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Signed edge majority domination numbers in graphs

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Abstract

The open neighborhood $N_G(e)$ of an edge e in a graph G is the set consisting of all edges having a common end-vertex with e and its closed neighborhood is $N_G[e] = N_G(e) \cup \{e\}$. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1, 1\}$. If $\sum_{x \in N_G[e]} f(x) \geq 1$ for at least a half of the edges $e \in E(G)$, then f is called a signed edge majority dominating function of G. The minimum of the values of $\sum_{e \in E(G)} f(e)$, taken over all signed edge majority dominating functions f of G, is called the signed edge majority domination number of G and is denoted by $\gamma'_{sm}(G)$. In this paper we initiate the study of signed edge majority domination in graphs. We first use an existing upper bound for the majority domination numbers of graphs to present an upper bound for signed edge majority domination numbers of graphs. Then we establish a sharp lower bound for the signed edge majority domination number of a graph.

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [5] for terminology and notation which are not defined here. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G. It is easy to see that $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e \in S} f(e)$. The *edge-neighborhood* $E_G(v)$ of a vertex $v \in V(G)$ is the set of all edges at vertex v. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e \in E_G(v)} f(e)$. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge majority dominating function* (SEMDF) of G, if $f(N_G[e]) \geq 1$ for at least a half of the edges $e \in E(G)$. The minimum of the values of $f(E(G))$, taken over all signed edge majority dominating functions f of G, is called the *signed edge majority domination number* of G and is denoted by $\gamma'_{sm}(G)$. The signed edge majority dominating function f of G with $f(E(G)) = \gamma'_{sm}(G)$ is called $\gamma'_{sm}(G)$-*function*. The authors also defined [4] the *signed edge majority total domination number* of a graph and established a sharp lower bound for the signed edge majority total domination number of forests.

A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge dominating function* (SEDF) of G, if $f(N_G[e]) \geq 1$ for each edge $e \in E(G)$. The minimum of the values of $f(E(G))$, taken over all signed edge dominating functions f of G, is called the *signed edge domination number* of G. The signed edge domination number was introduced by Xu in [6] and denoted by $\gamma'_s(G)$. The signed edge domination number has been studied by several authors [3, 6, 7, 8].

An opinion function on a graph G is a function $f : V(G) \longrightarrow \{-1, 1\}$. By the vote of a vertex v we mean $\sum_{w \in N[v]} f(w)$. A k-*subdominating* function [2] of a graph G is an opinion function for which the votes of at least k vertices are positive. The k-subdominating number of G is the minimum of the values of $\sum_{v \in V(G)} f(v)$, taken over all k-subdominating functions f of G. In the special case [1] when $k = \lceil \frac{|V|}{2} \rceil$, we have the *majority domination number* $\gamma_{maj}(G)$.

Here are some well-known results on $\gamma_{maj}(G)$ and $\gamma'_s(G)$.

Theorem 1. [1] For any connected graph G of order n,

$$\gamma_{maj}(G) \leq \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

The proof of the following theorem is straightforward and therefore omitted.

Theorem 2. For any graph G of order $n \geq 2$ which has no isolates,

$$\gamma'_{sm}(G) = \gamma_{maj}(L(G)).$$

Theorems 1 and 2 lead to:

Corollary 3. For any connected graph G of size $m \geq 1$ which has no isolates,

$$\gamma'_{sm}(G) \leq \begin{cases} 1 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \text{ is even.} \end{cases}$$

Theorem 4. [7] For any positive integer m, define

$$\Psi(m) = \min\{\gamma'_s(G) \mid G \text{ is a graph of size } m\}.$$

Then

$$\Psi(m) = 2\lceil\frac{1}{3}\lceil\frac{\sqrt{24m+25}+6m+5}{6}\rceil\rceil - m.$$

We make use of the following lemma in the next section. The proof of this lemma is straightforward.

Lemma 5. Let Ψ be as in Theorem 4.

1. $m \geq \Psi(m)$ for every positive integer m, and
2. $\Psi(a) + \Psi(b) \geq \Psi(a+b)$ for each pair of positive integers a and b.

2 A lower bound for SEMDN of graphs

For a graph G, let $\omega(G)$ denote the number of components of G and $T(G) = \{u \in V(G) \mid \deg(u) \leq 2\}$. Let f be an SEMDF of G. An edge e is said to be a $+1$ edge if $f(e) = 1$ and it is said to be a -1 edge if $f(e) = -1$. In this section we prove that for any simple graph G of order $n \geq 3$ and size m, $\gamma'_{sm}(G) \geq \Psi(t) - (m-t)$ for some integer $\lceil\frac{m}{2}\rceil \leq t \leq m$. Moreover, we show that this bound is sharp for $t = \lceil\frac{m}{2}\rceil$.

Theorem 6. Let G be a simple graph of order $n \geq 3$ and size m. Then

$$\gamma'_{sm}(G) \geq \Psi(t) - (m-t)$$

for some integer $\lceil\frac{m}{2}\rceil \leq t \leq m$. Furthermore, this bound is sharp when $t = \lceil\frac{m}{2}\rceil$.

Proof. The statement holds for all simple graphs of size $m = 1, 2, 3$. Now assume $m \geq 4$. Let, to the contrary, G be a simple graph of size $m \geq 4$ such that $\gamma'_{sm}(G) < \Psi(t) - (m-t)$ for every integer $\lceil\frac{m}{2}\rceil \leq t \leq m$. Choose such a graph G with as few edges as possible for which $\omega(G) + |T(G)|$ is maximum. Without loss of generality we may assume G has no isolated vertices. Let f be a $\gamma'_{sm}(G)$-function. Define $P = \{e \in E(G) \mid f(e) = 1\}$, $M = \{e \in E(G) \mid f(e) = -1\}$ and $X = \{e \in E(G) \mid f(N[e]) \geq 1\}$. Let $G_1, \ldots, G_{\omega(G)}$ be the connected components of G. If $G_i \simeq K_2$ for each $1 \leq i \leq \omega(G)$, then obviously

$$\gamma'_{sm}(G) \geq \lceil\frac{m}{2}\rceil - \lfloor\frac{m}{2}\rfloor \geq 0 \geq \Psi(\lceil\frac{m}{2}\rceil) - \lfloor\frac{m}{2}\rfloor.$$

Let G have a component H of size at least 2.

Claim 1. $E(H) \cap M \subseteq X$.

Let $e \in E(H) \cap M$. Suppose that, to the contrary, $e \notin X$. Assume G' is obtained from $G - e$ by adding a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 1\}$ by $g(u_0v_0) = -1$ and $g(e) = f(e)$ if $e \in E(G) \setminus \{e\}$. Obviously, g is an SEMDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$. This contradicts the assumptions on G. Thus $e \in X$.

Claim 2. For every non-pendant edge $e = uv \in E(H) \cap M$ we have $\deg(u) = \deg(v) = 2$.

If $f(u) \geq 1$ (the case $f(v) \geq 1$ is similar) and G' is obtained from $G - e$ by adding a pendant edge uv', then obviously $g : E(G') \longrightarrow \{-1, 1\}$, which is defined by $g(uv') = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$, is an SEMDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$. This contradicts the assumptions on G. Hence, $f(u) = f(v) = 0$. Therefore $\deg(u)$ and $\deg(v)$ are even. Let $\deg(u) \geq 4$ (the case $\deg(v) \geq 4$ is similar). Then there is a $+1$ edge $e' = uw$ at u. Assume G' is obtained from $G - \{e, e'\}$ by adding a new vertex z and two new edges vz and wz. Define $g : E(G') \longrightarrow \{-1, 1\}$ by $g(vz) = -1$, $g(wz) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, e'\}$. Obviously, g is an SEMDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Hence, $\deg(u) = \deg(v) = 2$.

Claim 3. Let $e = uv \in E(H) \cap M$ be a non-pendant edge and $uu', vv' \in E(G)$. Then $uu', vv' \in X$.

Let, to the contrary, $uu' \notin X$ (the case $vv' \notin X$ is similar). Since $e \in X$, $f(uu') = f(vv') = 1$. Suppose that $\deg(u') = 1$ and G' is obtained from $G - \{e, uu'\}$ by adding a pendant edge vv_1 and a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 1\}$ by $g(vv_1) = -1$, $g(u_0v_0) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, uu'\}$. Then g is an SEMDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradictions. Therefore $\deg(u') \geq 2$. Similarly, we can see that $\deg(v') \geq 2$.

First let $u' = v'$. Since $uu' \notin X$, we have $vv' \notin X$. Suppose that there exists a -1 pendant edge $u'z$ at u'. By Claim 1, $u'z \in X$, which implies that $f(u') \geq 1$. Let G' be the graph obtained from $G - \{e\}$ by adding a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 1\}$ by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G)\setminus\{e\}$. Obviously, g is an SEMDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G')+|T(G')| > \omega(G)+|T(G)|$, a contradiction. Therefore, there is no -1 pendant edge at $u' = v'$. If there exists a -1 non-pendant edge at u', then an argument similar to that described in Claim 2 shows that $\deg(u') = 2$, a contradiction. Thus every edge at u' is a $+1$ edge. This forces $uu' \in X$, a contradiction.

Now let $u' \neq v'$. Since we have assumed $uu' \notin X$ it follows that $f(u') \leq 1$. If there is a -1 pendant edge $u'w$ at u', then by Claim 1 we have $u'w \in X$ and hence, $f(u') = f[u'w] \geq 1$. If there is a -1 non-pendant edge at u', then $\deg(u') = 2$ by Claim 2 and hence, $f(u') = 0$. It follows that $f(u') = 0, 1$.

When $f(u') = 1$, define G' to be the graph obtained from $G - \{e\}$ by adding a new component u_0v_0. Then $g : E(G') \longrightarrow \{-1, 1\}$ defined by $g(u_0v_0) = -1$ and

$g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$ is an SEMDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore $f(u') = 0$ and hence, there exists a -1 edge $u'u''$ at u'. If $\deg(u'') = 1$, define G' to be the graph obtained from $G - \{u'u''\}$ by adding a new component u_0v_0. Then $g : E(G') \longrightarrow \{-1, 1\}$ defined by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G)\setminus\{u'u''\}$ is an SEMDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Hence, $\deg(u'') = 2$ (see Claim 2). Let G' be obtained from $G - \{e, uu', u'u''\}$ by adding a new component u_0v_0 and two new edges $u''z, zv$. Then $g : E(G') \longrightarrow \{-1, 1\}$ defined by $g(u_0v_0) = -1$, $g(u''z) = -1$, $g(zv) = 1$ and $g(x) = f(x)$ if $x \in E(G)\setminus\{e, uu', u'u''\}$ is an SEMDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore $uu' \in X$, a contradiction.

Claim 4. $E(H) \cap P \subseteq X$.

Let $e = uv \in E(H) \cap P$. If there is a -1 non-pendant edge at u or at v, then by Claim 3 we have $e \in X$. If there exists a -1 pendant edge e' at u, then $e' \in X$ by Claim 1 and hence, $f(u) = f[e'] \geq 1$. If all the edges at u are $+1$ edges, then $f(u) \geq 1$. Similarly, if there is no -1 non-pendant edge at v, we see that $f(v) \geq 1$. Hence, $e \in X$.

Let $G_1, \ldots, G_s$ be the connected components of G for which $E(G_i) \subseteq X$. Thus, $f|_{G_i}$ is a γ'_s-function on G_i for each $1 \leq i \leq s$. Now by Claims 1 and 3, $X \cap [\cup_{i=s+1}^{w(G)} E(G_i)] = \emptyset$. Let $|E(G_i)| = m_i$ for each $1 \leq i \leq w(G)$. Then $|X| = \sum_{i=1}^{s} m_i \geq \lceil \frac{m}{2} \rceil$ and $\sum_{i=s+1}^{w(G)} m_i \leq \lfloor \frac{m}{2} \rfloor$. Then by Lemma 5,

$$\begin{aligned} \gamma'_{sm}(G) &= \textstyle\sum_{i=1}^{s} \gamma'_s(G_i) - \sum_{i=s+1}^{w(G)} m_i \\ &\geq \textstyle\sum_{i=1}^{s} \Psi(m_i) - \sum_{i=s+1}^{w(G)} m_i \\ &\geq \Psi(\textstyle\sum_{i=1}^{s} m_i) - \sum_{i=s+1}^{w(G)} m_i \\ &\geq \Psi(t) - (m - t) \end{aligned}$$

where $t = \sum_{i=1}^{s} m_i \geq \lceil \frac{m}{2} \rceil$.

In order to prove that the lower bound is sharp when $t = \lceil \frac{m}{2} \rceil$, let H_1 be a graph of size $\lceil \frac{m}{2} \rceil$ with $\gamma'_s(H) = \Psi(\lceil \frac{m}{2} \rceil)$ (see [7]) and let H_2 be a graph of size $\lfloor \frac{m}{2} \rfloor$ such that $V(H_1) \cap V(H_2) = \emptyset$. Suppose $G = H_1 \cup H_2$ and f is a $\gamma'_s(H_1)$-function. Then $g : E(G') \longrightarrow \{-1, 1\}$ defined by $g(e) = f(e)$ if $e \in E(H_1)$ and $g(e) = -1$ if $e \in E(H_2)$, is an SEMDF of G with $g(E(G)) = \Psi(\lceil \frac{m}{2} \rceil) - \lfloor \frac{m}{2} \rfloor$. This completes the proof. □

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