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Perfect k-domination in graphs

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Abstract

Let k be a positive integer. A vertex subset D of a graph $G = (V, E)$ is a perfect k-dominating set of G if every vertex v of G, not in D, is adjacent to exactly k vertices of D. The minimum cardinality of a perfect k-dominating set of G is the perfect k-domination number $\gamma_{kp}(G)$. In this paper, we generalize perfect domination to perfect k-domination, where many bounds of $\gamma_{kp}(G)$ are obtained. We prove that the perfect k-domination problem is NP-complete for general graphs.

1 Introduction

All graphs considered here are finite, undirected with no loops and multiple edges. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph

G, respectively. In general, we use $G[X]$ to denote the subgraph induced by the set of vertices X, and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. Let $\deg(v)$ be the degree of vertex v and as usual $\delta(G) = \delta$, the minimum degree of G, and $\Delta(G) = \Delta$, the maximum degree of G. For graph-theoretical terminology and notation not defined here we follow [10].

In [8] Fink and Jacobson generalized the concept of dominating sets. Throughout the paper, k will be a positive integer. A subset D of vertices in a graph $G = (V, E)$ is a *k-dominating set* if every vertex of $V - D$ is adjacent to at least k vertices in D. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. Hence for $k = 1$, 1-dominating sets are the classical dominating sets. For a review of the topic of domination and its related parameters, see [1], [11] and [12].

In this paper we are interested in a generalization of a variation of domination called perfect domination. A vertex subset D of a graph G is said to be a perfect dominating set of G if any vertex of G not in D is adjacent to exactly one vertex of D. Perfect domination was introduced by Cockayne et al. in [5]. We note that sets that are both perfect dominating and independent are called perfect codes by Biggs in [3] or efficient dominating sets by Bange, Barkauskas and Slater in [2]. In [2] the authors gave a linear-time algorithm for the problem of efficient domination on trees. For more details on perfect domination and perfect codes, see [6], [13] and [14]. We also note that perfect domination has been studied by Fellow and Hoover in [7] with a different terminology which they called semiperfect domination. Fellow and Hoover [7] proved that determining whether a graph G has a semiperfect or efficient dominating set is an NP-Complete problem. In [4] Chellali, Khelladi and Maffray studied another variation, namely exact double dominating sets, where every vertex of the graph is dominated exactly twice. They showed that the complexity of the problem of deciding whether a graph admits an exact double dominating set is NP-Complete.

For a positive integer k, a vertex subset D of a graph G is called a perfect k-dominating set of G if any vertex v of V not in D is adjacent to exactly k vertices of D. The minimum cardinality of a perfect k-dominating set of G is the perfect k-domination number $\gamma_{kp}(G)$. Thus a perfect 1-dominating set of G is a perfect dominating set. Note that every nontrivial graph G has a perfect k-dominating set, since the entire vertex set is such a set and there are graphs whose only perfect k-dominating set is $V(G)$, for example the stars $K_{1,t}$ for $1 < k < t$. A graph G for which $\gamma_{kp}(G) < n$ is called a perfect k-dominating graph, abbreviated PkD-graph, and a tree T for which $\gamma_{kp}(T) < n$ is called a PkD-tree.

A possible application to perfect k-domination is provided by a specialist giving radiation (or some powerful drug) to a patient. In order to be effective there must be precisely k units administered to the neighboring cells (any more is very dangerous). The cells where the drug is given directly are, unfortunately, weakened or harmed, and thus we wish to minimize the number of spots/cells where it is given. Thus we would want a minimum perfect k-dominating set.

2 Graphs with perfect k-dominating sets

We begin by making a couple of observations.

Observation 1 *For every graph G and positive integer k, every vertex with degree at most $k-1$ belongs to every (perfect) k-dominating set.*

Observation 2 *Any perfect k-dominating set is a k-dominating set, and hence $\gamma_k(G) \leq \gamma_{kp}(G)$ for every graph G and positive integer k.*

However, the difference $\gamma_{kp}(G) - \gamma_k(G)$ can be arbitrarily large, as may be seen by considering the following graph G obtained from k disjoint stars $K_{1,k}$ for $k \geq 2$ by adding a new vertex attached to the centers of the stars. Then $n = k^2 + k + 1$, $\gamma_k(G) = k^2+1$ and $\gamma_{kp}(G) = k^2+k$. Moreover, the same graph G shows that equality between the two parameters can occur; indeed, $\gamma_{k+1}(G) = \gamma_{(k+1)p}(G) = k^2 + 1$.

Since every vertex v not in a perfect k-dominating set D of a graph G should be adjacent to k vertices in D, it follows that G is not a PkD-graph for $k > \Delta(G)$. Hence it is natural to study PkD-graphs for k such that $1 \leq k \leq \Delta$.

We present next some sufficient conditions for graphs G to be PkD-graphs. A bipartite graph is called k-semiregular if every vertex in one of the two partite sets has degree k. The following lower bound on the k-domination number is due to Fink and Jacobson [8].

Theorem 3 (Fink and Jacobson [8]) *For every graph G with n vertices and m edges and every positive integer $k, \gamma_k(G) \geq n - \frac{m}{k}$. Furthermore, if $m \neq 0$, then $\gamma_k(G) = n - \frac{m}{k}$ if and only if G is a bipartite k-semiregular graph.*

Clearly from Theorem 3, all graphs G with $m \geq 1$ attaining the lower bound are PkD-graphs. The converse is false as can be seen by considering the double star $S_{k,k}$, where $S_{k,k}$ is a PkD-tree with $n = 2k + 2$ but $\gamma_k(S_{k,k}) = \gamma_{kp}(S_{k,k}) = 2k > \dfrac{(k-1)n+1}{k} = 2k - \dfrac{1}{k}$.

Theorem 4 *Let A be the degree sequence of a graph G. If k is an integer such that $k \in A$, then G is a PkD-graph.*

Proof. Let $k \in A$ and v be any vertex of G with degree k. Then $D = V(G) - \{v\}$ is a perfect k-dominating set of G and therefore G is a PkD-graph. □

We note that the converse of the above theorem is not true. For example, we consider the double star $S_{k,k}$. Then $S_{k,k}$ is a PkD-graph but k does not belong to the degree sequence of such a double star.

Theorem 5 *Let G be a graph with a complete graph $G[u_1, u_2, ..., u_r]$ as an induced subgraph and* $\deg(u_i) = k$*, $1 \leq i \leq r$. Then G is a PsD-graph whenever $k - r + 1 \leq s \leq k$.*

Proof. Let $S' = \{u_1, u_2, ..., u_{k-s+1}\}$ and $k - r + 1 \leq s \leq k$. Then $V(G) - S'$ is a perfect s-dominating set of G. Therefore G is a PsD-graph whenever $k - r + 1 \leq s \leq k$. □

The following bound on the perfect k-domination number will be useful for the next theorem. Recall that the independence number $\beta_0(G)$ of a graph G is the maximum cardinality of a subset of vertices of G that is both independent and dominating.

Proposition 6 *For any graph G and positive integer k, $\gamma_{kp}(G) \leq n - \beta_0(G[S_k])$, where S_k is the set of vertices in G of degree k.*

Proof. Let S_k be the set of vertices of degree k in G and M a maximum independent set of the subgraph induced by S_k. Clearly then $V(G) - M$ is a perfect k-dominating set of G and so $\gamma_{kp}(G) \leq n - |M| = n - \beta_0(G[S_k])$. □

For the particular cases $k = \Delta$ and $\Delta - 1$ we have the following.

Theorem 7 *Every graph G with $\Delta \geq 1$, is $P\Delta D$-graph. Furthermore, $\gamma_\Delta(G) = \gamma_{\Delta p}(G) = n - \beta_0(G[S_\Delta])$.*

Proof. By Theorem 4, G is a $P\Delta D$-graph. Since every Δ-dominating set is a perfect Δ-dominating set, it follows that $\gamma_\Delta(G) = \gamma_{\Delta p}(G)$. Now if S is any perfect Δ-dominating set, then $V(G)-S$ is an independent set whose vertices have degree Δ. Hence $\beta_0(G[S]) \geq |V(G) - S| = n-\gamma_{\Delta p}(G)$. The equality follows from Proposition 6. □

Theorem 8 *Let $k = \Delta(G) - 1$. Then G is a PkD-graph if and only if G satisfies one of the following conditions:*

(i) *There exist at least two adjacent vertices u, v such that* $\deg(u) = \deg(v) = \Delta(G)$.

(ii) *There exists a vertex u such that* $\deg(u) = \Delta(G) - 1$.

Proof. Suppose G is a PkD-graph for $k = \Delta(G) - 1$. If there are no vertices of degree $\Delta(G) - 1$ in G, then we have to prove condition (i). Suppose D is a perfect k-dominating set of G. Then there exists at least one vertex $v \in V - D$ with $\Delta(G)-1$ neighbors in D. But by our assumption no vertex in G is of degree $\Delta(G) - 1$ and so $\deg(v) > \Delta(G)-1$. Therefore $\deg(v) = \Delta(G)$. Thus there exists a vertex $w \in V - D$ adjacent to v and clearly w has degree $\Delta(G)$. Therefore (i) holds.

Conversely, if there exist two adjacent vertices u, v of degree $\Delta(G)$, then taking all other vertices as elements of D, we obtain a perfect $(\Delta(G)-1)$- dominating set of G. Now, if there exists a vertex u of degree $\Delta(G)-1$, then $V(G)-\{u\}$ is a perfect $(\Delta(G)-1)$-dominating set of G. In either case, G is a PkD-graph for $k=\Delta(G)-1$. □

We note that the conditions expressed in Theorem 8 can be checked in polynomial time and so we can decide whether G is a PkD-graph with $k=\Delta(G)-1$ in polynomial time.

Theorem 9 *Let T be a PkD-tree. Then at least one of the following holds:*

(i) *There exists at least one vertex of degree k in T.*

(ii) *There exist at least two vertices u and v of degree $k+1$ such that every interior vertex in the $u-v$ path has degree greater than $k+1$.*

Proof. Let D be a perfect k-dominating set of a PkD-tree T. Suppose (i) does not hold. Then we have to prove that (ii) holds. Hence, we consider the following cases:

Case 1. There exists at least one vertex of degree $k+1$ in $V-D$. Let u be a vertex of degree $k+1$ in $V-D$. Since $u \in V-D$ and $\deg(u)=k+1$, u is adjacent to a vertex $u_1 \in V-D$. Again since $u_1 \in V-D$, $\deg(u_1) \geq k+1$. If $\deg(u_1)=k+1$, then case (ii) holds. If not, u_1 is adjacent to at least one vertex $u_2 \in V-D$. Continuing the same arguments and since T is a finite tree we can conclude that (ii) holds.

Case 2. $V-D$ does not have a vertex of degree $k+1$. Let u be a vertex in $V-D$ of degree greater than $k+1$. Then u has at least two neighbors in $V-D$. Let u_1 be one such neighbor. Since $u_1 \in V-D$ and $\deg(u_1)>k+1$, u_1 has a neighbor $u_2 \in V-D$ and $\deg(u_2)>k+1$. Clearly this argument never ends and so it is a contradiction to the fact that T is finite. Consequently the only possibility is, Case 1 for which (ii) holds. □

Note that the converse of Theorem 9 is not true. It can be seen by the tree T obtained from a path $P_3 = a$-b-c by attaching 13 new vertices, four at each of a and c, and five at b. Then for $k=4$, vertices a and c have degree $k+1=5$ and satisfies Condition (ii) of Theorem 9 but T is not a $P4D$-tree.

Theorem 10 *Let G be a graph with $\gamma_{ap}(G)=\gamma_{bp}(G)=t$, provided $t<n$ and $a \neq b$. If D_1 is a perfect a-dominating set and D_2 is a perfect b-dominating set of G, then the following conditions are satisfied:*

(i) $D_1 \cap D_2 \neq \emptyset$.

(ii) $D_1 \neq D_2$.

Proof. Let D_1 be a perfect a-dominating set and D_2 a perfect b-dominating set of G with $|D_1| = \gamma_{ap}(G)$ and $|D_2| = \gamma_{bp}(G)$. Then, consider the following cases.

Case 1. If possible, let $D_1 \cap D_2 = \emptyset$. Since D_1 is a perfect a-dominating set of G, there will be $a\,|D_2| = at$ edges from D_2 to D_1 and since D_2 is a perfect b-dominating set of G, there will be $b\,|D_1| = bt$ edges from D_1 to D_2. So $a = b$, which contradicts our assumption. Hence $D_1 \cap D_2 \neq \emptyset$ follows.

Case 2. If possible, suppose $D_1 = D_2$. As $a \neq b$, it follows from the definition of perfect a-dominating and perfect b-dominating sets that $D_1 \neq D_2$. □

As an example of graphs for Theorem 10, let us consider the cycle $C_4 = a$-b-c-d-a. Then the cycle C_4 admits four minimum perfect 1-dominating sets $\{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}$ and two minimum perfect 2-dominating sets $\{a,c\}, \{b,d\}$. Clearly Conditions (i) and (ii) of Theorem 10 are satisfied for any minimum perfect 1-dominating sets and perfect 2-dominating sets. On the other hand to see that the condition $\gamma_{ap}(G) = \gamma_{bp}(G) = t$ is necessary for Theorem 10, consider the bipartite graph G with partite sets A and B with $|A| = 6$, where every vertex of A has degree two, and $|B| = 3$ and every vertex of B has degree four. Then A is a minimum perfect 4-dominating set, B is a minimum perfect 2-dominating set and clearly $A \cap B = \emptyset$.

Our next two results are upper bounds on the perfect k-domination number in terms of the order, the minimum degree and the diameter of a graph G.

Theorem 11 *For any PkD-graph G, $\gamma_{kp}(G) \leq n - \delta(G) + k - 1$.*

Proof. Let D be a minimum perfect k-dominating set of G and let v be any vertex of $V - D$. The number of vertices to which v is adjacent in $V - D$ will be less than or equal to $|(V - D) - \{v\}|$. Hence $\deg(v) - k \leq n - \gamma_{kp}(G) - 1$. Since $\delta(G) \leq \deg(v)$, so $\delta(G) - k \leq n - \gamma_{kp}(G) - 1$ and the upper bound follows. □

Theorem 12 *Let G be a connected PkD-graph. If D is a $\gamma_{kp}(G)$-set, then*

$$diam(G) \leq 3n - 2\gamma_{kp}(G) - (t + 1),$$

where t is the number of vertices in D which are adjacent to vertices in $V - D$.

Proof. Let D be a $\gamma_{kp}(G)$-set of a PkD-graph G. Suppose that $|V - D| = r$ and let $a_1, a_2, ..., a_r$ denote the vertices in $V - D$. Define $S_i = \{u \in D : u \in N(a_i)\}$, $1 \leq i \leq r$ and let $S = S_1 \cup S_2 \cup ... \cup S_r$. Then $t = |S|$ (by hypothesis). Now the diameter of G is maximum, if S_i's are mutually disjoint. Therefore, diam$(G) \leq (2r) + (r-1) + (n-r-t) = 2r + n - (t+1) = 2(n - \gamma_{kp}(G)) + n - (t+1) = 3n - 2\gamma_{kp}(G) - (t+1)$. □

The following observation is straightforward.

Observation 13 *Let G be a non trivial graph with $\gamma_{kp}(G) = k$. Then G has $K_{k,n-k}$ as a spanning subgraph, thus $\Delta(G) \geq$ Max. $\{k, n-k\}$.*

For the special case $k = 2$ we have the following.

Theorem 14 *Let G be a P2D-graph of order n. Then $\gamma_{2p}(G) = 2$ if and only if G satisfies the one the following conditions:*

(i) there exist two adjacent vertices u, v such that deg(u) = deg(v) $= n - 1$.

(ii) there exist two non-adjacent vertices u, v such that deg(u) = deg(v) $= n - 2$.

Proof. Let $D = \{u, v\}$ be a perfect 2-dominating set of G. Then both u and v should be adjacent to every other vertex in $V - D$. Hence deg(u)=deg(v)=$n - 1$ or $n - 2$ depending on whether u, v are adjacent or not, respectively.

Conversely, if G satisfies (i) or (ii), clearly $\{u, v\}$ is a perfect 2-dominating set of a graph G and therefore $\gamma_{2p}(G) = 2$. □

Theorem 15 *For any connected graph G, $\gamma_{2p}(G) \geq \lceil diam(G)/2 \rceil + 1$.*

Proof. Let diam(G)=r, and u, v be two vertices with $d(u, v) = r$. Let P be a path of length r in G connecting u and v. Then $\gamma_{2p}(P) = \lceil \text{diam}(P)/2 \rceil + 1$. Now G can be re-constructed from the path P by adding vertices and edges. Suppose $\gamma_{2p}(G)$ decreases while doing that. For that to happen, a vertex w in a perfect 2-dominating set D of the path P has to be included in $V - D$. But it will happen only if either of the following cases holds.

Case 1. If w becomes adjacent to two vertices in a perfect 2-dominating set of path P, then it will result in reduction of $d(u, v)$ which is a contradiction to our assumption.

Case 2. If w becomes adjacent to two vertices not in path P and those vertices are in a perfect 2-dominating set of that graph, then in this case $\gamma_{2p}(G)$ is not decreased since we have added two more vertices to perfect 2-dominating set of P.

From the above cases, it follows that $\gamma_{2p}(G) \geq \lceil \text{ diam}(P)/2 \rceil + 1$. □

3 Complexity results

Note that replacing the requirement of exactly one neighbor in a perfect dominating set S by exactly k neighbors in a perfect dominating set S allows the interesting question to be asked, whether such a minimal set S exists at all in the graph. Our main view of this section is to study the complexity of the following decision problem, to which we shall refer as perfect k-dominating set:

Instance: A graph G and a positive integer p.

Question: Does G have a perfect k-dominating set of cardinality p or less?

In particular, we show that this problem is NP-complete by reducing the following well known NP-complete problem EXACT 3-COVERS ($X3C$) to our problem.

Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

Question: Does C contain an exact cover for X, that is, a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C' ?

Note that if C' exists, then its cardinality is precisely q. For more details on computational complexity, we refer to [9].

Theorem 16 *The perfect k-dominating set is NP-complete.*

Proof. The perfect k-dominating set is in NP. If given a set S, $|S| < p$ as a witness to yes instance, the neighborhoods of all vertices $v \in (V - S)$ could be checked to ensure that $|N(v) \cap S| = k$.

A graph $G = (V, E)$ and a positive integer p, constructed from an instance of $X3C$ will have an exact cover if and only if the graph G has a perfect k- dominating set of cardinality at most p. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of $X3C$, where $|X| = 3q$, $|C| = t$ and $p = t + k - 1$.

The construction of a graph G is given by creating a vertex x_i for each element $x_i \in X$ and a component consisting of a path P_2 with vertices labeled c_j and s_j for every subset $C_j \in C$. Add new vertices $\{u_1, u_2, \ldots, u_{k-1}\}$ adjacent to all c_j's, x_j's and s_j's. Also add the edges $E' = \{(x_i c_j) : x_i \in C_j\}$.

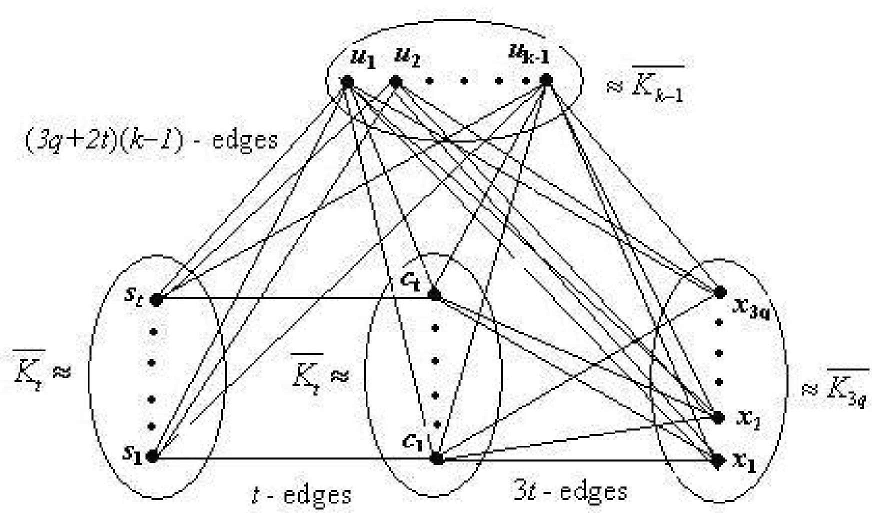

Figure 1: NP-completeness constructions of k-perfect domination

Now, we have to show that C has an exact cover if and only if the graph G has a perfect k-dominating set of cardinality at most p. Suppose C' is an exact cover of C. Then $S = \{c_j : C_j \in C'\} \cup \{s_j : C_j \notin C'\} \cup \{u_1, u_2, \ldots, u_{k-1}\}$ is a perfect k-dominating set for G with $|S| = t + k - 1$.

Now suppose that S, $|S| \leq t + k - 1$, is a perfect k-dominating set for G. In the graph G, the degree of s_j is k. So either $s_j \in S$ or all the neighbors of s_j are in S. Hence for each path $P_2 = s_j c_j$, one of its vertices is in S. Now since $|S| \leq t + k - 1$, $\{s_1, s_2, \ldots, s_t\} \nsubseteq S$ and so $\{u_1, u_2, \ldots, u_{k-1}\} \subset S$, $\{c_1, c_2, \ldots, c_t\} \cap S \neq \emptyset$. It follows that none of the vertices $x_i \in S$. Thus each x_i has only one neighbor in $S \cap \{c_1, c_2, \ldots, c_t\}$. Therefore $C' = \{C_j : c_j \in S\}$ is an exact cover for C.

From the figure it is clear that this graph will have $(3q + 2t + k - 1)$-vertices and $(4t + (3q + 2t)(k - 1))$- edges. Thus it forms a polynomial transformation. $\square$

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