On the Spectrum of the Closed-Set Lattice of a Graph

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Abstract. Let $\mathcal{L}(G)$ be the closed-set lattice of a graph G, and let $l(\Gamma)$ denote the length of a chain Γ in $\mathcal{L}(G)$. The *spectrum* of $\mathcal{L}(G)$ is defined as the set

$$\mathcal{S}(\mathcal{L}(G)) = \{l(\Gamma) \mid \Gamma \text{ is a maximal chain in } \mathcal{L}(G)\}.$$

For every nontrivial graph $G, \mathcal{S}(\mathcal{L}(G))$ is a finite set of natural numbers greater than one. We prove in this paper that (*) *for any finite set A of natural numbers greater than one, there exists a graph G such that $\mathcal{S}(\mathcal{L}(G)) = A$.*

A set S of vertices of a graph G is said to be *k-independent* if $d(u, v) \geq k$ for all distinct members u, v in S. A k-independent set of G is said to be *maximal* if it is not properly contained in any k-independent set of G. To prove the result (*), we first establish the following : Given any finite set B of natural numbers, there exists a graph G such that

$$\{| S || S \text{ is a maximal 3-independent set of } G\} = B.$$

Key Words : Closed-set lattice of a graph
Lexicographic extension of graphs
k-independent set of a graph
Spectrum of a lattice

AMS Subject Classification (1980) : 05C99
06B99

1. Introduction.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For each a in $V(G)$, let $N(a) = \{x \in V(G) \mid ax \in E(G)\}$ be the set of *neighbours* of a. A subset S of $V(G)$ is called a *closed* set of G if, for each pair of distinct elements a, b in $S, N(a) \cap N(b) \subseteq$ S. Let $\mathcal{L}(G)$ be the family of closed sets of G, inclusive of the empty set $\emptyset$. It is evident that the family $\mathcal{L}(G)$ forms under set-inclusion a lattice with least element $\emptyset$ and greatest element $V(G)$ in which the *meet* $A \wedge B$ is the set-intersection $A \cap B$ and the *join* $A \vee B$ is the closed set of G generated by $A \cup B$ in G(i.e., the intersection of all closed sets containing $A \cup B$) for any pair of members A, B in $\mathcal{L}(G)$. The lattice $\mathcal{L}(G)$, which was first introduced by N. Sauer (see [13]), is called the *closed-set lattice* of the graph G.

In $[3, 4, 6-11]$ we investigate the relation between the graph structure of G and the lattice structure of $\mathcal{L}(G)$. In [5,12] we study the lengths of maximal chains in the lattice $\mathcal{L}(G)$ when G is a tree. The *length* of a chain Γ in a finite lattice, denoted by $l(\Gamma)$, is defined by $l(\Gamma) = \mid \Gamma \mid -1$. The *spectrum* of $\mathcal{L}(G)$, denoted by $\mathcal{S}(\mathcal{L}(G))$, is defined as the set

$$\mathcal{S}(\mathcal{L}(G)) = \{l(\Gamma) \mid \Gamma \text{ is a maximal chain in } \mathcal{L}(G)\}.$$

For every nontrivial graph G, the set $\mathcal{S}(\mathcal{L}(G))$ is a finite set of natural numbers greater than one. A set J of natural numbers is said to be *dense* if, whenever $a \leq x \leq b$ where x is a natural number and $a, b \in J$, then $x \in J$. In [5,12], we show that if the graph G is a tree, then the set $\mathcal{S}(\mathcal{L}(G))$ is always dense. The result is however no longer true if G is not a tree. It is the aim of this paper to establish the following result.

Theorem A. Given any finite set A of natural numbers greater than one, there always exists a graph G such that

$$\mathcal{S}(\mathcal{L}(G)) = \text{A}.$$

Throughout this note every graph is assumed to be finite and connected. A set B is said to be of *order* k if $\mid B \mid = $ k. For a subset V of $V(G)$, we shall denote by $< V >$ the closed set of G *generated* by V and $[V]$ the subgraph of G *induced* by V. For each u in $V(G)$, $d(u, V) = \min\{d(u,v) \mid v \in V\}$, where $d(u, v)$ is the *distance* between u and v in G. The *diameter* of the induced subgraph $[V]$ of G, denoted by diam($[V]$), is defined by $\text{diam}([V]) = \max\{d_{[V]}(u, v) \mid u, v \in V\}$, where $d_{[V]}(u, v)$ is the distance between u and v in $[V]$. For all terminology on graphs and lattices not explained here, we refer to [1] and [2] respectively.

2. Lexicographic Extension of Graphs.

In the remainder of this paper, let G be a graph of order $n(\geq 2)$ with $V(G) = \{x_1, x_2, \ldots, x_n\}$. Let $\{H_1, H_2, \ldots, H_n\}$ be a family of n graphs. The *G-lexicographic extension* of $H_1, H_2, \ldots, H_n$, denoted by $G(H_1, H_2, \ldots, H_n)$, is the graph with

$$V(G(H_1, H_2, \ldots, H_n)) = \dot{\bigcup}\{V(H_i) \mid i = 1, 2, \ldots, n\}$$

and

$$E(G(H_1, H_2, \ldots, H_n)) = \dot{\bigcup}\{E(H_i) \mid i = 1, \ldots, n\}$$
$$\dot{\bigcup}\{uv \mid u \in V(H_j), v \in V(H_k), x_j x_k \in E(G)\}.$$

We shall denote by $\mathcal{K}(G)$ the class of all G-lexicographic extensions of n nontrivial complete graphs. That is,

$$\mathcal{K}(G) = \{G(H_1, H_2, \ldots, H_n) \mid \text{each } H_i \text{ is a complete graph of order at least 2}\}.$$

A subset S of $V(G)$ is called an *r-independent set* of a graph G if $d(x, y) \geq r$ for any pair of distinct elements x, y in S. An r-independent set S of G is said to be *maximal* if it is not properly contained in any r-independent set of G. In what follows we shall show that the order of a maximal 3-independent set of G determines the length of a maximal chain in $\mathcal{L}(H)$ for each H in $\mathcal{K}(G)$.

First of all we have the following observation.

Lemma 1. Let H be in $\mathcal{K}(G)$ and x, y be in $V(H)$. Then $<\{x,y\}>= V(H)$ if and only if $d(x,y) = 1$ or 2.

Proof. It is clear that $<\{x,y\}>= V(H)$ if $d(x,y) = 1$ or 2. If $d(x,y) \geq 3$, then $<\{x,y\}>= \{x,y\} \neq V(H)$. □

Let $H \in \mathcal{K}(G)$ and let A be a proper closed set of the graph $H = G(H_1, H_2, \ldots, H_n)$. By Lemma 1, $d(a,b) \geq 3$ for every pair of distinct elements a, b in A. Thus $\mid A \cap H_i \mid \leq 1$ for each $i = 1, 2, \ldots, n$. Let $I = \{i \mid A \cap H_i \neq \emptyset\}$. Then $\{x_i \mid i \in I\}$ is a 3-independent set of G. Conversely, if $\{x_{\alpha(1)}, x_{\alpha(2)}, \ldots, x_{\alpha(k)}\}$ is a 3-independent set of G, let a_j be an element in $H_{\alpha(j)}$ for each $j = 1, 2, \ldots, k$. Then $A = \{a_1, a_2, \ldots, a_k\}$ is a proper closed set of H. The following result thus follows.

Lemma 2. Let H be in $\mathcal{K}(G)$. Then H has a proper (resp., maximal proper) closed set of order k if and only if G has a 3-independent (resp., maximal 3-independent) set of order k. □

Lemma 3. Let H be in $\mathcal{K}(G)$. Then the lattice $\mathcal{L}(H)$ has a maximal chain of length $k+1$ if and only if G has a maximal 3-independent set of order k.

Proof. Let $\{x_{\alpha(1)}, x_{\alpha(2)}, \ldots, x_{\alpha(k)}\}$ be a maximal 3-independent set of G. For each $j = 1, 2, \ldots, k$, let a_j be an element in $H_{\alpha(j)}$ and $A_j = \{a_t \mid t \leq j\}$. By Lemma 2, $\emptyset = A_0 -< A_1 -< \ldots -< A_k -< V(H)$ is a maximal chain of length $k+1$ in $\mathcal{L}(H)$.

Conversely, if $\emptyset = A_0 -< A_1 -< \ldots -< A_k -< V(H)$ is a maximal chain in $\mathcal{L}(H)$, then by Lemma 1, A_k is of order k. By Lemma 2, G has a corresponding maximal 3-independent set of order k. □

As an immediate consequence of Lemma 3 we have :

Corollary. Let H be in $\mathcal{K}(G)$. Then

$$\mathcal{S}(\mathcal{L}(H)) = \{k+1 \mid G \text{ has a maximal 3-independent set of order } k\}. \square$$

3. Two Fundamental Constructions.

The corollary to Lemma 3 suggests a way to prove Theorem A. Given a finite set A of natural numbers greater than one, let $A' = \{k-1 \mid k \in A\}$. If a graph G can be constructed in such a way that the set of orders of its maximal 3-independent sets is equal to A', then $\mathcal{S}(\mathcal{L}(H)) = A$ for any H in $\mathcal{K}(G)$. In this section we shall introduce two methods of construction which enable us to construct graphs whose sets of orders of all maximal 3-independent sets are equal to two special sets of natural numbers. These will then be applied to prove our main result in the final section.

Lemma 4. For each natural number r, there exists a graph P such that every maximal 3-independent set of P is of order r.

Proof. Construct a graph P with

$$V(P) = \cup\{V_i \mid i = 1, 2, \ldots, r\}$$

such that the following conditions hold :

(i) $V_i \cap V_j = \emptyset$ if $i \neq j$,

(ii) $\mathrm{diam}([V_i]) = 2$ for each $i = 1, 2, \ldots, r$, and

(iii) for each $i = 1, 2, \ldots, r$, there exists a vertex u_i in V_i such that

$$d(u_i, V_j) \geq 3 \text{ for each } j = 1, 2, \ldots, r \text{ with } j \neq \text{i}.$$

Such a graph P can easily be constructed (see the example in Figure 1). We claim that every maximal 3-independent set of P is of order r.

Let S be a maximal 3-independent set of P. By (ii), $\mid S \cap V_i \mid \leq 1$ for each $i = 1, 2, \ldots, r$. If $\mid S \cap V_j \mid = 0$ for some $j = 1, 2, \ldots, r$, then by (iii), $S \cup \{u_j\}$ is a 3-independent set of P, which however contradicts the maximality of S. Thus $\mid S \cap V_i \mid = 1$ for each $i = 1, 2, \ldots, r$. Hence by (i), $\mid S \mid = r$, as required. □

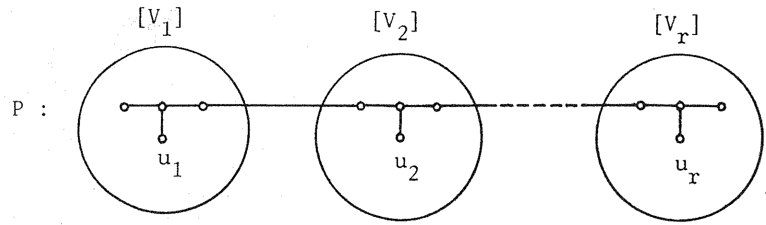

Figure 1. A graph in which every maximal 3-independent set is of order r

Lemma 5. Let $\{n_1, n_2, \ldots, n_q\}$ be a set of natural numbers where $q \geq 1$. There exists a graph Q such that the set of orders of all maximal 3-independent sets of Q is $\{1, n_1 + 1, n_2 + 1, \ldots, n_q + 1\}$.

Remark. Lemma 5 simply says that every finite set of natural numbers including one is the set of orders of all maximal 3-independent sets of a graph.

Proof. We may assume $n_1 < n_2 < \ldots < n_q$. Let K_p be the complete graph of order $p = 1 + q + n_q$ with $V(K_p) = \{a_1, a_2, \ldots, a_q, b_1, b_2, \ldots, b_{n_q}, w\}$ and let

$\{c_1, c_2, \ldots, c_{n_q}\}$ be a set of vertices disjoint from $V(K_p)$. Define a graph Q with

$$V(Q) = V(K_p) \dot{\cup} \{c_1, c_2, \ldots, c_{n_q}\}$$

and

$$E(Q) = (E(K_p) - \bigcup_{i=1}^{q} \{a_i b_j \mid j = 1, 2, \ldots, n_i\}) \dot{\cup} \{b_i c_i \mid i = 1, 2, \ldots, n_q\}.$$

We claim that the set of orders of all maximal 3-independent sets of Q is $\{1, n_1 + 1, \ldots, n_q + 1\}$.

Let S be a maximal 3-independent set of Q. There are four cases to be considered.

Case (i). $a_i \in S$ for some $i = 1, 2, \ldots, q$.

In this case, $S = \{a_i, c_1, c_2, \ldots, c_{n_i}\}$ and hence $\mid S \mid = n_i + 1$.

Case (ii). $b_i \in S$ for some $i = 1, 2, \ldots, n_q$.

We have $S = \{b_i\}$ and hence $\mid S \mid = 1$.

Case (iii). $c_i \in S$ for some $i = 1, 2, \ldots, n_q$.

If $S \cap \{a_1, a_2, \ldots, a_q\} = \emptyset$, then $S \subset \{a_q, c_1, c_2, \ldots, c_{n_q}\}$, which contradicts the maximality of S since the latter is a 3-independent set of Q. Thus $a_i \in S$ for some $i = 1, 2, \ldots, q$, which has already been dealt with in case (i).

Case (iv). $w \in S$.

We then have $S = \{w\}$ and hence $\mid S \mid = 1$.

The proof of Lemma 5 is thus complete. □

4. The Main Result.

We are now in a position to prove Theorem A. By the corollary to Lemma 3, this is equivalent to proving the following result.

Theorem B. Let $\{s_0, s_1, \ldots, s_q\}$ be a set of natural numbers where $q \geq 0$. There exists a graph R such that the set of orders of all maximal 3-independent sets of R is $\{s_0, s_1, \ldots, s_q\}$.

Proof. By Lemma 3, the result is clearly true if $q = 0$. Hence we may assume $q \geq 1$ and $s_0 < s_1 < \ldots < s_q$.

If $s_0 = 1$, then by letting $n_i = s_i - 1$ for each $i = 1, 2, \ldots, q$, we obtain a sequence of natural numbers $n_1 < n_2 < \ldots < n_q$. By Lemma 5, there exists a graph Q such that the set of orders of its maximal 3-independent sets is $\{1, n_1 + 1, n_2 + 1, \ldots, n_q + 1\}$, i.e., $\{s_0, s_1, \ldots, s_q\}$.

Assume now $s_0 \geq 2$ and let $n_i = s_i - s_0$ for each $i = 1, 2, \ldots, q$. Then a graph Q, whose set of orders of all maximal 3-independent sets is $\{1, n_1+1, n_2+1, \ldots, n_q+1\}$, can be constructed as given in the proof of Lemma 5. Further, let $r = s_0 - 1 \geq 1$. Then there exists by Lemma 4, a graph P in which every maximal 3-independent set is of order r. We now refer to the construction of P as given in the proof of Lemma 4. For each $i = 1, 2, \ldots, r$, let $F_i = \{v \mid v \in V_i, d(v, u_i) = 2\}$ and let $F = \dot{\bigcup}\{F_i \mid i = 1, 2, \ldots, r\}$. Define a graph R such that

$$V(R) = V(P) \dot{\cup} V(Q) \dot{\cup} \{z\} \text{ and } E(R) = E(P) \dot{\cup} E(Q) \dot{\cup} \{wz\} \dot{\cup} Z,$$

where w is the vertex of Q defined in the proof of Lemma 5, z is a new vertex and Z is any nonempty subset of the set $\{az \mid a \in F\}$.

We claim that the set of orders of all maximal 3-independent sets of R is $\{s_0, s_1, \ldots, s_q\}$.

Let S be a maximal 3-independent set of R. By applying a similar argument as developed in the proof of Lemma 4, we have $| S \cap V_i | = 1$ for each $i = 1, 2, \ldots, r$ and thus $| S \cap V(P) | = r$. Let $S' = S \cap (V(Q) \cup \{z\})$. There are five cases to be considered.

Case (i). $a_i \in S'$ for some $i = 1, 2, \ldots, q$.

In this case, $S' = \{a_i, c_1, c_2, \ldots, c_{n_i}\}$ and hence $| S' | = n_i + 1$.

Case (ii). $b_i \in S'$ for some $i = 1, 2, \ldots, n_q$.

We have $S' = \{b_i\}$ and hence $| S' | = 1$.

Case (iii). $z \in S'$.

In this case, $S' = \{z, c_1, c_2, \ldots, c_{n_q}\}$ and hence $| S' | = n_q + 1$.

Case (iv). $c_i \in S'$ for some $i = 1, 2, \ldots, n_q$.

If $S' \cap \{z, a_1, a_2, \ldots, a_q\} = \emptyset$, then $S' \subset \{a_q, c_1, c_2, \ldots c_{n_q}\}$ and hence $S \subset (S \cap V(P)) \cup \{a_q, c_1, c_2, \ldots, c_{n_q}\}$, which contradicts the maximality of S since the latter is a 3-independent set of R. Thus $S' \cap \{z, a_1, a_2, \ldots, a_q\} \neq \emptyset$, which has already been dealt with in either case (i) or case (iii).

Case (v). $w \in S'$.

We then have $S' = \{w\}$ and hence $| S' | = 1$.

Now, we have

$$| S | = | S \cap V(P) | + | S \cap (V(Q) \cup \{z\}) |$$

and thus either $| S | = r + 1 = s_0$

or $| S | = r + (n_i + 1) = s_i$, where $i = 1, 2, \ldots, q$.

The proof of Theorem B is thus complete. □

Remark. Combining Theorem B with the corollary to Lemma 3, we actually arrive at the following result which is somewhat stronger than Theorem A : *Given* *a finite set A of natural numbers greater than one, there always exists a graph G such that $\mathcal{S}(\mathcal{L}(H)) = A$ for every $H \in \mathcal{K}(G)$.*

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