L'_r if $n \in \{||L'_r||, ||L_r||\}$ or a realization of $\tau' = (a_1, \ldots, a_p, ||L_r|| - n)$ in L_r otherwise. This way, since L_r and L'_r are subgraphs of M_r, we receive the desired packing. ■

Theorem 15 *Let $G = C_{a_1} \cup \cdots \cup C_{a_p}$ be a simple 2-regular graph of order $n = \sum_{i=1}^{p} a_i$, where $a_1, \ldots, a_p$ are even numbers. Then $c(G) = \lceil\sqrt{2n}\rceil - 1$ if $\frac{r^2}{2} < n \leqslant \binom{r+1}{2}$ for some odd r and $c(G) = \lceil\sqrt{2n}\rceil$ in all other cases with one exception $c(C_4 \cup C_4) = 5$.*

Proof. Since L_r is a maximal, in terms of size, Eulerian subgraph of M_r with even number of edges, it is obvious, we cannot pack closed trails of lengths $a_1, \ldots, a_p$ in M_r if $n > ||L_r||$. Therefore, by Theorem 14, we have $c(G) = r$ ($c(C_4 \cup C_4) = 5$), where r is the smallest number such that $n \leqslant ||L_r||$. The solution of this problem yields the described result. ■

The problem remains still open if we admit odd cycles as components of 2-regular graphs. The main obstacle here is the fact that closed trails of length three cannot contain loops. On the other hand, we may assume, while looking for a realization of some τ in L_n, that in τ there is enough a_i's of proper lengths to cover all the loops of L_n. Observe however, that even though the sequence $\tau = (3,3,6,6)$ comply with this condition for L_6 (each closed trail of length 6 may contain 3 loops), there exists no L_6-realization of τ.

Another problem is the technique of decomposing graphs we used, basing on Theorem 4 for bipartite graphs, in which only closed trails of even lengths exist. However, using a slightly different approach to the subject of packing, together with some additional condition for closed trails of length three, should result in determining the irregular coloring number for all 2-regular graphs, but it is still to come.

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(Received 3 Dec 2006)