A Census of 2-(9,3,3) Designs

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Abstract

There are 22521 nonisomorphic 2-(9,3,3) designs of which 9218 are decomposable and 395 resolvable. Computational methods used to find and analyse these designs are discussed. All cubic multigraphs on 8 vertices are displayed and their role in the generation process is outlined. Statistics are presented concerning neighborhood graphs, multiple blocks, parallel classes, subdesigns, group orders and fragments. It is verified that all 2-(9,3,3) designs are 2- and 3-resolvable and that 3 of them have mutually orthogonal resolutions. Two families of designs together with some of their properties are listed.

1. Introduction.

A *2-(v,k,λ) design* is a pair (V, B) where V is a v-set of *elements,* and B is a set of k-subsets of V called *blocks* such that every 2-subset of V is contained in exactly λ blocks. A design is called *simple* if it does not contain repeated blocks. A central problem in design theory concerns the enumeration of all nonisomorphic solutions for a given set of parameters. Catalogues of designs are important for the study of various properties and for the formulation and verification of conjectures. The enumeration of 2-(9,3,3) designs is of particular interest for several reasons. To establish the fine structure of 3-fold triple systems it is essential to know the spectrum of repeated blocks for systems of small orders as ingredients for recursive constructions [2]. 2-(9,3,3) designs are also important for the study of decomposability and α-resolvability of triple systems [1,4,11]. There have been several attempts to enumerate all simple 2-(9,3,3) designs. The first search has been carried out in 1980 by Harnau [7] who found 329 nonisomorphic designs. In 1985 Ivanov [8] obtained 330 solutions using a different computer algorithm. The final count has been settled in 1987 by Harms, Colbourn and Ivanov [6] who established that there are 332 nonisomorphic designs without repeated blocks.

In this paper we are interested in a complete enumeration of 2-(9,3,3) designs when repeated blocks are permitted. The paper is organized as follows. In the next section we describe our generation procedure. As in [6] neighborhood graphs are employed for a partial rejection of isomorphic solutions. In Section 3 we highlight computational results and display statistics concerning multiple blocks, neighborhood graphs, parallel classes, subdesigns and fragments. The last section is concerned with decomposability and α–resolvability. In addition to various statistics we exhibit some interesting designs related to generalized Room squares. In the Appendices we display all cubic multigraphs on 8 vertices and list the homogeneous and resolvable indecomposable designs.

2. Method of generation

From now on we will assume that a 2-(9,3,3) design (V, B) has blocks $B = \{B_0, B_1, ..., B_{35}\}$ on the 9-element set $V = \{0,1,...,8\}$. Clearly, there are 36 blocks of size 3, each element appears in exactly 12 blocks and each 2-subset of V appears in exactly 3 blocks. The set of blocks containing a given element $x \in V$ will be denoted by B(x). A *neighborhood graph* G(x) on an element x is the graph (X, E) with vertices $X = V \setminus \{x\}$ and edges $E = \{(y,z) \mid (x,y,z) \in B(x)\}$. In a 2-(9,3,3) design each G(x) is a cubic multigraph on 8 vertices. There are exactly 32 such cubic multigraphs which are displayed in Appendix I. They have been found by a simple exhaustive backtrack search with isomorph rejection. For each graph we have determined a set of invariants, listed in Table 1, which can be used for easy identification. The column headed |G| contains the order of the automorphism group while columns headed Mt, Md, T, and Q contain the number of triple edges, double edges, triangles and quadrangles, respectively. Neighborhood graphs serve as a powerful design invariant and will be used to help with isomorph rejection.

Table 1. Invariants of cubic multigraphs on 8 vertices

No.	|G|	Mt	Md	T	Q	No.	|G|	Mt	Md	T	Q
0	384	4	0	0	0	16	4	0	2	2	1
1	32	2	2	0	1	17	4	0	2	2	0
2	192	2	0	4	0	18	2	0	2	1	0
3	12	1	3	0	0	19	4	0	2	0	3
4	16	1	2	2	0	20	4	0	2	0	1
5	8	1	2	0	2	21	8	0	1	3	2
6	8	1	1	2	0	22	4	0	1	2	1
7	12	1	0	2	3	23	2	0	1	2	1
8	24	1	0	0	9	24	2	0	1	1	2
9	32	0	4	0	2	25	8	0	1	0	5
10	8	0	4	0	0	26	1152	0	0	8	0
11	8	0	3	2	0	27	16	0	0	4	0
12	4	0	3	1	0	28	4	0	0	2	2
13	2	0	3	0	1	29	12	0	0	1	2
14	12	0	3	0	0	30	48	0	0	0	6
15	96	0	2	4	1	31	16	0	0	0	4

We are now in a position to describe our algorithm for generating 2-(9,3,3) designs. To facilitate the search we use two multisets *S* and *P* for the available elements and pairs, respectively. Initially, *S* contains 12 copies of each element and *P* 3 copies of each pair. In the first step of the algorithm we input 12 blocks corresponding to B(0) with a specified neighborhood graph G(0). Each of the 32 nonisomorphic graphs is considered in turn, starting from type 0 and ending with type 31. The remaining 24 blocks are found recursively as follows. On the i-th level we use available elements $x,y,z \in S$, $x<y<z$, and $(x,y),(x,z),(y,z) \in P$ to form a candidate block. The candidates are generated in increasing lexicographical order. Whenever an element in *S* is completely exhausted we compare the type of its neighborhood graph to that of G(0). If it is smaller than G(0) we turn to the next candidate otherwise set $B_i = (x,y,z)$, update *S* and *P* and recurse to level i+1. In case there are no candidates left on the i-th level, we backtrack

to the next candidate on level i-1, after recovering S and P. When a block B_{35} on level 35 is accepted, the set B of all blocks forms a 2-(9,3,3) design. At this point we carry out a partial rejection of isomorphic solutions. For this we use the vector of neighborhood graph types **g** = (|G(0)|, |G(1)|, ..., |G(8)|), where |G(x)| denotes the type of G(x). For every automorphism π of G(0) we permute the elements in B and calculate π**g**. A new design is rejected if π**g** is lexicographically smaller than **g**, otherwise it is accepted. For a given input graph G(0) the search terminates if all candidates on level 12 are exhausted.

To compute automorphism groups and canonical orderings we have employed the program *nauty* of Brendan McKay [10]. From the 34460 designs generated by our algorithm we obtained exactly 22521 nonisomorphic solutions. To check the correctness of our algorithm we have relabeled the types of input neighborhood graphs and repeated the search. This yielded a different set of designs but the same canonical solutions as before. In addition, we have generated the decomposable 2-(9,3,3) designs directly by concatenating the unique 2-(9,3,1) affine plane to each of the 36 2-(9,3,2) designs from [9,11] in all 840 possible ways . After rejecting isomorphic solutions we ended up with the same 9218 decomposable designs. As a final check we have compared the 332 simple designs extracted from our set with those listed in [6]. Again there was a perfect match.

We end this section with a few comments concerning the way our designs are presented and stored. The blocks of any 2-(9,3,λ) design are contained in the set of all 3-subsets of our 9-set V. By using a character to encode each of the 84 3-subsets we can represent a 2-(9,3,3) design compactly as a string of 36 characters. Together with each design we store its graph vector **g**, automorphism group order, number of distinct fragments, parallel classes and affine planes as well as information about decomposability and resolvability, requiring another 15 characters. The design elements are permuted so that the corresponding graph vectors have components arranged in nondecreasing order. Finally, we sort the designs lexicographically by their vectors and blocks. A simple program can use this list to extract designs with any desired properties and calculate various statistics.

3. Results and analysis

Since it is technically impossible to list all 22521 nonisomorphic designs we will compile statistics concerning designs with specific properties. We start with the spectrum of repeated blocks. Table 2 displays the number of designs with a given number of double blocks (first column) and triple blocks (first row). There are no designs with 5,7,8,9,10 and 11 blocks repeated three times. The (0,0)-entry is 332 and corresponds to the number of simple designs; the sum of all entries in the table is 22521. Next we study the distribution of neighborhood graphs in the set of all designs.

Table 2. Frequency of double and triple blocks

↓2/3→	0	1	2	3	4	5	6	12
0	332	26	1	2	1	0	0	1
1	1319	71	5	0	0	0	0	0
2	2774	186	6	0	1	0	0	0
3	4021	263	12	0	0	0	0	0
4	4299	344	21	1	2	0	0	0
5	3649	335	18	0	0	0	0	0
6	2485	246	19	2	2	0	1	0
7	1253	143	13	1	1	0	0	0
8	440	51	8	0	1	0	0	0
9	113	20	3	2	0	0	0	0
10	19	2	1	0	0	0	0	0
11	2	1	0	0	0	0	0	0
12	2	0	0	0	0	0	0	0

Table 3. Frequency of neighborhood graphs

No.	Nd	Ng	No.	Nd	Ng
0	8	16	16	7545	9017
1	87	99	17	7433	8867
2	27	29	18	13068	20596
3	544	675	19	7445	9358
4	555	632	20	4776	5396
5	995	1304	21	5864	6452
6	1219	1795	22	10067	13199
7	784	997	23	10787	14561
8	178	205	24	16782	30615
9	327	341	25	6471	7653
10	1918	2197	26	48	48
11	2760	3076	27	4231	4584
12	5221	6526	28	12761	22504
13	8761	13014	29	6432	8081
14	2164	2393	30	1846	1990
15	220	223	31	5274	6246

In Table 3 the column headed Nd displays the number of designs which contain a graph of type No. and column Ng contains the total number of times a graph of type No. occurs in the set.of all designs. Hence, the Nd-columns sum to 22521 and the Ng-columns sum to 8 times more. As we can see from the table, graph No.0 occurs least and graph No.24 most frequently.

A design is called *homogeneous* if all its neighborhood graphs are of the same type and *heterogeneous* if no two graphs are of the same type. Among the 22521 designs 14 are homogeneous and 726 heterogeneous.

Table 4. Frequencies of graph vectors, distinct parallel classes and affine planes, and group orders.

N	Nv	Np	Na	Ngp	N	Nv	Np*	Na	Ngp*
0	0	0	13303	0	14	2	2757	0	0
1	14859	1	8077	21534	15	0	2416	0	0
2	2049	1	759	792	16	1	1904	0	2
3	487	6	354	83	17	1	1355	0	0
4	208	25	23	39	18	0	996	0	4
5	97	57	0	0	19	0	530	0	0
6	38	187	4	41	20	0	377	0	0
7	31	344	0	0	21	0	143	0	0
8	12	671	0	4	22	1	84	0	0
9	7	1060	0	3	23	0	24	0	0
10	4	1742	0	0	24	0	19	0	5
11	2	2283	0	0	25	0	2	0	0
12	3	2792	1	7	26	0	2	0	0
13	0	2740	0	0	27	0	1	0	0

Table 5. Spectrum of distinct fragments

	0	1	2	3	4	5	6	7	8	9
10	1	0	0	0	0	0	1	0	0	0
20	0	0	0	0	1	2	1	0	1	1
30	0	2	0	1	3	0	1	1	0	3
40	5	5	4	5	5	3	5	7	13	13
50	10	19	12	11	18	30	27	38	28	47
60	38	58	74	78	74	84	75	113	157	148
70	156	147	151	197	222	212	270	261	283	314
80	317	289	366	354	367	381	428	439	426	445
90	454	418	459	461	470	479	479	435	469	439
100	471	481	450	475	435	405	393	453	426	424
110	347	323	346	313	339	377	326	312	271	207
120	229	230	235	193	182	164	197	179	162	161
130	147	97	97	77	97	92	84	51	51	43
140	63	41	57	31	37	29	24	15	11	13
150	13	11	4	2	3	0	5	1	4	2
160	0	1	0	0	0	0	0	0	0	0
170	0	0	0	0	0	1	0	0	0	0

The homogeneous designs are listed in Appendix II (see end of Section 4 for an explanation of the first line of each design). Table 4 gives statistics for graph vectors, parallel classes, affine planes and group orders. Its columns have the following meaning: Nv graph vectors occur N times, Np designs contain N distinct parallel classes, Na designs contain N distinct affine planes (i.e. 2-(9,3,1) subdesigns) and Ngp designs have an automorphism group of order N. It is obvious that the sum of the Nv-column entries taken with multiplicity N, and the sums of each remaining column again equal to 22521, except for a few entries which did not fit into the table (*). These are: one design with 30 and one with 37 parallel classes and one design with each of the group orders 32, 36, 48, 54, 108, 432 and 1296.

A *fragment* (or a Pasch configuration) in a triple system 2-(v,3,λ) is a set of four blocks of the form {(a,c,d), (a,e,f), (b,c,e), (b,d,f)}. Fragments play an important role as design invariants and can be used to transform one design into another [3]. In Table 5 we display the number of designs containing a given number of distinct fragments which is specified in the first row (units) and first column (tens). For example, there are 354 designs containing 83 fragments.

Finally, we assess the strength of some invariants considered in this section to distinguish nonisomorphic designs. The *efficiency* of an invariant on a set *D* of nonisomorphic designs is the ratio of the number of values it takes on *D* to the cardinality of *D*. Table 6 displays the efficiency of four combinations of invariants.

Table 6. Efficiency of selected invariants

V	VMGDR	VF	VMGDRF
17802	20875	21531	22190
0.790	0.927	0.956	0.985

The letters V, M, G, D, R, and F stand for graph vector, block multiplicities, group order, decomposability, resolvability and the number of distinct fragments, respectively. The first row displays the number of designs with different invariants, the second row contains the corresponding efficiencies. As we can see from the table, graph vectors in conjunction with fragment numbers are capable of distinguishing over 95% of nonisomorphic designs.

4. Decomposability and resolvability

A 2-(v,k,λ) design (*V*,*B*) is said to be *decomposable* if there is a proper subset *B'* of *B* for which (*V*,*B'*) is a 2-(v,k,λ') subdesign. A design with no such subset is called *indecomposable*. It is easy to see that the complementary blocks *B**B'* form a 2-(v,k,λ–

λ') subdesign. Of interest are also partitions of a 2-(v,k,λ) design into n indecomposable 2-(v,k,λ_i) designs with $\lambda_1 + \ldots + \lambda_n = \lambda$ (see [1]). The 2-(9,3,λ) family of designs begins with a unique design with $\lambda=1$ which has the structure of an affine plane (of order 3). Of the 36 designs with $\lambda=2$, 9 are decomposable into 2 disjoint affine planes. For $\lambda=3$ the situation is a little more complicated. Any decomposable design D contains at least one affine plane. D can be one of 3 types depending on the structure of the 2-(9,3,2) subdesign complementary to an affine plane. D is of type 1 or 2 if the complement to every affine subplane is indecomposable or decomposable, respectively; D is of type 3 if it contains both subplanes with decomposable and indecomposable complements. In the set of 22521 designs with $\lambda=3$ there are 9218 decomposable ones. These are partitioned into 8854 designs of type 1, 347 of type 2 and 17 of type 3.

A subset C of blocks in a 2-(v,k,λ) design (V,B) is called an α-*class* if each element of V occurs in exactly α blocks. A design is α-*resolvable* if its blocks can be partitioned into α-classes [4]. For $\alpha=1$ we have the usual notion of resolvability into parallel classes. Two α-resolutions of (V,B) are said to be *orthogonal* if any α-class of one of them has at most one block in common with any α-class of the other. A design is called *m-tuply* α-*resolvable* if it has m mutually orthogonal α-resolutions. There are 395 resolvable designs of which exactly 3 contain orthogonal resolutions. We display these 3 designs (No. 0, 1809, 22197) and their orthogonal resolutions by listing each block together with a tuple from 12 letters signifying the parallel classes it belongs to in various resolutions. The i-th coordinate of each tuple corresponds to the i-th resolution.

012aaa	012bbb	012ccc	034ddd	034eee	034fff	056ggg	056hhh	056iii
078jjj	078kkk	078lll	135jlk	135kjl	135lkj	147gih	147hgi	147ihg
168dfe	168edf	168fed	238ghi	238hig	238igh	246jkl	246klj	246ljk
257def	257efd	257fde	367acb	367bac	367cba	458abc	458bca	458cab

012aaa	012bbb	012ccc	034ddd	035eee	038fff	046ggg	047hhh	056iii
057jjj	068kkk	078lll	134lkj	135klg	138jih	146fel	147efi	156dhf
157gdk	168hjd	178ige	234ijk	235hgl	238ghi	246jle	247kif	256lfh
257fkd	268edj	278deg	367acb	367bac	367cba	458abc	458bca	458cab

012aa	013bb	014cc	023dd	024ee	034ff	056gg	057hh	058ii
067jj	068kk	078ll	125jk	126ih	135lj	137gi	145kl	148hg
167fe	168df	178ed	236hl	237kg	246li	248gj	257cf	258fb
278bc	347ik	348jh	356ec	358ae	368ca	456bd	457da	467ab

A probabilistic approach based on simulated annealing has been suggested by Gordon Royle to verify that every 2-(9,3,3) design is both 2- and 3-resolvable. His algorithm starts with an arbitrary partition of blocks into 6 (or 4) equal sets and swaps blocks from different subsets to minimize the number of conflicts via simulated annealing. The final partition is in most cases a 2-resolution (or 3-resolution). Processing all 22521 designs took less than an hour of CPU time on a SUN SPARCstation. From the 3 designs which are multiply resolvable only 2 (No. 1809, 22197) are doubly 2-resolvable. We display them as squares in which the rows correspond to one 2-resolution and columns to another.

012	458	367	246	138	057
367	012	458	035	247	168
458	367	012	178	056	234
034	156	278	068	235	147
157	238	046	134	078	256
268	047	135	257	146	038

012	467	248	137	358	056
236	013	278	456	057	148
368	347	014	058	126	257
457	125	067	023	348	168
145	068	356	178	024	237
078	258	135	246	167	034

It would be interesting to know whether there exist 3 mutually orthogonal 2-resolutions. This would be a maximal set since there are no mutually orthogonal Latin squares of order 6. We note that for obvious reasons there are no orthogonal 3-resolutions of a 2-(9,3,3) design.

Finally, a summary is given in Table 7 of results concerning 2-(9,3,λ) designs for λ=1,2,3. The columns headed M, N, S, D and R contain the numbers of distinct, nonisomorphic, simple, decomposable and resolvable designs, respectively.

Table 7. The family of 2-(9,3,λ) designs

λ	M	N	S	D	R
1	840	1	1	0	1
2	4 409 916	36	13	9	9
3	7 974 771 700	22521	332	9218	395

In Appendix III we present a list of all resolvable indecomposable 2-(9,3,3) designs. The first line gives the design No., vector of graph types, group order, number of distinct fragments, triple blocks, double blocks, number of distinct affine planes with decomposable complements, with indecomposable complements, number of distinct parallel classes and an indicator of resolvability.

Acknowledgements

Research of the first author is supported by NSERC Canada under grant number A8651. Research of the second author is supported by NSERC Canada and the Department of Computer Science, University of Toronto. We would like to thank Gordon Royle for making available to us his program to find α-resolutions.

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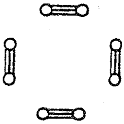
Appendices

I. Cubic multigraphs of order 8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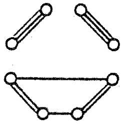
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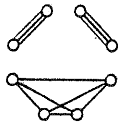
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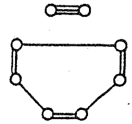
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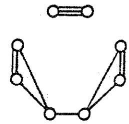
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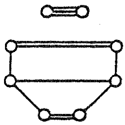
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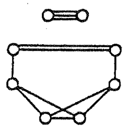
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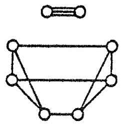
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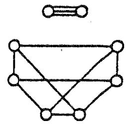
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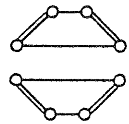
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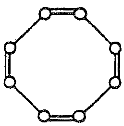
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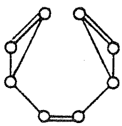
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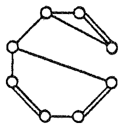
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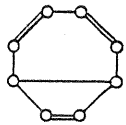
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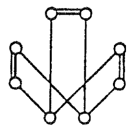
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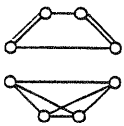
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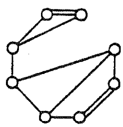
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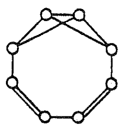
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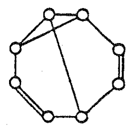
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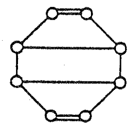
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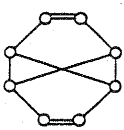
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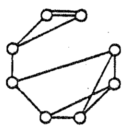
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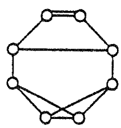
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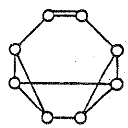
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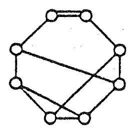
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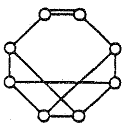
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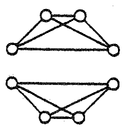
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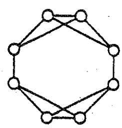
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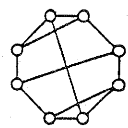
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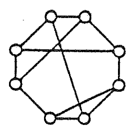
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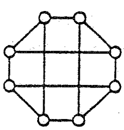
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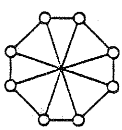
31

II. Homogeneous (9,3,3) designs (14)

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168 168 168 238 238 238 246 246 246 257 257 257 367 367 367 458 458 458

120 3 3 3 3 3 3 3 3 3 108 18 3 9 2 0 7 1
012 012 012 034 034 035 046 057 057 068 068 078 134 138 138 147 147 156
156 157 168 235 235 238 246 246 247 256 278 278 367 367 367 458 458 458

1809 8 8 8 8 8 8 8 8 8 1296 54 3 0 12 0 37 1
012 012 012 034 035 038 046 047 056 057 068 078 134 135 138 146 147 156
157 168 178 234 235 238 246 247 256 257 268 278 367 367 367 458 458 458

2130 10 10 10 10 10 10 10 10 10 18 33 0 12 2 0 8 1
012 012 013 024 035 035 046 046 057 068 078 078 125 134 134 146 158 158
167 167 178 237 237 238 248 248 256 256 267 347 356 368 368 457 457 458

5866 12 12 12 12 12 12 12 12 12 9 63 0 9 0 0 3 0
012 013 014 027 027 035 036 048 048 056 056 078 123 123 145 148 157 157
167 168 168 238 245 246 246 256 258 278 346 347 347 358 358 367 457 678

9185 13 13 13 13 13 13 13 13 13 9 36 0 9 3 0 12 1
012 013 013 024 024 035 046 056 057 068 078 078 127 128 134 146 146 157
157 158 168 235 238 238 247 256 256 267 345 347 367 367 368 458 458 478

17653 18 18 18 18 18 18 18 18 18 18 63 0 6 3 0 10 1
012 013 017 023 024 035 046 046 058 058 067 078 126 126 138 138 145 147
148 156 157 234 235 248 257 257 268 278 347 347 356 367 368 456 458 678

21991 24 24 24 24 24 24 24 24 24 6 99 0 3 0 1 20 0
012 012 013 024 035 037 045 048 056 067 068 078 126 135 138 145 147 148
157 167 168 235 237 238 247 248 256 258 267 346 346 347 368 456 578 578

21992 24 24 24 24 24 24 24 24 24 18 99 0 3 3 0 20 1
012 012 013 024 035 037 045 048 056 067 068 078 126 137 138 145 147 148
156 157 168 235 237 238 247 248 256 258 267 345 346 346 368 467 578 578

22437 28 28 28 28 28 28 28 28 28 6 132 0 0 0 1 12 0
012 013 018 024 026 035 037 045 046 057 068 078 123 124 137 145 147 156
158 167 168 236 238 247 256 257 258 278 345 346 348 358 367 468 478 567

22438 28 28 28 28 28 28 28 28 28 18 126 0 0 3 0 12 1
012 013 018 024 026 035 037 045 046 057 068 078 124 127 135 136 147 148
156 158 167 236 237 238 245 257 258 268 345 346 348 378 467 478 567 568

22439 28 28 28 28 28 28 28 28 28 9 135 0 0 0 3 12 0
012 013 018 024 026 035 037 045 046 057 068 078 124 125 134 138 147 156
157 167 168 236 237 238 247 256 258 278 345 348 356 367 458 467 468 578

22440 28 28 28 28 28 28 28 28 28 6 165 0 0 0 1 12 0
012 013 018 024 026 035 037 045 046 057 068 078 123 125 136 145 146 147
158 167 178 234 238 248 256 257 267 278 347 348 356 357 368 458 467 568

22520 29 29 29 29 29 29 29 29 29 54 99 0 0 3 0 30 1
012 013 018 024 025 034 035 046 057 067 068 078 123 126 137 146 147 148
156 157 158 234 235 247 258 267 268 278 348 356 367 368 378 456 457 458

III. Indecomposable resolvable (9,3,3) designs (22)

614 3 7 7 24 24 24 24 24 24 6 75 1 3 0 0 18 1
012 356 478 012 358 467 012 378 456 034 158 267 034 167 258 035 168 247
046 157 238 057 136 248 057 148 236 068 137 245 068 145 237 078 134 256

1397 5 6 7 18 20 24 24 28 29 1 69 1 3 0 0 18 1
012 348 567 012 356 478 012 368 457 035 167 248 035 178 246 037 168 245
046 137 258 046 158 237 047 156 238 058 134 267 068 134 257 078 145 236

1421 5 6 7 20 20 24 24 28 28 2 71 1 3 0 0 18 1
012 356 478 012 368 457 012 378 456 035 147 268 035 167 248 037 168 245
046 138 257 046 158 237 048 157 236 058 134 267 067 134 258 078 156 234

1545 6 6 6 14 24 24 24 29 30 3 65 1 3 0 0 18 1
012 348 567 012 358 467 012 368 457 036 145 278 036 158 247 037 156 248
045 168 237 046 178 235 048 167 235 057 134 268 058 137 246 078 134 256

1755 6 7 7 24 24 28 28 31 31 2 92 1 1 0 0 20 1
012 356 478 012 357 468 012 378 456 034 158 267 034 167 258 035 168 247
046 157 238 057 136 248 058 147 236 067 138 245 068 145 237 078 134 256

1791 7 7 7 28 28 28 29 30 30 3 98 1 0 0 0 21 1
012 346 578 012 356 478 012 378 456 034 158 267 035 168 247 037 156 248
046 138 257 048 167 235 057 134 268 058 147 236 067 145 238 068 137 245

2126 9 22 22 23 24 24 24 24 25 1 80 0 4 0 0 16 1
013 258 467 014 237 568 014 238 567 026 148 357 026 158 347 027 138 456
035 126 478 035 167 248 045 127 368 068 157 234 078 125 346 078 136 245

12481 13 18 19 20 24 24 25 27 28 1 82 0 4 0 0 18 1
014 256 378 014 267 358 018 247 356 025 138 467 025 168 347 028 167 345
036 127 458 036 157 248 037 145 268 046 123 578 057 123 468 078 156 234

13288 13 23 24 24 24 24 24 28 29 1 93 0 3 0 0 19 1
014 236 578 014 237 568 015 234 678 025 137 468 025 167 348 028 146 357
036 128 457 036 158 247 038 127 456 047 138 256 067 135 248 078 126 345

13588 14 17 19 20 22 24 25 28 28 1 80 0 4 0 0 18 1
014 256 378 014 268 357 017 238 456 025 167 348 025 178 346 027 168 345
036 124 578 036 158 247 037 156 248 048 135 267 058 123 467 068 123 457

16578 17 18 18 24 24 24 28 29 31 1 90 0 3 0 0 19 1
015 246 378 015 268 347 017 236 458 023 148 567 023 168 457 025 138 467
034 156 278 046 137 258 048 136 257 067 124 358 068 127 345 078 124 356

19515 18 23 24 24 25 28 28 29 29 1 106 0 2 0 0 20 1
012 356 478 012 357 468 015 246 378 027 136 458 034 128 567 034 167 258
035 168 247 046 157 238 058 147 236 067 138 245 068 145 237 078 134 256

19550 18 24 24 24 24 28 28 28 31 1 104 0 2 0 0 20 1
014 257 368 014 268 357 016 278 345 023 158 467 023 167 458 025 136 478
037 128 456 047 138 256 056 127 348 058 124 367 068 157 234 078 135 246

19555 18 24 24 24 24 28 28 29 31 2 100 0 2 0 0 22 1
014 258 367 014 267 358 015 278 346 023 157 468 023 168 457 026 178 345
037 156 248 047 138 256 056 127 348 058 123 467 068 124 357 078 136 245

19575 18 24 24 24 25 28 28 28 29 1 98 0 2 0 0 20 1
012 356 478 012 357 468 017 236 458 025 146 378 034 128 567 034 158 267
038 157 246 047 168 235 056 134 278 058 136 247 067 145 238 068 137 245

21184 22 22 24 28 28 28 29 29 31 1 114 0 1 0 0 19 1
012 345 678 012 358 467 015 248 367 027 138 456 034 126 578 036 147 258
038 146 257 047 135 268 048 156 237 056 178 234 057 148 236 068 137 245

21348 22 23 24 27 28 28 30 31 31 1 115 0 1 0 0 19 1
012 358 467 012 378 456 013 256 478 024 168 357 034 157 268 036 147 258
045 138 267 057 136 248 058 127 346 067 148 235 068 145 237 078 156 234

21517 22 24 24 27 28 28 29 30 31 1 115 0 1 0 0 19 1
012 345 678 012 356 478 013 267 458 028 157 346 035 147 268 038 146 257
046 137 258 047 158 236 048 156 237 056 124 378 057 168 234 067 138 245

21897 23 24 24 28 28 28 29 29 31 1 113 0 1 0 0 21 1
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045 167 238 046 138 257 048 157 236 057 124 368 068 135 247 078 156 234

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038 156 247 045 127 368 046 137 258 047 123 568 067 148 235 078 134 256

22511 28 28 28 28 28 29 29 31 31 2 120 0 0 0 0 24 1
012 368 457 013 258 467 014 237 568 026 135 478 027 156 348 034 168 257
038 157 246 045 178 236 056 137 248 058 124 367 067 128 345 078 146 235

22513 28 28 28 28 28 29 29 31 31 2 126 0 0 0 0 24 1
013 257 468 014 267 358 015 238 467 024 168 357 025 136 478 028 147 356
034 126 578 037 128 456 056 127 348 067 158 234 068 137 245 078 145 236