Some New Families of Simple t-Designs

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ABSTRACT

Applying some methods of construction on existing t-designs, we obtain some infinite families of new simple designs. We give a table which contains many new simple designs in small cases. The main method which is called the union method, involves taking the union of every two blocks in a given design. We also combine this method with some other well known ones. Some of the new designs obtained are the following. If there exists a Hadamard matrix of order $4m$, then there exists a simple 2-$(4m-1, m, m(m-1)/2)$ design, a 2-$(4m, m, (2m-1)(m-1))$ design, a simple 3-$(4m, 2m-1, (m-1)(2m-3))$ design, and a simple 3-$(4m, 2m-2, (2m-3)(m-2)(m-1))$ design. Finally if q is a prime power, then there exists a simple 2-$(q^2(q+2), q(2q+1), q(q+1)(2q+1)(2q-1)/2)$ design. We show that the number of non-isomorphic simple 2-$(15, 4, 6)$ designs is at least 10.

1. Introduction

In this paper we apply some methods of construction for t-designs to obtain new families of designs, from existing ones. In this section we give some basic definitions. Terminologies not defined here, can be found in all standard textbooks, for example in [**H**].

A t-(v, k, λ) *design*, or simply a t-*design* of *order* v, *block size* k and *index* λ is a pair $(V, \mathcal{B})$. V is a set of v *elements*, and $\mathcal{B}$ is a collection of k-subsets of V called *blocks*. Every t-subset appears in precisely λ of the blocks. When $\mathcal{B}$ contains no repeated blocks, the t-design is *simple*. We are concerned here mainly with simple t-designs. It is a well known fact that any t-design is also a t'-design for $t' < t$. The number of appearances of each i-subset of $V (i \leq t)$, is denoted by λ_i. Thus $\lambda_t = \lambda$.

The number of blocks in a t-design is usually denoted by b ($b = \lambda_0$). The number of blocks containing each given element is denoted by r ($r = \lambda_1$). A 2-design is also referred to as a *balanced incomplete block design* (BIBD). And a BIBD is called *symmetric* if $b = v$.

A *pairwise balanced design* (PBD) is a collection of subsets of a set V called blocks, whose cardinalities are from the set $\{k_1, k_2, ..., k_m\}$, such that every pair of elements of V appears in exactly λ of the blocks. A PBD will be denoted by $(v; k_1, ..., k_m; \lambda)$ PBD, and λ is known as its *index of pairwise balance*.

2. Methods of Construction

In this section we discuss theorems and lemmas, which are the main tool for our constructions.

Theorem 2.1

(a) If there is a t-(v, k, λ) design $(V, \mathcal{B})$ and a t-(k, k', λ') design $(V', \mathcal{B}')$, then there is a t-$(v, k', \lambda\lambda')$ design $(V, \mathcal{B}'')$.

(b) If $(V', \mathcal{B}')$ is simple and $k' > \max\{|B_i \cap B_j| \ : B_i, B_j \in \mathcal{B}, i \neq j\}$, then $(V, \mathcal{B}'')$ is simple.

Proof.

(a) Define a t-(k, k', λ') design on each block of $(V, \mathcal{B})$. $\mathcal{B}''$ consists of all the blocks of these t- (k, k', λ') designs.

(b) Since $(V', \mathcal{B}')$ is simple, no two blocks of $(V, \mathcal{B}'')$ defined on the same blocks of $(V, \mathcal{B})$ contain the same elements. The given condition on k' guarantees no two blocks of $(V, \mathcal{B}'')$ defined on different blocks of $(V, \mathcal{B})$ contain the same elements. ∎

Theorem 2.1(a) generalizes a well known result of Haim Hanani from $t = 2$ to arbitrary t.

Example 1. As an example, if we take any of the five nonisomorphic 2-(15, 7, 3) designs [N] with a 2-(7, 4, 2) design, and apply Theorem 2.1, we obtain five simple 2-(15, 4, 6) designs. By using a test discussed in [BM], these designs are nonisomorphic. They are all listed in Appendix 2 (designs number (6)–(10)).

We now state a theorem which is important for our later constructions.

Theorem 2.2. If there is a 2-(v,k,λ) design $(V,\mathcal{B})$, then there is a PBD $(V,\mathcal{B}')$ with v elements and index of pairwise balance $\lambda b - \frac{\lambda(\lambda+1)}{2} + (r-\lambda)^2$.

Proof. We define $\mathcal{B}' = \{B_i \cup B_j \mid B_i, B_j \in \mathcal{B}, i \neq j\}$. Let x and y be any pair of elements of V. For simplicity we will denote the set $\{x,y\}$, by xy. Assume that $B_1, B_2, ..., B_\lambda$ are λ blocks of $\mathcal{B}$, which contain xy, and let $B_{\lambda+1}, ..., B_v$ be the rest of the blocks in $\mathcal{B}$. Then xy is contained in the following $\lambda b - \frac{\lambda(\lambda+1)}{2}$ blocks of $\mathcal{B}'$:

$$B_i \cup B_j, \quad i = 1,2,...,\lambda; \quad j = i+1,...,v.$$

Now there are $r-\lambda$ blocks of $\mathcal{B}$ which contain x but not y, and $r-\lambda$ blocks which contain y but not x. These will produce $(r-\lambda)^2$ more blocks in $\mathcal{B}'$, which also contain xy. Thus every pair xy appears $\lambda b - \frac{\lambda(\lambda+1)}{2} + (r-\lambda)^2$ times in the blocks of $\mathcal{B}'$. ∎

Corollary 2.3 (Morgan) [M]. If there is a symmetric 2-(v, k, λ) design, then there is a 2-$(v, 2k-\lambda, \lambda v - \frac{\lambda(\lambda+1)}{2} + (k-\lambda)^2)$ design.

Proof. It is well known that, in any symmetric 2-(v,k,λ) design, every pair of blocks have λ elements in common. Clearly the blocks of $(V,\mathcal{B}')$ in the proof of Theorem 2.2 all contain precisely $2k-\lambda$ elements. ∎

Professor R.G. Stanton has pointed out that S.A. Lonz in his master's thesis [Lo] has discussed a method on construction of designs, which involves taking the intersection of pairs of blocks in a symmetric design. Applying this method on the complement of a symmetric design, and taking the complement of each block again, amounts to the same method as that of Corollary 2.3.

Example 2. There are five nonisomorphic 2-(15, 7, 3) designs [N]. Applying Corollary 2.3, we obtain five 2-(15, 11, 55) designs, whose complements are five more nonisomorphic simple 2-(15, 4, 6) designs. These designs are also different from the ones obtained in Example 1. They are listed in Appendix 2 (designs number (1)–(5)).

Next we discuss a lemma which indicates that from a symmetric BIBD, by applying Corollary 2.3, one can always obtain a simple design (possibly with a complementation).

Lemma 2.4. In Corollary 2.3, if $v \geq 2k$, then the resulting design $(V,\mathcal{B}')$ is simple.

Proof. We proceed by contradiction. Suppose for two blocks in $\mathcal{B}'$, say $A \cup B$ and $C \cup D$ where $A, B, C, D \in \mathcal{B}$, we have

$$A \cup B = C \cup D, \ A \neq B, \ C \neq D. \qquad (1)$$

We may assume that at least one of the blocks on the left side is distinct from the ones on the right side. Let E be distinct from C and D. Then (1) implies that $A \cup B = A \cup C \cup D$. Thus, $|A \cup B| = |A \cup C \cup D|$, and then
$|A| + |B| - |A \cap B| = |A| + |C| + |D| - |A \cap C| - |A \cap D| - |C \cap D| + |A \cap C \cap D|$.

Substituting for the values of known cardinalities, we obtain

$$2k - \lambda = 3k - 3\lambda + |A \cap C \cap D|$$

or

$$|A \cap C \cap D| = 2\lambda - k.$$

This implies that

$$2\lambda - k \geq 0. \tag{2}$$

But in any symmetric BIBD, we have

$$\lambda(v-1) = k(k-1). \tag{3}$$

From (2) and (3) we obtain $v \leq 2k-1$, which is in contradiction with the assumption of the lemma. ∎

The next lemma will be used later in the construction of some interesting designs.

Lemma 2.5. In Corollary 2.3, in the resulting design $(V, \mathcal{B}')$, every pair of blocks have at least λ and at most $\max\{4\lambda, k+\lambda\}$ elements in common.

Proof. Since the intersection of every pair of blocks in a symmetric design $(V, \mathcal{B})$ is λ, the lower bound is trivial. To prove the upper bound, let $A \cup B$ and $C \cup D$ be two blocks of $\mathcal{B}'$, where $A, B, C, D \in \mathcal{B}$. Without loss of generality, we need consider only the following two cases:

Case 1. All four blocks A, B, C and D are distinct. Then

$$|(A \cup B) \cap (C \cup D)| = |(A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)| \leq 4\lambda$$

Case 2. $A = C$. Then

$$|(A \cup B) \cap (A \cup D)| = |A \cup (B \cap D)| \leq |A| + |B \cap D| = k + \lambda. \blacksquare$$

Example 3. There are exactly 78 nonisomorphic 2-$(25, 9, 3)$ designs [D]. If we apply Corollary 2.3, we obtain 78 simple 2-$(25, 15, 105)$ designs. From each of these designs and a trivial 2-$(15, 14, 13)$ design, there will result a 2-$(25, 14, 1365)$ design, by applying Theorem 2.1. By complementing these designs, we obtain simple 2-$(25, 11, 825)$ designs. Existence of designs with these parameters was previously unknown [CCK]. Whether all of these designs are nonisomorphic is under investigation.

3. Some Families of Simple Designs

In this section we introduce some infinite families of 2-designs and 3-designs. First we recall two well known facts. If there exists a Hadamard matrix of order $4m$, then there exists a symmetric 2-$(4m-1, 2m-1, m-1)$ design. We will call it a *Hadamard design of order m*. Also, if there exists a finite projective plane of order n, then it is a symmetric 2-$(n^2+n+1, n+1, 1)$ design.

3.1. A Family of Simple 2-$(4m-1, m, \frac{m(m-1)}{2})$ Designs

If there exists a Hadamard design of order m, then we may apply Corollary 2.3 to obtain a 2-design $(V, \mathcal{B}')$. The complement of this design will be the desired design.

3.2. A Family of 2-$(4m, m, (2m-1)(m-1))$ Designs

If there is a Hadamard design of order m, then it is well known that, it can be extended to a 3-$(4m, 2m, m-1)$ design. Applying the method of Theorem 2.2 to this design, we obtain a 2-$(4m; 3m, 4m; \mu)$ PBD, where $\mu = 18m^2 - 11m + 2$. In this PBD, there are exactly $4m-1$ blocks of size $4m$, which can be omitted to obtain a 2-$(4m, 3m, 3(2m-1)(3m-1))$ design. The complement of this design is a 2-$(4m, m, (2m-1)(m-1))$ design. These designs sometimes are simple. For a discussion on this see [R].

3.3. Designs from Projective Planes

If there exists a finite projective plane of order n, we may apply Corollary 2.3 to the corresponding design and obtain a simple 2-$(n^2+n+1, 2n+1, n(2n+1))$ design [M].

3.4. Other Families of Simple 2-Designs

By a theorem in [H], page 311, for any prime power q there exists a symmetric 2-$(q^2(q+2), q(q+1), q)$ design. Applying Corollary 2.3 to this design we obtain a simple 2-$(q^2(q+2), q(2q+1), q(q+1)(2q+1)(2q-1)/2)$ design.

By another theorem in [H], page 316, if q and q^2+q+1 are prime powers, then there exists a symmetric 2-$(q^3+3q^2+4q+3, q^2+2q+2, q+1)$ design. Applying Corollary 2.3 to this design we obtain a simple 2-$(q^3+3q^2+4q+3, 2q^2+3q+3, \frac{1}{2}(2q^2+3q+3)(2q^2+3q+2))$ design.

3.5. Three Classes of Simple 3-Designs

3.5.1. Let $(V, \mathcal{B})$ be a 3-$(4m, 2m, m-1)$ design, which is the extension of a Hadamard design of order m. Then by applying Theorem 2.1 on $(V, \mathcal{B})$ and a trivial 3-$(2m, 2m-1, 2m-3)$ design, we obtain a simple 3-$(4m, 2m-1, (m-1)(2m-3))$ design.

3.5.2. If we apply Theorem 2.1 to $(V, \mathcal{B})$ of 3.5.1 and a trivial 3-$(2m, 2m-2, (2m-3)(m-2))$ design, then we obtain a simple 3-$(4m, 2m-2, (2m-3)(m-2)(m-1))$ design.

3.5.3. For every prime power q, and any integer $d \geq 2$, there exists a 3-$(q^d + 1, q+1, 1)$ design (see, for example [**HK**, page 201]). This and a trivial 3-$(q+1, q, q-2)$ design, by applying Theorem 2.1, will result in a simple 3-$(q^d + 1, q, q-2)$ design.

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5. Appendix 1: A Table of Some Simple Designs with $v \leq 30$

In [CCK], a set of tables is presented surveying existence and nonexistence results for simple t-designs of small order ($v \leq 30$). Here we present a list of simple t-designs obtained by employing Theorem 2.1 or Corollary 2.3 or one of several other methods. With the exception of the second listed design, all of those we list are previously unknown according to [CCK]. Several of the entries in our table are obtained using the following permutation lemma (abbrev. P.L.) of [**GPT**]: If a t-(v, k, λ) design $(V, \mathcal{B})$ exists, then it can be chosen to be disjoint from D, a given collection of k-subsets of V, when $v! > |\mathcal{B}||\mathcal{D}|k!(v-k)!$. Several others are obtained using the method of combining (abbrev. Comb.) of [**Li**]: If there exist a t-$(v-1, k-1, \lambda')$ design and a t-$(v-1, k, \lambda'')$ design such that $\lambda'_{t-1} = \lambda' + \lambda''$, then there exists a t-$(v, k, \lambda' + \lambda'')$ design. Note that λ'_{t-1} here is the number of $(t-1)$-subsets in the first design, and also that, if the small designs are both simple, then the resulting design will also be simple.

Newly obtained simple t-(v, k, λ) designs

t-(v, k, λ)	Remarks
2-(21, 7, 756)	Theorem 2.1, 2-(21, 9, 36), 2-(9, 7, 21)
2-(21, 9, 36) (Not New)	Sec. (3.3)
2-(25, 4, 9)	Theorem 2.1, 2-(25, 9, 3), 2-(9, 4, 3)
2-(25, 4, 18)	Theorem 2.1, 2-(25, 9, 3), 2-(9, 4, 6)
2-(25, 4, 21)	Derived of 3-(26, 5, 21) design
2-(25, 4, 27)	Theorem 2.1, 2-(25, 9, 3), 2-(9, 4, 9)
2-(25, 4, 63)	Theorem 2.1, 2-(25, 9, 3), 2-(9, 4, 21)
2-(25, 5, 105)	Theorem 2.1, 2-(25, 9, 3), 2-(9, 5, 35)
2-(25, 10, 45)	Union, 2-(25, 9, 3)
2-(25, 11, 825)	See Example 3.
2-(25, 11, 825s), $s = 2, 3$	P.L. with 2-(25, 11, 825) design
2-(26, 5, 24)	3-(26, 5, 3) design as a 2-design
2-(26, 5, 168)	Derived of 3-(27, 6, 168) design
2-(26, 12, 1980s), $1 \leq s \leq 3$	Comb. 2-(25, 11, 825s), 2-(25, 12, 1155s)
2-(27, 6, 150)	Comb. 2-(26, 5, 24), 2-(26, 6, 126)
2-(27, 6, 1050)	4-(27, 6, 21) design as a 2-design
2-(27, 7, 21)	Sec. (3.1)
2-(27, 7, 21s), $2 \leq s \leq 8$	P.L. with 2-(27, 7, 21) design
2-(27, 11, 330)	Derived of 3-(28, 12, 330) design
2-(27, 11, 330s), $s = 2, 3$	P.L. with 2-(27, 11, 330) design
2-(27, 12, 66)	Derived of 3-(28, 13, 66) design
2-(27, 12, 66s), $2 \leq s \leq 142$	P.L. with 2-(27, 12, 66) design
2-(27, 13, 150s), $1 \leq s \leq 142$	Comb. 2-(26, 12, 66s), 2-(26, 13, 84s)
2-(28, 13, 156)	3-(28, 13, 66) design as a 2-design
2-(28, 13, 156s), $2 \leq s \leq 66$	P.L. with 2-(28, 13, 156) design
2-(29, 13, 105s), $1 \leq s \leq 16$	Comb. 2-(28, 12, 429s), 2-(28, 13, 624s)
3-(26, 5, 3)	Sec. (3.5.3)
3-(26, 5, 21)	Derived of 4-(27, 6, 21) design
3-(26, 6, 147)	Residual of 4-(27, 6, 21) design
3-(27, 6, 168)	4-(27, 6, 21) design as a 3-design
3-(28, 12, 330)	Sec. (3.5.2)
3-(28, 12, 660)	P.L. with 3-(28, 12, 330) design
3-(28, 13, 66)	Sec. (3.5.1)
3-(28, 13, 66s), $2 \leq s \leq 66$	P.L. with 3-(28, 13, 66) design
3-(29, 13, 858)	Comb. 3-(28, 12, 330), 3-(28, 13, 528)
3-(29, 13, 1716)	Comb. 3-(28, 12, 660), 3-(28, 13, 1056)
4-(27, 6, 21)	Theorem 2.1, 4-(27, 7, 7), 4-(7, 6, 3)

6. Appendix 2: A table of 10 nonisomorphic 2-(15, 4, 6) designs

(1)

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111111111111111111111111111122222222222222222222333
22222233333344445555667788aab3333334444555667789aa444
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3333333333333444444444444455555555556666666677777889
455566778889a5556667788889b6666777888777889ac889ad9cb
d9ab9cbd9abab9ab79a9b9accd7adeacc9cd89cacabd9dabebec
fefcfdfeedceccde8fbafbfede8eefcdfbeffbebdfdfaedffffe
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(2)

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887fbe768afd7deffedeccecdfdf678afe6dcfdcfceecdedff8ce

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455566778889a5556667788889b6666777888777889ad88aac9cb
d9abbe9c9abab9ab79b9a9accd79ceacd9cd89cacace9dbddbec
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(3)

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(4)

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455566778889b5556666777888666677788877788ac889d9cabb
d9ab9aab9abce9ab79ac9bd9ac7ace9cd9cd89c9dbdacaebedcc
fcfedecdedcffedc8fbdafebfe8bdfafebeffbeaeffbdffffeed
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(5)

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33333333333334444444444444555555555566666677799999aac
445556667778885556667778886666777888777888888aabbbbd
ce9ab9ab9ab9ab9ab9ab9ab9ab7ace9cd9cd9cd9cdacecdcdcee
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(6)

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(7)

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(8)

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(9)

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fbad9affdeecfbefffeeedcccdddcebabfffcdeffedfeefdffef
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(10)

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