AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **41** (2008), Pages 37–44

Resolvable modified group divisible designs with higher index

Peter Danziger*

Department of Mathematics
Ryerson University
Toronto, ON M5B 2K3
Canada
danziger@ryerson.ca

Chengmin Wang

School of Science
Jiangnan University
Wuxi 214122
China
chengmin_wang@163.com

Abstract

A resolvable modified group divisible design (RMGD) is a modified group divisible design whose blocks can be partitioned into parallel classes. We show that the necessary conditions for the existence of a 3-RMGDD$_\lambda$ of type g^u, namely $g \geq 3$, $u \geq 3$, $gu \equiv 0 \bmod 3$ and $\lambda(g-1)(u-1) \equiv 0 \bmod 2$, are sufficient with the two exceptions of $(g, u, \lambda) \in \{(6, 3, 1), (3, 6, 1)\}$.

1 Introduction

A group divisible design K-GDD$_\lambda$ is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a finite set of points, $\mathcal{G}$ is a partition of X into subsets (called groups), and $\mathcal{B}$ is a family of subsets of X with sizes from K (called blocks) such that any pair of distinct points of X which are not from the same group occur together in exactly λ blocks of $\mathcal{B}$. We use the usual "exponential" notation to describe the type of a GDD; a GDD$_\lambda$ of type $\prod g_i^{u_i}$ means that it has u_i groups of size g_i. When $K = \{k\}$, we simply write k for K.

A K-GDD$_\lambda$ is called resolvable, denoted by K-RGDD$_\lambda$, if its blocks can be partitioned into classes, called parallel classes, such that every point of X occurs in exactly one block of each class. The following result appears in [1, 4, 5]; see also [2].

* supported by NSERC research grant #OGP0170220

Theorem 1.1. *A 3-RGDD of type g^u exists if and only if $u \geq 3$, $gu \equiv 0 \pmod 3$, $g(u-1) \equiv 0 \pmod 2$ and $(g,u) \notin \{(2,3),\ (2,6),\ (6,3)\}$.*

A transversal design, $\mathrm{TD}_\lambda(k,n)$, is a k-GDD_λ of type n^k. A resolvable transversal design, denoted by $\mathrm{RTD}_\lambda(k,n)$, is a $\mathrm{TD}_\lambda(k,n)$ whose blocks can be partitioned into parallel classes. It is well known that an $\mathrm{RTD}(k,n)$ is equivalent to $k-1$ mutually orthogonal Latin squares of order n. The following result is given in [2].

Theorem 1.2. *An $RTD_2(7,t)$ exists if and only if $t \geq 7$.*

A GDD_λ is called *incomplete*, denoted IGDD_λ, if it contains a *hole*, that is a set of points $H \subseteq X$ such that no pair of points from H appears in any block of the GDD_λ. If the blocks of an IGDD can be partitioned into parallel classes and partial parallel classes, which patition $X \backslash H$ it is called resolvable, denoted IRGDD. As with GDDs we use an exponential notation to denote these designs. By a k-IRGDD_λ of type $\prod(g_i,e_i)^{u_i}$ we mean an IRGDD_λ with blocks of size k and u_i groups of size g_i which intersect the hole in e_i points. We note that when $H=\emptyset$ a k-IRGDD_λ of type $\prod(g_i,0)^{u_i}$ is just a k-RGDD of type $\prod g_i^{u_i}$.

2 Resolvable Modified Group Divisible Designs

A resolvable modified group divisible design k-RMGD_λ of type g^u is a quadruple $(X,\mathcal{M},\mathcal{N},\mathcal{B})$ where

1. X is a set of gu points.
2. $\mathcal{M}$ and $\mathcal{N}$ are collections of subsets of X, called groups. Each member of $\mathcal{M}$ is of size g and each member of $\mathcal{N}$ is of size u. $\mathcal{M}$ and $\mathcal{N}$ each partition the points of X, and $|M \cap N| = 1$ for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$.
3. $\mathcal{B}$ is a collection of subsets of X of size k, called blocks, such that for any pair of points of X exactly one of the following holds:

   i they are both in $\mathcal{M}$,

   ii they are both in $\mathcal{N}$, or

   iii they appear together in a block $B \in \mathcal{B}$ exactly λ times.

Further the blocks may be partitioned into resolution classes such that each point of X occurs exactly once in each resolution class.

As with GDDs an RMGD_λ is called incomplete if it contains a *hole*. The symmetry of the roles of u and g in the definition of RMGDD's implies the following fact that will be employed extensively in what follows.

Lemma 2.1. *A k-$RMGDD_\lambda$ of type g^u exists if and only if a k-$RMGDD_\lambda$ of type u^g exists.*

It is straightforward to obtain the following necessary conditions.

Theorem 2.2. *The necessary conditions for the existence of a k-RMGDD$_\lambda$ of type g^u are $g \geq k$, $u \geq k$, $gu \equiv 0 \bmod k$ and $\lambda(g-1)(u-1) \equiv 0 \bmod (k-1)$.*

The following is evident by taking multiple copies of the blocks of a design.

Lemma 2.3. *If there exists a k-RMGDD$_\lambda$ of type g^u then there exists a k-RMGDD$_{\lambda\mu}$ of type g^u for every $\mu > 0$.*

Recently, in [3] the necessary conditions for a 3-RMGDD$_\lambda$ of type g^u were shown to be sufficient when $\lambda = 1$. Applying Lemma 2.3 to the result of [3] gives the following Theorem.

Theorem 2.4. *There exists a 3-RMGDD$_\lambda$ of type g^u whenever $g \geq 3$, $u \geq 3$, $gu \equiv 0 \bmod 3$ and $(g-1)(u-1) \equiv 0 \bmod 2$ except when $(g,u,\lambda) = (3,6,1)$ or $(6,3,1)$.*

In this paper we consider the case when $\lambda > 1$. Considering the necessary conditions from Theoerem 2.2, Lemma 2.3 and the result of Theorem 2.4 it suffices to solve the case of a 3-RMGDD$_2$ of type g^u with $g \equiv 0 \bmod 6$ and u even as well as 3-RMGDD$_\lambda$ of type 6^3 for $\lambda > 1$.

We begin by giving the constructions to be used, we then give some small designs which are required for the recursion. Finally, we prove the main result, that the necessary conditions above are sufficient.

3 Constructions

In this section we give some constructions that we shall use to create the designs. We start by giving several weighting or multiplicative constructions. Here $I_n = \{0, 1, 2, \ldots, n-2\}$, the independent set on n vertices.

Lemma 3.1. *If there is an l-RGDD$_\mu$ of type m^g, a k-RMGD$_\lambda$ of type m^g and a k-RMGDD$_\lambda$ of type n^l, then there exists a k-RMGDD$_{\lambda\mu}$ of type $(mn)^g$.*

Proof. Let X be the point set of the l-RGDD$_\mu$ of type m^g. The point set of the new design will be $X \times I_n$. On each expanded block of size l, B say, we place a k-RMGD$_\lambda$ of type n^l in such a way that the groups of size n are the expansion of each point and the groups of size l are $\{(x,j) \mid x \in B\}$ for $j = 1 \ldots n$. The resolution classes of the original RGDD give rise to resolution classes in the new design. Each pair of the form $(x,i)(y,j)$ $i \neq j$, x and y not in the same group, occurs λ times in the expansion of the block containing x, y. There are no pairs of the form $(x,i)(y,i)$, $x, y \in X$, $i \in I_n$, as these are the size l groups of the RMGDs. For each $i \in I_n$ we fill this set of points with the k-RMGD$_\lambda$ of type m^g. □

The following is an adaptation of Lemma 3.4 of [3] to cover the case of arbitrary λ. The details of this extension are straightforward and are omitted. The reader is refered to [3] for the proof of the original Theorem.

Lemma 3.2. *Suppose that there exists an $RTD\mu(u+1,t)$. If there exist a k-IRMGDD$_\lambda$ of type $(m+e_1,e_1)^u$ and a k-IRGDD$_\lambda$ of type $(m+e_i,e_i)^u$ for any i, $2 \le i \le t$, then there exists a k-IRMGDD$_\lambda\mu$ of type $(mt+e,e)^u$ with $e=\sum_{i=1}^t e_i$. Furthermore, if a k-RMGDD$_\lambda\mu$ of type e^u also exists, then so does a k-RMGDD$_\lambda\mu$ of type $(mt+e)^u$.*

4 Small Cases

In this section we give some small designs that will be needed. We begin by settling the case of a 3-RMGDD$_\lambda$ of type 6^3 for $\lambda > 1$.

Lemma 4.1. *A 3-RMGDD$_\lambda$ of type 6^3 exists for all $\lambda > 1$*

Proof. By Lemma 2.3 it suffices to give designs for $\lambda = 2, 3$ which we give below.

$\lambda = 2$ We take point set $Z_3 \times Z_6$, groups are given by $Z_3 \times \{i\}$ and $\{j\} \times Z_6$, $i \in Z_6$, $j \in Z_3$. Each line below gives a parallel class when developed mod $(3, -)$.

{ (0, 0) (1, 1) (2, 2) }	{ (0, 3) (1, 4) (2, 5) }
{ (0, 0) (1, 1) (2, 3) }	{ (0, 2) (1, 4) (2, 5) }
{ (0, 0) (2, 1) (1, 4) }	{ (0, 2) (1, 3) (2, 5) }
{ (0, 0) (2, 1) (1, 5) }	{ (0, 2) (1, 3) (2, 4) }
{ (0, 0) (2, 2) (1, 3) }	{ (0, 1) (2, 4) (1, 5) }
{ (0, 0) (1, 2) (2, 4) }	{ (0, 1) (2, 3) (1, 5) }
{ (0, 0) (1, 2) (2, 5) }	{ (0, 1) (2, 3) (1, 4) }
{ (0, 0) (2, 3) (1, 4) }	{ (0, 1) (1, 2) (2, 5) }
{ (0, 0) (1, 3) (2, 5) }	{ (0, 1) (2, 2) (1, 4) }
{ (0, 0) (2, 4) (1, 5) }	{ (0, 1) (2, 2) (1, 3) }

$\lambda = 3$ We take $(\mathbb{Z}_5 \cup \infty) \cup \mathbb{Z}_3$ as the point set, $\{(\mathbb{Z}_5 \cup \{\infty\}) \times \{i\} \mid i \in \mathbb{Z}_3\}$ as the groups of size 6, and $\{\{i\} \times \mathbb{Z}_3 \mid i \in \mathbb{Z}_5\}$ together with $\{(\infty,0),(\infty,1),(\infty,2)\}$ as the groups of size 3. We list the base blocks of three parallel classes, develop the blocks mod(5,-) to obtain the desired design.

Class I:
$\{(\infty,0),(2,1),(4,2)\}$, $\{(\infty,1),(4,0),(0,2)\}$, $\{(\infty,2),(3,0),(4,1)\}$,
$\{(1,0),(0,1),(2,2)\}$, $\{(0,0),(1,1),(3,2)\}$, $\{(2,0),(3,1),(1,2)\}$,

Class II:
$\{(\infty,0),(1,1),(4,2)\}$, $\{(\infty,1),(3,0),(2,2)\}$, $\{(\infty,2),(4,0),(3,1)\}$,
$\{(0,0),(2,1),(1,2)\}$, $\{(1,0),(0,1),(3,2)\}$, $\{(2,0),(4,1),(0,2)\}$,

Class III:
$\{(\infty,0),(3,1),(2,2)\}$, $\{(\infty,1),(4,0),(1,2)\}$, $\{(\infty,2),(3,0),(1,1)\}$,
$\{(0,0),(2,1),(3,2)\}$, $\{(1,0),(4,1),(0,2)\}$, $\{(2,0),(0,1),(4,2)\}$,

□

Lemma 4.2. *If $u \in \{4,6,8,10,14,22,26\}$, there exists a 3-RMGDD$_2$ of type 6^u.*

Proof. We take $(\mathbb{Z}_{u-1} \cup \{x\}) \times (\mathbb{Z}_5 \cup \{y\})$ as the point set. We take $\{i\} \times (\mathbb{Z}_5 \cup \{y\})$, $i \in \mathbb{Z}_{u-1} \cup \{x\}$ as the groups of size 6 and $(\mathbb{Z}_{u-1} \cup \{x\}) \times \{i\}$, $i \in \mathbb{Z}_5 \cup \{y\}$ as the groups of size u. For the stated values of u, a 3-RMGDD$_2$ of type 6^u contains $5(u-1)$ parallel classes. In each case the given blocks form a parallel class, develop this class mod$(u-1, 5)$ with the convention that $x + i = x$, $i \in \mathbb{Z}_{u-1}$ and $y + j = y$, $j \in \mathbb{Z}_5$, to obtain the desired design.

$\mathbf{u = 4}$:
$\{(0, y), (x, 0), (1, 2)\}$, $\{(1, y), (x, 1), (0, 2)\}$, $\{(2, y), (0, 4), (1, 1)\}$,
$\{(x, 2), (2, 4), (1, 0)\}$, $\{(x, 3), (0, 1), (2, 2)\}$, $\{(x, 4), (0, 0), (2, 3)\}$,
$\{(x, y), (1, 3), (2, 1)\}$, $\{(0, 3), (1, 4), (2, 0)\}$.

$\mathbf{u = 6}$:
$\{(0, y), (x, 0), (3, 3)\}$, $\{(1, y), (x, 1), (0, 0)\}$, $\{(2, y), (4, 4), (1, 0)\}$,
$\{(3, y), (4, 0), (1, 1)\}$, $\{(4, y), (0, 2), (1, 3)\}$, $\{(x, 2), (2, 3), (0, 4)\}$,
$\{(x, 3), (3, 4), (1, 2)\}$, $\{(x, 4), (4, 2), (0, 1)\}$, $\{(x, y), (1, 4), (2, 1)\}$,
$\{(0, 3), (2, 2), (3, 0)\}$, $\{(2, 0), (3, 1), (4, 3)\}$, $\{(2, 4), (3, 2), (4, 1)\}$.

$\mathbf{u = 8}$:
$\{(0, y), (x, 0), (3, 1)\}$, $\{(1, y), (x, 1), (4, 4)\}$, $\{(2, y), (6, 3), (1, 2)\}$,
$\{(3, y), (5, 1), (4, 0)\}$, $\{(4, y), (3, 4), (2, 2)\}$, $\{(5, y), (2, 0), (3, 2)\}$,
$\{(6, y), (1, 4), (0, 0)\}$, $\{(x, 2), (1, 3), (4, 1)\}$, $\{(x, 3), (0, 1), (1, 0)\}$,
$\{(x, 4), (0, 3), (1, 1)\}$, $\{(x, y), (4, 2), (6, 4)\}$, $\{(0, 2), (3, 3), (5, 4)\}$,
$\{(0, 4), (3, 0), (5, 3)\}$, $\{(2, 1), (4, 3), (6, 2)\}$, $\{(2, 3), (5, 0), (6, 1)\}$,
$\{(2, 4), (5, 2), (6, 0)\}$.

$\mathbf{u = 10}$:
$\{(0, y), (x, 0), (5, 1)\}$, $\{(1, y), (x, 1), (2, 4)\}$, $\{(2, y), (7, 4), (0, 1)\}$,
$\{(3, y), (0, 2), (4, 1)\}$, $\{(4, y), (1, 1), (8, 3)\}$, $\{(5, y), (3, 0), (7, 2)\}$,
$\{(6, y), (1, 0), (0, 4)\}$, $\{(7, y), (6, 4), (0, 0)\}$, $\{(8, y), (2, 3), (7, 0)\}$,
$\{(x, 2), (7, 1), (1, 4)\}$, $\{(x, 3), (1, 2), (5, 4)\}$, $\{(x, 4), (2, 1), (5, 2)\}$,
$\{(x, y), (6, 2), (8, 1)\}$, $\{(8, 0), (7, 3), (4, 4)\}$, $\{(2, 2), (6, 1), (0, 3)\}$,
$\{(2, 0), (1, 3), (8, 2)\}$, $\{(5, 3), (3, 1), (4, 0)\}$, $\{(3, 3), (4, 2), (6, 0)\}$,
$\{(3, 2), (4, 3), (8, 4)\}$, $\{(3, 4), (5, 0), (6, 3)\}$.

$\mathbf{u = 14}$:
$\{(0, y), (x, 0), (7, 2)\}$, $\{(1, y), (x, 1), (7, 3)\}$, $\{(2, y), (7, 4), (3, 0)\}$,
$\{(3, y), (6, 4), (8, 3)\}$, $\{(4, y), (12, 2), (6, 1)\}$, $\{(5, y), (1, 3), (9, 4)\}$,
$\{(6, y), (4, 1), (12, 3)\}$, $\{(7, y), (3, 1), (6, 2)\}$, $\{(8, y), (2, 4), (12, 0)\}$,
$\{(9, y), (7, 1), (10, 3)\}$, $\{(10, y), (7, 0), (0, 1)\}$, $\{(11, y), (6, 0), (0, 3)\}$,
$\{(12, y), (9, 0), (11, 3)\}$, $\{(x, 2), (10, 1), (11, 0)\}$, $\{(x, 3), (3, 4), (9, 2)\}$,
$\{(x, 4), (10, 0), (8, 2)\}$, $\{(x, y), (5, 2), (11, 4)\}$, $\{(6, 3), (9, 1), (8, 0)\}$,
$\{(5, 1), (1, 0), (9, 3)\}$, $\{(2, 2), (8, 1), (12, 4)\}$, $\{(11, 1), (3, 2), (8, 4)\}$,
$\{(2, 3), (5, 4), (10, 2)\}$, $\{(2, 0), (3, 3), (4, 4)\}$, $\{(1, 1), (5, 0), (10, 4)\}$,
$\{(0, 2), (4, 3), (1, 4)\}$, $\{(0, 0), (1, 2), (12, 1)\}$, $\{(0, 4), (2, 1), (4, 2)\}$,
$\{(4, 0), (5, 3), (11, 2)\}$.

$\mathbf{u = 22}$:

$\{(0,y),(x,0),(7,4)\}$, $\{(1,y),(x,1),(12,2)\}$, $\{(2,y),(7,2),(13,4)\}$,
$\{(3,y),(16,1),(9,0)\}$, $\{(4,y),(8,2),(1,4)\}$, $\{(5,y),(0,3),(1,0)\}$,
$\{(6,y),(18,2),(4,3)\}$, $\{(7,y),(20,4),(5,0)\}$, $\{(8,y),(17,2),(3,4)\}$,
$\{(9,y),(5,2),(10,3)\}$, $\{(10,y),(20,2),(18,4)\}$, $\{(11,y),(10,1),(19,4)\}$,
$\{(12,y),(3,3),(14,2)\}$, $\{(13,y),(2,1),(6,0)\}$, $\{(14,y),(13,3),(16,2)\}$,
$\{(15,y),(9,3),(12,1)\}$, $\{(16,y),(20,3),(1,1)\}$, $\{(17,y),(10,4),(18,1)\}$,
$\{(18,y),(0,4),(6,3)\}$, $\{(19,y),(3,1),(13,2)\}$, $\{(20,y),(2,2),(6,1)\}$,
$\{(x,2),(18,0),(14,3)\}$, $\{(x,3),(14,1),(2,0)\}$, $\{(x,4),(16,3),(19,1)\}$,
$\{(x,y),(16,4),(17,1)\}$, $\{(2,4),(13,0),(18,3)\}$, $\{(20,0),(8,4),(0,1)\}$,
$\{(10,0),(4,1),(8,3)\}$, $\{(4,0),(6,2),(11,4)\}$, $\{(15,3),(1,2),(17,4)\}$,
$\{(3,0),(9,1),(19,3)\}$, $\{(2,3),(20,1),(17,0)\}$, $\{(3,2),(12,3),(14,4)\}$,
$\{(12,0),(4,4),(19,2)\}$, $\{(5,1),(12,4),(0,2)\}$, $\{(11,2),(8,0),(5,4)\}$,
$\{(1,3),(13,1),(14,0)\}$, $\{(15,0),(7,3),(6,4)\}$, $\{(10,2),(11,0),(15,1)\}$,
$\{(11,1),(15,4),(19,0)\}$, $\{(7,0),(9,4),(11,3)\}$, $\{(4,2),(7,1),(17,3)\}$,
$\{(8,1),(0,0),(9,2)\}$, $\{(5,3),(15,2),(16,0)\}$.

$\mathbf{u = 26}$:

$\{(0,y),(x,0),(18,3)\}$, $\{(1,y),(x,1),(0,4)\}$, $\{(2,y),(23,3),(20,2)\}$,
$\{(3,y),(9,3),(20,4)\}$, $\{(4,y),(14,1),(7,0)\}$, $\{(5,y),(15,0),(16,1)\}$,
$\{(6,y),(13,2),(23,4)\}$, $\{(7,y),(5,2,(8,3)\}$, $\{(8,y),(16,2),(21,3)\}$,
$\{(9,y),(21,0),(15,4)\}$, $\{(10,y),(23,0),(6,3)\}$, $\{(11,y),(18,4),(19,1)\}$,
$\{(12,y),(16,0),(7,3)\}$, $\{(13,y),(0,2),(12,0)\}$, $\{(14,y),(15,3),(23,2)\}$,
$\{(15,y),(10,3),(12,1)\}$, $\{(16,y),(10,2),(19,3)\}$, $\{(17,y),(11,0),(7,4)\}$,
$\{(18,y),(23,1),(7,2)\}$, $\{(19,y),(24,0),(21,1)\}$, $\{(20,y),(11,3),(22,2)\}$,
$\{(21,y),(0,0),(19,2)\}$, $\{(22,y),(8,1),(12,4)\}$, $\{(23,y),(14,0),(12,3)\}$,
$\{(24,y),(8,4),(21,2)\}$, $\{(x,2),(9,1),(19,4)\}$, $\{(x,3),(14,4),(8,2)\}$,
$\{(x,4),(22,0),(7,1)\}$, $\{(x,y),(9,4),(17,1)\}$, $\{(6,2),(22,4),(1,3)\}$,
$\{(8,0),(4,4),(1,2)\}$, $\{(19,0),(18,2),(2,1)\}$, $\{(3,0),(15,1),(17,3)\}$,
$\{(11,1),(21,4),(17,0)\}$, $\{(5,3),(12,2),(24,4)\}$, $\{(9,0),(14,2),(11,4)\}$,
$\{(3,4),(15,2),(10,0)\}$, $\{(4,3),(5,4),(2,0)\}$, $\{(18,0),(14,3),(16,4)\}$,
$\{(6,0),(4,1),(24,3)\}$, $\{(0,1),(16,3),(24,2)\}$, $\{(4,2),(10,4),(18,1)\}$,
$\{(0,3),(6,1),(1,0)\}$, $\{(5,1),(11,2),(2,3)\}$, $\{(20,1),(6,4),(5,0)\}$,
$\{(1,4),(10,1),(13,3)\}$, $\{(13,4),(24,1),(3,3)\}$, $\{(3,2),(4,0),(17,4)\}$,
$\{(3,1),(13,0),(20,3)\}$, $\{(20,0),(13,1),(2,2)\}$, $\{(1,1),(22,3),(9,2)\}$,
$\{(2,4),(17,2),(22,1)\}$.

□

Lemma 4.3. *There exists a* 3*-RMGD*$_2$ *of type* 6^u *for every* $u \in \{12, 16, 18, 20, 28, 30\}$

Proof. We use the multiplicative construction given in Lemma 3.1, with $\lambda = 2$, $\mu = 1$, $k = l = 3$ and $g = 6$. In each case $u = mn$ for some factors m and n, so that a 3-RGDD of type m^6 (Theorem 1.1), a 3-RMGD$_2$ of type m^6 (Lemma 4.2) and a 3-RMGD$_2$ of type n^3 (Theorem 2.4 or Lemma 4.1) exist. Apply Theorem 3.1, using Lemma 2.1 as appropriate, to these designs to get the result.

The requisite values of n and m are given in the table below.

u	12	16	18	20	24	28	30
m	4	4	6	4	6	4	6
n	3	4	3	5	4	7	5

□

We can summarize the last two lemmas by the following:

Lemma 4.4. *There exists a 3-RMGD$_2$ of type 6^u for every even u with $4 \le u \le 30$.*

5 Results

We now present our results. We start by solving the case where one group is of size 6.

Lemma 5.1. *A 3-RMGD$_2$ of type 6^u exists for every even $u \ge 4$.*

Proof. All cases where $u < 32$ are covered by Lemma 4.4.

For $u \ge 32$, write $u = 4t + 4$ or $4t + 6$ as appropriate. $t \ge 7$, so an RTD$_2(7, t)$ exists by Lemma 1.2. Apply Lemma 3.2 with $k = 3$, $m = 4$, $e_1 = e_2 = 2$ and if $u \equiv 0 \bmod 4$ set $e_i = 0$ for $3 \le i \le t$, otherwise when $u \equiv 2 \bmod 4$ set $e_3 = 2$ and $e_i = 0$ for $4 \le i \le t$.

A 3-IRGDD of type $(4, 0)^6$ is equivalent to a 3-RGDD of type 4^6 and so exists by Lemma 1.1; a 3-IRGDD of type $(6, 2)^6$ is given in [3]. A 3-RMGDD of type 4^6 is equivalent to a 3-RMGDD$_2$ of type 6^4 by Lemma 2.1; the latter and a 3-RMGDD of type 6^6 are given in Lemma 4.2. □

We are now ready to prove our main result.

Theorem 5.2. *A 3-RMGD$_\lambda$ of type g^u exists if and only if $g, u \ge 3$, $gu \equiv 0 \bmod 3$ and $\lambda(g-1)(u-1) \equiv 0 \bmod 2$, with the two exceptions of $(g, u, \lambda) \in \{(6, 3, 1), (3, 6, 1)\}$.*

Proof. If either g or u is odd the result is given by Theorem 2.4 or Lemma 4.1. We now assume that $g \equiv 0 \bmod 6$ and u is even. Let $g = 3n$, where n is even, if $n = 2$, so $g = 6$, the result follows from Lemma 5.1. If $n > 2$, so $n \ne 6$ we may apply Lemma 3.1 using a 3-RGDD$_2$ of type 3^u, a 3-RMGDD of type 3^u and a 3-RGDD of type n^3, which exist by Theorem 2.4. □

References

[1] A. M. Assaf and A. Hartman, Resolvable group divisible designs with block size 3, *Discrete Math.* **77** (1989), 5–20.

[2] C. J. Colbourn and J. H. Dinitz (eds.), CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton FL, 1996.

[3] C. Wang, Y. Tang and P. Danziger, Resolvable Modified Group Divisible Designs with Block Size Three, *J. Combin. Designs* (to appear).

[4] R. S. Rees, Two new direct product type constructions for resolvable group divisible designs, *J. Combin. Designs* **1** (1993), 15–26.

[5] R. S. Rees and D. R. Stinson, On resolvable group-divisible designs with block size 3, *Ars Combin.* **23** (1987), 107–120.

(Received 29 Nov 2006; revised 23 Oct 2007)