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Embeddings of canonical Kirkman packing designs

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Abstract

For a given positive integer v with $v \equiv 4 \pmod 6$ and $v > 4$, let CKPD(v) denote a canonical Kirkman packing design of order v. It is proved in this paper that any CKPD(v) can be embedded in a CKPD(u) if $u \equiv v \equiv 4 \pmod 6$, $v \geq 82$ and $u \geq 3.5v$.

1 Introduction

A packing of order v is a pair $(X, \mathbf{A})$ where X is a v-set and $\mathbf{A}$ is a collection of subsets(called blocks) of X such that each 2-subset of X is contained in at most one block of $\mathbf{A}$. The leave of $(X, \mathbf{A})$ is a graph $(X, \mathbf{E})$ where $\{x, y\} \in \mathbf{E}$ if and only if $\{x, y\}$ is not contained in any block of $\mathbf{A}$. A packing is called resolvable if its block set $\mathbf{A}$ admits a partition into parallel classes, each parallel class being a partition of the v-set X.

Let $v \equiv 3 \pmod 6$; then the maximum possible number of parallel classes in a resolvable packing of a v-set by triples cannot exceed $(v-1)/2$. A resolvable packing of a v-set by triples that achieves this bound is called a Kirkman triple system, and is denoted by $KTS(v)$. It is well known that a $KTS(v)$ exists if and only if $v \equiv 3 \pmod 6$.

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Similarly, if $v \equiv 3 \pmod 6$, then a resolvable packing of a v-set by triples with $v/2-1$ parallel classes is called a nearly Kirkman triple system, denoted by NKTS(v). It is also well known that an NKTS(v) exists if and only if $v \equiv 0 \pmod 6$ and $v \geq 18$, and the leave of an NKTS(v) is a one-factor.

For a given $v \equiv 4 \pmod 6$ and $v > 4$, following [2], we define a canonical Kirkman packing design of order v, denoted by CKPD(v), to be a resolvable packing with $(v-4)/2$ parallel classes such that:

(i) each parallel class consists of a K_4 and $(v-4)/3$ triples;

(ii) the leave consists of the vertex-disjoint union of a K_4 and $(v-4)/2$ edges.

The existence of canonical Kirkman packing designs has been completely determined:

Theorem 1.1 [2,3,10] *Let $v \equiv 4$ (mod 6). Then there exists a CKPD(v) if and only if $v \geq 22$.*

For given positive integers u and v with $u \equiv v \equiv 4 \pmod 6$, let $(X, \mathbf{A})$ be a CKPD(v) and $(Y, \mathbf{B})$ be a CKPD(u). If $X \subset Y$, $\mathbf{A} \subseteq \mathbf{B}$, each parallel class of $(X, \mathbf{A})$ is a part of some parallel class of $\mathbf{B}$, and the leave of $(X, \mathbf{A})$ is a subgraph of the leave of $(Y, \mathbf{B})$, then $(X, \mathbf{A})$ is said to be embedded in $(Y, \mathbf{B})$, or $(X, \mathbf{A})$ is a subsystem of $(Y, \mathbf{B})$. Removing all the blocks of $\mathbf{A}$ from $\mathbf{B}$ gives an incomplete canonical Kirkman packing design. Formally, we give the following definition:

Let $u \equiv v \equiv 4 \pmod 6$. An incomplete canonical Kirkman packing design of order u with a hole of size v, denoted by ICKPD(u, v), is a triple $(X, Y, \mathbf{B})$ where X is a point set of u elements, Y (called hole) is a v-subset of X, and $\mathbf{B}$ is a collection of subsets (blocks) of X such that:

(i) $|B \cap Y| \leq 1$ for each $B \in \mathbf{B}$;

(ii) any pair of distinct elements in X occurs together in Y or in at most one block;

(iii) $\mathbf{B}$ admits a partition into $(u-v)/2$ parallel classes on X, each of which contains one block of size 4 and $(v-4)/3$ triples, and a further $(v-4)/2$ holey parallel classes of triples on $X \backslash Y$;

(iv) each element of $X \backslash Y$ is contained in exactly two blocks of size 4.

The embedding problem for various kinds of resolvable designs has been studied extensively (see, e.g. [1] and [4]) and completely solved for Kirkman triple systems [11, 12] and nearly Kirkman triple systems [5–7].

In this paper, we study the embedding problem for canonical Kirkman packing designs CKPD(v)s with $v \equiv 4 \pmod 6$. It is easy to prove the following necessary condition:

Lemma 1.2 *Let $u \equiv v \equiv 4$ (mod 6). If a CKPD(v) can be embedded in a CKPD(u), then $u \geq 3v + 4$.*

Our main purpose in this paper is to prove that for $v \equiv 4 \pmod 6$ and $v \geq 82$, any CKPD(v) can be embedded in a CKPD(u) if $u \equiv 4 \pmod 6$ and $u \geq 3.5v$.

2 The existence of an ICKPD(u, v) for $v \in \{4, 10\}$

In this section, we will mainly use Kirkman frames to construct ICKPD(u, v)s for $v \in \{4, 10\}$. First, since a CKPD(u) is equivalent to an ICKPD$(u, 4)$, we have the following theorem:

Theorem 2.1 *Let $u \equiv 4$ (mod 6). Then there exists an ICKPD$(u, 4)$ if and only if $u \geq 22$.*

In order to deal with the case $v = 10$, we need some definitions.

A group divisible design (GDD) is a triple $(X, \mathbf{G}, \mathbf{B})$ where X is a set of points, $\mathbf{G}$ is a partition of X into groups, and $\mathbf{B}$ is a collection of subsets (blocks) of X so that any pair of distinct points occurs together in either one group or exactly one block, but not both. A K-GDD $(X, \mathbf{G}, \mathbf{B})$ of type ${g_1}^{t_1}{g_2}^{t_2}\cdots{g_s}^{t_s}$ has t_i groups of size g_i, $i = 1, 2, \cdots, s$, and $|B| \in K$ for every $B \in \mathbf{B}$. It is also sometimes called a K-GDD of type T where $T = \{|G| \mid G \in \mathbf{G}\}$. If $K = \{k\}$, and $|G| = m$ for every $G \in \mathbf{G}$, then this K-GDD is called uniform and denoted by GDD$(k, m; v)$. A GDD$(k, m; v)$ with $v = km$ is called a transversal design and denoted by TD(k, m). It is well known that the existence of a TD(k, m) is equivalent to the existence of $k - 2$ mutually orthogonal latin squares of order m.

A GDD$(X, \mathbf{G}, \mathbf{B})$ is called frame resolvable if its block set $\mathbf{B}$ admits a partition into holey parallel classes, each holey parallel class being a partition of $X - G$ for some $G \in \mathbf{G}$. A kirkman frame is a frame resolvable GDD in which all the blocks have size three. The groups in a Kirkman frame are often referred as holes. Kirkman frames were formally introduced by Stinson [12], who established their spectrum in the case where all the holes have the same size.

Theorem 2.2 [13] *There exists a Kirkman frame of type t^u if and only if $t \equiv 0$ (mod 2), $u \geq 4$ and $t(u - 1) \equiv 0$ (mod 3).*

The following theorem gives a powerful construction for Kirkman frames from group divisible designs.

Theorem 2.3 [12] *Let $(X, \mathbf{G}, \mathbf{B})$ be a group divisible design. Let $w : X \to Z^+ \cup \{0\}$ be a weight function on X. Suppose that for each block $B \in \mathbf{B}$, there exists a Kirkman frame of type $\{w(x) : x \in B\}$. Then there is a Kirkman frame of type $\{\sum_{x \in G} w(x) : G \in \mathbf{G}\}$.*

The following "filling in holes" construction is analogous to [13, Theorem 1]. It provides a powerful tool for the existence of incomplete canonical Kirkman packing designs:

Theorem 2.4 *Suppose there is a Kirkman frame of type T on v points. If, for some a, there exists an ICKPD$(t + a, a)$ for all $t \in T$, then there is an ICKPD$(v + a, a)$ and an ICKPD$(v + a, t + a)$ for every $t \in T$.*

Lemma 2.5 [3, 10] *There exists an ICKPD*$(u, 10)$ *for each* $u \in \{34, 40\}$

Lemma 2.6 *There exists an ICKPD*$(u, 10)$ *for each* $u \in \{46, 52, 58, 64, 70, 76, 88, 94, 100, 112, 118\}$.

Proof: For each $u \in \{46, 52, 58, 64, 70, 76, 88, 94, 100, 112, 118\}$, we will construct an ICKPD$(u, 10)(X, Y, \mathbf{B})$ directly, where $X \backslash Y = Z_{(u-10)/2} \times \{1, 2\}$; $Y = \{\infty_1, \infty_2, \ldots, \infty_{10}\}$ if $u \in \{46, 64, 70, 88, 94, 112, 118\}$, otherwise, $Y = \{a_0, a_1, a_2\} \cup \{\infty_1, \infty_2, \ldots, \infty_7\}$; three holey parallel classes are obtained by developing $0_1 4_1 5_1$ and $0_2 4_2 5_2$ mod$(u/2, -)$; $(u-10)/2$ parallel classes are obtained by developing a base parallel class mod$(u/2, -)$; subscripts on a are evaluated mod 3; and the leave on $X \backslash Y$ (a one-factor) is generated by the pure differences $u/4$ on orbits 1 and 2 if $u/2 \equiv 0 \pmod 2$.

(1) ICKPD$(46, 10)$

A base parallel class:

$0_1 2_1 3_2 9_2$, $1_1 7_1 5_2$, $3_1 0_2 2_2$, $5_1 8_1 15_1$, $12_2 15_2 4_2$, $6_1 6_2 \infty_1$, $9_1 11_2 \infty_2$, $12_1 17_2 \infty_3$, $10_1 16_2 \infty_4$, $11_1 1_2 \infty_5$, $4_1 14_2 \infty_6$, $14_1 7_2 \infty_7$, $16_1 10_2 \infty_8$, $13_1 8_2 \infty_9$, $17_1 13_2 \infty_{10}$

(2) ICKPD$(52, 10)$

Leave: $0_1 6_2$ mod $(21, -)$

A base parallel class:

$0_1 6_1 0_2 9_2$, $2_1 9_1 1_2$, $1_1 10_1 20_2$, $12_1 13_2 19_2$, $13_1 3_2 17_2$, $4_1 14_1 17_1$, $5_2 15_2 18_2$, $3_1 5_1 a_0$, $8_1 10_2 a_1$, $4_2 6_2 a_2$, $15_1 2_2 \infty_1$, $19_1 14_2 \infty_2$, $7_1 12_2 \infty_3$, $16_1 7_2 \infty_4$, $20_1 16_2 \infty_5$, $18_1 11_2 \infty_6$, $11_1 8_2 \infty_7$

(3) ICKPD$(58, 10)$

A base parallel class:

$0_1 3_1 0_2 6_2$, $5_1 12_1 13_2$, $1_1 10_1 19_2$, $6_1 17_1 4_2$, $8_1 15_2 22_2$, $13_1 17_2 21_2$, $20_1 1_2 11_2$, $9_1 15_1 23_1$, $12_2 20_2 23_2$, $2_1 4_1 a_0$, $7_1 9_2 a_1$, $3_2 5_2 a_2$, $11_1 21_2 \infty_1$, $19_1 18_2 \infty_2$, $22_1 10_2 \infty_3$, $18_1 14_2 \infty_4$, $16_1 8_2 \infty_5$, $21_1 16_2 \infty_6$, $14_1 7_2 \infty_7$

(4) ICKPD$(64, 10)$

Leave: $0_1 24_2$ mod $(24, -)$

A base parallel class:

$0_1 3_1 4_2 10_2$, $2_1 4_1 23_2$, $1_1 10_1 0_2$, $5_1 18_1 3_2$, $6_1 8_2 15_2$, $7_1 13_2 22_2$, $11_1 19_2 2_2$, $20_1 26_1 9_1$, $8_1 15_1 23_1$, $12_2 14_2 26_2$, $6_2 9_2 17_2$, $21_1 21_2 \infty_1$, $22_1 18_2 \infty_2$, $13_1 16_2 \infty_3$, $25_1 20_2 \infty_4$, $19_1 24_2 \infty_5$, $14_1 7_2 \infty_6$, $17_1 1_2 \infty_7$, $16_1 5_2 \infty_8$, $12_1 25_2 \infty_9$, $24_1 11_2 \infty_{10}$

(5) ICKPD$(70, 10)$

A base parallel class:

$0_1 2_1 3_2 9_2$, $1_1 4_1 0_2$, $3_1 12_1 1_2$, $5_1 15_1 21_2$, $6_1 19_1 27_2$, $7_1 7_2 20_2$, $8_1 12_2 19_2$, $9_1 14_2 23_2$, $13_1 15_2 25_2$, $14_1 22_1 28_1$, $11_1 18_1 29_1$, $24_2 26_2 8_2$, $29_2 2_2 10_2$, $24_1 4_2 \infty_1$, $21_1 18_2 \infty_2$, $20_1 5_2 \infty_3$, $27_1 22_2 \infty_4$, $26_1 13_2 \infty_5$, $17_1 11_2 \infty_6$, $10_1 28_2 \infty_7$, $23_1 16_2 \infty_8$, $16_1 6_2 \infty_9$, $25_1 17_2 \infty_{10}$

(6) ICKPD$(76, 10)$

Leave: $0_1 28_2$ mod $(33, -)$

A base parallel class:

$0_1 3_1 7_2 13_2$, $1_1 8_1 0_2$, $6_1 15_1 4_2$, $5_1 16_1 2_2$, $12_1 26_1 8_2$, $9_1 24_1 25_2$, $21_1 24_2 27_2$, $14_1 22_2 1_2$, $31_1 10_2 21_2$, $11_1 16_2 29_2$, $20_1 31_2 14_2$, $13_1 23_1 29_1$, $10_1 22_1 30_1$, $11_2 19_2 26_2$, $20_2 30_2 6_2$, $17_1 17_2 \infty_1$, $25_1 18_2 \infty_2$, $19_1 28_2 \infty_3$, $32_1 23_2 \infty_4$, $18_1 32_2 \infty_5$, $27_1 15_2 \infty_6$, $28_1 12_2 \infty_7$, $2_1 4_1 a_0$, $7_1 9_2 a_1$, $3_2 5_2 a_2$

(7) ICKPD$(88, 10)$

A base parallel class:

$0_1 2_1 0_2 6_2$, $3_1 15_1 2_2$, $1_1 14_1 37_2$, $5_1 19_1 1_2$, $6_1 21_1 31_2$, $4_1 20_1 36_2$, $8_1 26_1 38_2$, $7_1 8_2 22_2$, $9_1 11_2 23_2$, $11_1 14_2 24_2$, $10_1 21_2 32_2$, $13_1 20_2 33_2$, $17_1 26_2 34_2$, $24_1 27_1 33_1$, $18_1 28_1 35_1$, $12_1 23_1 31_1$, $35_2 13_2 15_2$, $28_2 4_2 7_2$, $9_2 16_2 25_2$, $22_1 27_2 \infty_1$, $36_1 30_2 \infty_2$, $34_1 3_2 \infty_3$, $37_1 29_2 \infty_4$, $38_1 17_2 \infty_5$, $29_1 19_2 \infty_6$, $32_1 12_2 \infty_7$, $16_1 5_2 \infty_8$, $25_1 10_2 \infty_9$, $30_1 18_2 \infty_{10}$

(8) ICKPD$(94, 10)$

A base parallel class:

$0_1 2_1 0_2 7_2$, $1_1 9_1 10_2$, $3_1 13_1 2_2$, $4_1 15_1 1_2$, $5_1 17_1 40_2$, $10_1 24_1 4_2$, $6_1 21_1 25_2$, $7_1 25_1 27_2$, $8_1 18_2 33_2$, $12_1 26_2 38_2$, $11_1 14_2 22_2$, $14_1 20_2 29_2$, $16_1 24_2 34_2$, $19_1 35_2 6_2$, $18_1 31_1 3_2$, $20_1 37_1 40_1$, $22_1 28_1 41_1$, $23_1 32_1 39_1$, $17_2 19_2 37_2$, $39_2 13_2 16_2$, $36_2 5_2 11_2$, $29_1 41_2 \infty_1$, $36_1 32_2 \infty_2$, $34_1 9_2 \infty_3$, $35_1 30_2 \infty_4$, $33_1 12_2 \infty_5$, $31_1 23_2 \infty_6$, $26_1 8_2 \infty_7$, $30_1 21_2 \infty_8$, $27_1 15_2 \infty_9$, $38_1 28_2 \infty_{10}$

(9) ICKPD$(100, 10)$

Leave: $0_1 30_2$ mod $(45, -)$

A base parallel class:

$0_1 3_1 3_2 9_2$, $2_1 13_1 1_2$, $43_1 18_1 23_2$, $7_1 19_1 5_2$, $42_1 15_1 16_2$, $10_1 23_1 7_2$, $41_1 17_1 24_2$, $6_1 20_1 44_2$, $40_1 12_1 35_2$, $5_1 15_2 18_2$, $33_1 37_2 4_2$, $9_1 17_2 31_2$, $27_1 38_2 8_2$, $22_1 40_2 11_2$, $25_1 39_2 12_2$, $39_1 14_2 33_2$, $28_1 43_2 20_2$, $8_1 14_1 30_1$, $21_2 29_1 36_1$, $16_1 26_1 35_1$, $25_2 32_2 42_2$, $21_2 30_2 41_2$, $13_2 26_2 34_2$, $44_1 1_1 a_0$, $4_1 6_2 a_1$, $0_2 2_2 a_2$, $24_1 36_2 \infty_1$, $38_1 29_2 \infty_2$, $11_1 28_2 \infty_3$, $32_1 22_2 \infty_4$, $34_1 10_2 \infty_5$, $31_1 27_2 \infty_6$, $37_1 19_2 \infty_7$

(10) ICKPD$(112, 10)$

A base parallel class:

$0_1 2_1 0_2 6_2$, $50_1 6_1 49_2$, $4_1 12_1 1_2$, $49_1 11_1 45_2$, $1_1 15_1 36_2$, $46_1 10_1 40_2$, $5_1 22_1 42_2$, $48_1 17_1 39_2$, $7_1 30_1 46_2$, $47_1 20_1 34_2$, $45_1 4_2 12_2$, $3_1 5_2 14_2$, $40_1 41_2 2_2$, $8_1 11_2 25_2$, $36_1 44_2 8_2$, $9_1 18_2 35_2$, $41_1 48_2 15_2$, $23_1 28_2 50_2$, $25_1 37_2 10_2$, $21_1 24_1 43_1$, $19_1 31_1 37_1$, $13_1 29_1 38_1$, $18_1 28_1 39_1$, $47_2 19_2 21_2$, $13_2 29_2 32_2$, $23_2 30_2 43_2$, $17_2 27_2 38_2$, $16_1 31_2 \infty_1$, $33_1 26_2 \infty_2$, $14_1 33_2 \infty_3$, $32_1 22_2 \infty_4$, $34_1 7_2 \infty_5$, $27_1 9_2 \infty_6$, $26_1 3_2 \infty_7$, $35_1 16_2 \infty_8$, $42_1 20_2 \infty_9$, $44_1 24_2 \infty_{10}$

(11) ICKPD$(118, 10)$

A base parallel class:

$0_1 2_1 0_2 6_2$, $53_1 7_1 52_2$, $4_1 13_1 1_2$, $52_1 9_1 48_2$, $8_1 20_1 3_2$, $51_1 10_1 45_2$, $1_1 17_1 31_2$, $50_1 14_1 15_2$,

$3_1 22_1 32_2$, $48_1 15_1 30_2$, $6_1 28_1 46_2$, $47_1 50_2 4_2$, $5_1 10_2 21_2$, $44_1 53_2 11_2$, $16_1 23_2 36_2$, $45_1 47_1 8_2$, $18_1 26_2 42_2$, $30_1 43_2 7_2$, $46_1 17_2 39_2$, $25_1 37_2 12_2$, $33_1 2_2 25_2$, $11_1 34_1 37_1$, $24_1 39_1 49_1$, $21_2 27_1 41_1$, $19_1 26_1 43_1$, $33_2 5_2 35_2$, $24_2 41_2 44_2$, $13_2 27_2 34_2$, $19_2 29_2 38_2$, $29_1 51_2 \infty_1$, $38_1 28_2 \infty_2$, $23_1 49_2 \infty_3$, $31_1 20_2 \infty_4$, $36_1 9_2 \infty_5$, $32_1 16_2 \infty_6$, $12_1 40_2 \infty_7$, $42_1 22_2 \infty_8$, $40_1 18_2 \infty_9$, $35_1 14_2 \infty_{10}$

Lemma 2.7 [5, 8, 9] *(i) There exist* $\{4\}$*-GDDs of types* $6^5 9^1$, $6^6 9^1$*;*

(ii) There exists a $\{4\}$*-GDD of type* $g^4 m^1$ *with* $m > 0$ *if and only if* $g \equiv m \equiv 0$ *(mod 3) and* $0 < m \leq 3g/2$.

Theorem 2.8 *Let* $u \equiv 4$ *(mod 6). Then there exists an ICKPD*$(u, 10)$ *if and only if* $u \geq 34$.

Proof: Let $u = 6s + 10$. By [10], there exists a GDD on s points with block sizes from the set $\{k \in Z : k \geq 4\}$ and group sizes from the set $\{4, 5\}$ if $s \geq 32, s \neq 36, 37, 38, 39, 46, 47$. Apply weight 6 to the GDD. Using Theorems 2.2 and 2.3, we create a Kirkman frame with hole sizes in the set $\{24, 30\}$. Adjoin 10 ideal points and apply Theorem 2.4, filling in ICKPD$(6m + 10, 10)$s where $m \in \{24, 30\}$; we then obtain an ICKPD$(u, 10)$.

If $u \equiv 10$ (mod 24) and $u \geq 106$, then we take a Kirkman frame of type $24^{(u-10)/24}$ and adjoin 10 ideal points, filling in an ICKPD$(34, 10)$ on each hole together with the 10 ideal points. This yields an ICKPD$(u, 10)$ where the hole of size 10 is formed on the ideal points.

By Lemmas 2.5 and 2.6, we have to construct an ICKPD$(u, 10)$ for $u \in \{82, 124, 136, 142, 148, 160, 166, 170, 184, 190, 196, 232, 238, 144, 286, 292\}$. The process is as follows.

$u = 82$: Take a $\{4\}$-GDD of type $9^4 3^1$, give all the points weight 2 to obtain a Kirkman frame of type $18^4 6^1$, adjoin 4 ideal points and fill in ICKPD$(22, 4)$s.

$u = 17, 196$: Take a $\{4\}$-GDD of type $15^4 21^1$ or $18^4 21^1$, give all the points weight 2 to obtain a Kirkman frame of type $30^4 42^1$ or $36^4 42^1$, adjoin 10 ideal points, fill in ICKPD$(40, 10)$s, an ICKPD$(46, 10)$ and an ICKPD$(52, 10)$ to obtain an ICKPD$(u, 10)$. Similarly, for $u \in \{124, 142, 166, 190\}$, take a $TD(4, 5)$ or a $\{4\}$-GDD of type $6^s 9^1$ for $s \in \{4, 5, 6\}$, give all the points of the TD or the GDD weight 6 or 4.

$u = 136$: Take a TD$(5, 5)$ with one parallel class, delete 4 points from the TD to obtain a $\{4\}$-GDD of type $4^4 5^1$, give all the points of the $\{4\}$-GDD weight 6, adjoin 10 ideal points and apply Theorem 2.4.

$u = 292, 286$: Delete one or two points from a group of a TD$(6, 8)$ to obtain a $\{5, 6\}$-GDD of type $8^5 7^1$ or a $\{5, 6\}$-GDD of type $8^5 6^1$, give all points of the GDD weight 6, adjoin 10 ideal points and apply Theorem 2.4. Take a TD$(6, 5)$ or TD$(5, 8)$; we can deal with the cases $u = 232, 238, 244$ in a similar way.

$u = 148, 160$: Take a $\{4, 5\}$-GDD of type 5^5 and give all the points of the GDD weight 6, apply Theorem 2.4, adjoin 10 ideal points, and we then obtain an ICKPD$(160, 10)$. Take a TD$(6, 5)$, delete four points from one group and two points from another group to obtain a $\{4,5,6\}$-GDD of type $5^4 3^1 1^1$, give all the points of the GDD weight 6 to obtain a Kirkman frame of type $30^4 18^1 6^1$, adjoin 4 ideal points and apply Theorem 2.4. The proof is complete. ∎

3 Main results

For the existence of Kirkman frames, we require the following results:

Lemma 3.1 *[5, 6, 7]*

(i) There exist Kirkman frames of types $6^5 12^1$, $6^4 12^2$ *and* $12^{12} 18^1 24^1$.

(ii) If there exists a $TD(6,m)$, *then there exists a Kirkman frame of type* $(6m)^4(12m-6s)^1(6w)^1$ *for* $0 \le s \le m$ *and* $m \le w \le 2m$.

(iii) If $v \in \{126, 174, 222, 270\}$, $m = (v-18)/12+2$ *and* $0 \le w \le m$, *then there exists a Kirkman frame of type* $(6m)^4(12m-12)^1(12m-6w)^1$.

(iv) If $m \in \{14, 18, 22\}$ *and* $m \le w \le 2m$, *then there exists a Kirkman frame of type* $(6m)^4(12m)^1(6w)^1$.

Lemma 3.2 *If there exists a* $TD(6,m)$, *then for* $m \le w \le 2m$ *and* $t = 4$ *or 10 there exists an ICKPD*$(36m+6w+t, 12m+t)$.

Proof: Take the Kirkman frame in Lemma 3.1(2)(setting $s = 0$) and adjoin t ideal points. ∎

Let $T_6 = \{n | n \ge 5, n \in Z\} \backslash \{6, 10, 14, 18, 22\}$; then for each $n \in T_6$, there exists a $TD(6,n)$. By Lemmas 3.1 and 3.2, we have the following lemma:

Lemma 3.3 *(1) If* $v \equiv 4$ *(mod* 12*),* $v \ge 64$, $v \ne 76, 124, 172, 220, 268$, $u \equiv 4$ *(mod* 6*) and* $3.5v - 10 \le u \le 4v - 12$, *then there exists an ICKPD*(u,v).

(2) If $v \equiv 10$ *(mod* 12*),* $v \ge 70$, $v \ne 82, 130, 178, 226, 274$, $u \equiv 0$ *(mod* 6*) and* $3.5v - 25 \le u \le 4v - 30$, *then there exists an ICKPD*(u,v)

Lemma 3.4 *There exists an ICKPD*$(u, 16)$ *for* $u \in \{70, 118, 124\}$.

Proof: (i) $u = 70$, see [3].

(ii)$u = 118$

We present an ICKPD$(118, 16)$ as follows.

Point set: $Z_{51} \times \{1,2\} \cup \{a_0, a_1, a_2\} \cup \{\infty_1, \infty_2, \cdots, \infty_{13}\}$

Leave: $0_1 47_2$ mod $(51, -)$

Parallel classes: develop the following base parallel class mod $(51, -)$, where subscripts on a are evaluated mod 3.

$0_1 11_1 0_2 25_2$, $49_1 10_1 48_2$, $3_1 16_1 1_2$, $50_1 13_1 47_2$, $9_1 25_1 4_2$, $48_1 15_1 41_2$, $2_1 21_1 31_2$, $45_1 14_1 26_2$,

$5_1 26_1 33_2$, $43_1 44_2 5_2$, $7_1 10_2 23_2$, $46_1 50_2 13_2$, $12_1 17_2 32_2$, $40_1 46_2 12_2$, $18_1 27_2 45_2$, $32_1 3_2 22_2$, $35_1 43_2 15_2$, $6_1 28_1 31_1$, $20_1 29_1 44_1$, $19_1 36_1 42_1$, $18_2 29_2 38_2$, $16_2 37_2 40_2$, $49_2 14_2 20_2$, $1_1 8_1 a_0$, $4_1 6_2 a_1$, $2_2 9_2 a_2$, $23_1 34_2 \infty_1$, $30_1 24_2 \infty_2$, $27_1 42_2 \infty_3$, $38_1 30_2 \infty_4$, $22_1 39_2 \infty_5$, $37_1 28_2 \infty_6$, $17_1 36_2 \infty_7$, $47_1 35_2 \infty_8$, $41_1 11_2 \infty_9$, $33_1 19_2 \infty_{10}$, $34_1 7_2 \infty_{11}$, $24_1 8_2 \infty_{12}$, $39_1 21_2 \infty_{13}$

Holey parallel classes:

Develop the triples $0_1 4_1 5_1$, $0_2 4_2 5_2$, $0_1 2_1 10_1$ and $0_2 2_2 10_2$ mod $(51, -)$.

(iii)$u = 124$

We present an ICKPD$(124, 16)$ as follows.

Point set: $Z_{54} \times \{1, 2\} \cup \{a_0, a_1, a_2\} \cup \{\infty_1, \infty_2, \cdots, \infty_{13}\}$

Leave: $0_1 27_1, 0_2 27_2$ mod $(54, -)$

Parallel classes: develop the following base parallel class mod $(51, -)$, where subscripts on a are evaluated mod 3.

$0_1 20_1 0_2 26_2$, $53_1 11_1 51_2$, $2_1 13_1 1_2$, $51_1 10_1 48_2$, $3_1 17_1 53_2$, $52_1 14_1 47_2$, $9_1 26_1 3_2$, $48_1 12_1 41_2$, $6_1 27_1 52_2$, $47_1 15_1 38_2$, $49_1 50_2 7_2$, $5_1 8_2 20_2$, $29_1 33_2 46_2$, $22_1 27_2 42_2$, $21_1 29_2 45_2$, $7_1 17_2 34_2$, $37_1 39_2 4_2$, $28_1 44_2 11_2$, $45_1 10_2 30_2$, $16_1 39_1 42_1$, $19_1 34_1 43_1$, $25_1 31_1 50_1$, $14_2 28_2 37_2$, $18_2 21_2 43_2$, $25_2 31_2 49_2$, $33_1 40_2 \infty_1$, $32_1 22_2 \infty_2$, $24_1 35_2 \infty_3$, $35_1 24_2 \infty_4$, $23_1 36_2 \infty_5$, $36_1 23_2 \infty_6$, $18_1 32_2 \infty_7$, $38_1 19_2 \infty_8$, $41_1 5_2 \infty_9$, $37_1 15_2 \infty_{10}$, $46_1 13_2 \infty_{11}$, $40_1 16_2 \infty_{12}$, $44_1 12_2 \infty_{13}$, $1_1 8_1 a_0$, $4_1 6_2 a_1$, $2_2 9_2 a_2$

Holey parallel classes:

Develop the triples $0_1 4_1 5_1$, $0_2 4_2 5_2$, $0_1 2_1 10_1$ and $0_2 2_2 10_2$ mod $(54, -)$.

Lemma 3.5 *If* $v \in \{124, 172, 220, 268\}$, $u \equiv 4 \pmod 6$ *and* $3.5v \le u \le 4v + 24$, *then there exists an ICKPD*(u, v).

Proof: Let $v = 12m - 8$, Take a Kirkman frame of Lemma 3.1(3) and adjoin 4 ideal points. This gives an ICKPD(u, v) for $3.5v + 20 \le u \le 4v + 24$.

For $v \in \{172, 220, 268\}$, $3.5v \le u < 3.5v + 20$ and $u \equiv 4 (\text{mod } 6)$, take a Kirkman frame in Lemma 3.1(iv) and adjoin 4 ideal points. For $v = 124$ and $u \in \{436, 442\}$, take a $TD(6, 9)$ and give all the points of in four the groups weight 6, give all the points of in the fifth group weight 12 and give either 7 or 8 points in the sixth group weight 12 and all the remaining points weight 6 and adjoin 16 points(Note that an ICKPD$(112, 16)$ can be obtained by giving all the points in a $\{4\} - GDD$ of type $12^4 6^1$ weight 2 and adjoin 4 ideal points.) For $(u, v) = (448, 124)$, take a Kirkman frame of type 108^4 and adjoin 16 ideal points. The proof is complete. ■

Lemma 3.6 *If* $v \in \{82, 130, 178, 226, 274\}$, $u \equiv 4$ *(mod 6) and* $3.5v \le u \le 4v + 6$, *then there exists an ICKPD*(u, v).

Proof: Let $m = (v - 10)/12 + 1$, then $m \in \{7, 11, 15, 19, 23\}$. Take a $TD(6, m)$ and give all the points in four of the groups weight 6, give just one point in the fifth group weight 6 and all the remaining points weight 12, assign weight 6 or 12 to each point in the sixth group, and adjoin 6 ideal points, this gives an ICKPD(u, v), where $v = 12m - 2$ and $42m - 4 \leq u \leq 48m - 4$, i.e. $3.5v \leq u \leq 4v + 6$. The proof is complete. ■

Lemma 3.7 *If $v \equiv 10$ (mod 12), $v \geq 70$ and $u = 4v - 24$, $4v - 18$ or $4v - 12$, then there exists an ICKPD(u, v).*

Proof: If $v \notin \{118, 166, 214, 262\}$, let $m = (v + 2)/12$. Take a $TD(6, m)$ and give all the points in four of the groups weight 6, give just one point in the fifth group weight 6 and all the remaining points weight 12, assign either 5 or 4 or 3 points weight 6 in the sixth group and all the remaining points weight 12, adjoin 4 ideal points. For $v \in \{118, 166, 214, 262\}$, write $m = (v - 22)/12 + 3$, then $m \in \{11, 15, 19, 23\}$. Take a $TD(6, m)$, give all the points in four of the groups weight 6, give three points in the fifth group weight 6 and all the remaining points weight 12, either 9 or 10 or 11 points in the sixth group weight 6 and all the remaining points weight 12, adjoin 4 ideal points. The proof is complete. ■

Lemma 3.8 *(1)Suppose there exists a $TD(k, t)$, $k \geq 4$. Then there exists an ICKPD$(6s + 4, 6t + 4)$ if $4t \leq s \leq kt$ and $s \neq 4t + 1, 4t + 2$.*

(2) Suppose there exists a $TD(k, n)$, $k \geq 5$ and $3 \leq t \leq n$. Then there exists an ICKPD$(6s + 4, 6t + 4)$ if $4n + t \leq s \leq (k - 1)n + t$ and $s \neq 4n + t + 1, 4n + t + 2$.

Proof: We only prove (1), (2) can be dealt with in a similar way. We can write $s - 4t = m(5) + m(6) + \cdots + m(k)$ such that $m(i) = 0$ or $23 \leq m(i) \leq t$ for each $5 \leq i \leq k$, delete $t - m(i)$ points in the i-th group of the TD for each $5 \leq i \leq k$, then we obtain a $GDD(K, M; s)$ such that $m \geq 3$ for each $k \in K$, give all the points of the GDD weight 6 and adjoin 4 ideal points. ■

The following lemma can be easily checked:

Lemma 3.9 *Let $n \in T_6$ and $n \geq 7$. There exists a positive integer n_1 such that $n_1 > n$, $n_1 \in T_6$ and $4n_1 < 5n - 1$.*

By Lemmas 3.8 and 3.9, we have the following:

Lemma 3.10 *(i)Let $t \geq 3$, $n \in T_6$ and $n \geq max\{7, t\}$. Then there exists an ICKPD$(6s + 4, 6t + 4)$ if $s \geq 4n + t$ and $s \neq 4n + t + 1, 4n + t + 2$.*

(ii) Let $s \geq 7$ and $t \in T_6$. Then there exists an ICKPD$(6s + 4, 6t + 4)$ if $s \geq 4t$ and $s \neq 4t + 1, 4t + 2$.

Lemma 3.11 *Let $r = 1, 2, 3, 5, 6$ and $t \geq 6$. Then there exists an ICKPD$(6(5t + r) + 4, 6t + 4)$ if there exists a $TD(6, t + 1)$.*

Proof: For $r = 1, 2$ or 3, delete $4 - r$ points in one group of a $TD(5, t + 1)$, delete one point x in another group of the TD, take all the blocks and the group containing x as new groups to obtain a $GDD(\{4, 5, t + r - 3, t + 1\}, \{3, 4, t\}; 5t + r)$. Give all the points of the GDD weight 6 and adjoin 4 ideal points. For $r = 5, 6$, delete all the points of some block B of a $TD(6, t + 1)$, delete $t - r$ points in a group G of the TD, take all the blocks containing x for some $x \in B\backslash G$ and G' as new groups, where $G' = G_x\backslash\{x\}$ and G_x is the group of the TD containing x, to obtain a $GD(\{4, 5, 6, t, r\}, \{4, 5, t\}; 5t + r)$, Give all the points of the GDD weight 6 and adjoin 4 ideal points. The proof is complete. ■

Lemma 3.12 *For $s = 4t+1$ or $4t+2$ and $t \geq 5$, there exists an ICKPD$(6s+4, 6t+4)$.*

Proof: Take a $\{4\}-GDD$ of type $(3t-3)^4 12^1$ or a $\{4\}-GDD$ of type $(3t-3)^4 15^1$, give all the points of the GDD weight 2, we obtain a Kirkman frame of type of $(6t - 6)^4 24^1$ or a Kirkman frame of type of $(6t - 6)^4 30^1$, adjoin 10 ideal points, fill in ICKPD$(6t + 4, 10)$s and an ICKPD$(34, 10)$ or an ICKPD$(40, 10)$. The proof is complete. ■

Lemma 3.13 *For $t = 10, 14, 18, 22$, $4t \leq s \leq 5t$ and $s \neq 4t + 1, 4t + 2$, there is an ICKPD$(6s + 4, 6t + 4)$.*

Proof: Take a $TD(5, t/2)$, give all the points in four groups of the TD weight 12, give points in the fifth group weight 0, 6 or 12, obtain a Kirkman frame of type $(6t)^4(6(s - 4t))^1$, adjoin 4 ideal points. The proof is complete. ■

Lemma 3.14 *If $t \geq 7$ and $s \geq 4t$, then there is an ICKPD$(6s + 4, 6t + 4)$.*

Proof: If $t \geq 7$ and $t \in T_6$, apply Lemma 3.10(ii) and Lemma 3.12. If $t \geq 7$ and $t \notin T_6$, then the procedure is as follows: Take $n = t+1$ in Lemma 3.10(i), this covers $s \geq 5t + 4$ and $s \neq 5t + 5, 5t + 6$. For $s \in \{5t + 1, 5t + 2, 5t + 3, 5t + 5, 5t + 6\}$, apply Lemma 3.11. For $4t \leq s \leq 5t$, apply Lemmas 3.12-3.13. The proof is complete. ■

Lemma 3.15 *If $u \equiv v \equiv 4$ (mod 6), $v \geq 82$ and $u \geq 3.5v$, then there is an ICKPD(u, v).*

Proof: Write $v = 6t + 4$ and $u = 6s + 4$, then $t \geq 13$. If $3.5v \geq u \geq 4v - 12$, the conclusion then follows from Lemmas 3.3, 3.5-3.7. If $u \geq 4v - 18$, then $s \geq 4t$ and the conclusion follows from Lemma 3.13. The proof is complete. ■

Since a $CKPD(v)$ exists if $v \equiv 4 (\text{mod } 6)$ and $v \neq 10, 16$, the following theorem can be easily derived from Theorem 3.15.

Theorem 3.16 *Let $v \equiv 4$ (mod 6) and $v \geq 82$, then any $CKPD(v)$ can be embedded in a $CKPD(u)$ if $u \equiv 4$(mod 6) and $u \geq 3.5v$.*

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