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Further results on proper and strong set colorings of graphs

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Abstract

A *set coloring* α of a graph G is defined as an assignment of distinct subsets of a finite set X of *colors* to the vertices of G such that all the colors of the edges which are obtained as the symmetric differences of the sets assigned to their end-vertices are distinct. Additionally, if all the sets on the vertices and edges of G form the set of all nonempty subsets of X, then the coloring α is said to be a *strong set coloring*, and the graph G is called *strongly set colorable*. If all the nonempty subsets of X are obtained on the edges of G, then α is called a *proper set coloring*, and such a graph G is called *properly set colorable*. The *set coloring number* of a graph G, denoted by $\sigma(G)$, is the smallest cardinality of a set X such that G has a set coloring with respect to X.

This paper discusses the set coloring number of certain classes of graphs and the construction of strongly set colorable caterpillars which are also properly set colorable. An upper bound for b is found for $K_{3,b}$ to admit set coloring with set coloring number n.

1 Introduction

For all standard notation and terminology in graph theory we follow Harary [4] and West [7]. In this paper we consider only finite simple graphs.

The notion of set coloring of a graph was introduced by Hegde [5] in 2009. Acharya [1] initiated a general study of labeling of the vertices and the edges of a graph using subsets of a set and indicated their potential application in a variety of other areas of human enquiry. Given a graph $G = (V, E)$ and a nonempty set X of n colors, a function $f : V \to 2^X$ can be defined as the assignment of the colors $f(v)$, to each of the vertices $v \in V$, and given such a function f on the vertex set V, we define $f^{\oplus} : E \to 2^X$ which assigns colors to the edges $e = uv \in E$ as $f^{\oplus}(e) = f(u) \oplus f(v)$.

A graph G is said to be a *set colorable* graph if both f and $f^{\oplus}$ are injective. A graph G is said to be *properly set colorable* if it is set colorable with $f^{\oplus}(E) = 2^X \setminus \emptyset$, and G is said to be *strongly set colorable* if $f(V) \cup f^{\oplus}(E) = 2^X \setminus \emptyset$ and $f(V) \cap f^{\oplus}(E) = \emptyset$.

2 Set coloring number

The *set coloring number* $\sigma(G)$ of a graph G is the least cardinality of a set X with respect to which G has a set coloring.

For any graph $G(V, E)$, we have $\lceil \log_2\{|E| + 1\} \rceil \leq \sigma(G) \leq |V| - 1$, and the bounds are best possible as mentioned in Hegde [5]. In this section we find the set coloring number of some classes of cycles and complete bipartite graphs.

Theorem 1 *Given any positive integer $n \geq 3$, $\sigma(C_{2^n-3}) = n + 1 = \sigma(C_{2^n-2})$.*

Proof: Let the vertices of C_{2^n-3} be denoted by $v_1, v_2, \ldots, v_{2^n-3}$ such that $(v_i, v_{i+1}) \in E$ for all $1 \leq i \leq 2^n - 3$; then $\sigma(C_{2^n-3}) \geq n$. Let us assume that there exist a set coloring $(f, f^{\oplus})$ of C_{2^n-3} with respect to a set X of $|X| = n$. Then since both f and $f^{\oplus}$ are injective, the edges have distinct colors. That is,

$$f^{\oplus}(v_1, v_2) \oplus f^{\oplus}(v_2, v_3) \oplus \cdots \oplus f^{\oplus}(v_{2^n-3}, v_1) = \emptyset.$$

Let $A_1, A_2, \ldots, A_{2^n-1}$ be the distinct nonempty subsets of X. Assign $A_i = f^{\oplus}(v_i, v_{i+1})$ for $i = 1, 2, \ldots, 2^n - 3$; then we get

$$A_1 \oplus A_2 \oplus \cdots \oplus A_{2^n-3} = \emptyset. \tag{1}$$

But we know

$$A_1 \oplus A_2 \oplus \cdots \oplus A_{2^n-3} \oplus A_{2^n-2} \oplus A_{2^n-1} = \emptyset. \tag{2}$$

From (1) and (2) we get $A_{2^n-2} \oplus A_{2^n-1} = \emptyset \;\Rightarrow\; A_{2^n-2} = A_{2^n-1}$, which is a contradiction. Therefore $\sigma(C_{2^n-3}) > n$.

When $|X| = n+1$, by using the algorithm given in Molard and Payan [6], the subsets of X are assigned to the vertices of C_{2^n-3} to get the set coloring of C_{2^n-3}.

One can prove by using the same argument that $\sigma(C_{2^n-2}) = n + 1$. □

Theorem 2 *Given any positive integer $n \geq 3$, $\sigma(K_{3,2^{n-2}+1}) = n+1$.*

Proof: Let $\sigma(K_{3,b}) = n$. We shall prove that the maximum value that b can have is 2^{n-2}. Let V_1 (containing three vertices) and V_2 be the partition of the vertex set V, and let $A_1, A_2, \dots, A_{2^n}$ be the subsets of a nonempty set X of cardinality n. Assign the sets A_1, A_2, A_3 to the vertices of V_1 under the mapping $(f, f^{\oplus})$ defined for the sets $V(G)$ and $E(G)$ respectively.

If $A_1 \oplus A_2 = \{x_1, x_2, \dots, x_k\}$, then let $x_1, x_2, \dots, x_j \in A_1$ and $x_{j+1}, x_{j+2}, \dots, x_k \in A_2$. If A_k is any other set containing $\{x_1, x_2, \dots, x_{k-1}\}$ and A_r is a set containing x_k such that $A_k - \{x_1, x_2, \dots, x_{k-1}\} = A_r - \{x_k\}$, then $A_1 \oplus A_k = A_2 \oplus A_r$, which is a contradiction. Therefore, those sets containing x_k cannot be considered to assign the vertices of V_2 and the number of subsets of X not containing x_k are 2^{n-1} out of which A_1 is assigned to a vertex of V_1. Therefore the possible number of subsets that can be considered to assign the vertices of V_2 is $2^{n-1} - 1$. Again, if $A_2 \oplus A_3 = \{x'_1, x'_2, \dots, x'_m\}$, then by the similar argument mentioned above, sets containing one of x'_j, $j = 1, 2, \dots, m$, cannot be considered for the assignment to the vertices of V_2. The number of subsets out of $2^{n-1} - 1$ not containing x'_j is 2^{n-2}, out of which one set (A_2 or A_3) is assigned to a vertex of V_1. Since the set $A_1 \oplus A_2 \oplus A_3$ can be used for the assignment to a vertex of V_2, the maximum number of sets that are used for the assignment to the vertices of V_2 is 2^{n-2}. Thus $\sigma(K_{3,2^{n-2}}) = n$. Hence $\sigma(K_{3,2^{n-2}+1}) = n+1$. □

3 Properly and strongly set colorable graphs

In Hegde [5], it has been mentioned that a necessary condition for a (p, q)-graph G to be properly set colorable is that $q + 1 = 2^m$ for the positive integer $m = |X|$. This condition says that cycles of length not equal to $2^m - 1$ are not properly set colorable.

Molard and Payan [6] proved the following theorem, which says that the cycles of length $2^m - 1$ are properly set colorable.

Theorem 3 *For every integer $n \geq 2$, it is possible to label the vertices of the cycle C_{2^n-1} by all non zero vectors of the vector space $GF^n(2)$ of dimension n over the finite Galois field $GF(2)$ such that the vectors $x \oplus y$ with x and y adjacent, are also all distinct non-zero vectors of $GF^n(2)$, where $\oplus$ denotes addition in $GF^n(2)$.*

Using this result we shall prove the following.

Theorem 4 *For any two positive integers k_1 and k_2, $C_{2^{k_1}-1} + C_{2^{k_2}-1}$ is properly set colorable.*

Proof: Consider $G = C_{2^{k_1}-1} + C_{2^{k_2}-1}$ where k_1, k_2 are integers. Then the total number of edges in G is $2^{k_1} - 1 + 2^{k_2} - 1 + (2^{k_1} - 1)(2^{k_2} - 1) = 2^{k_1+k_2} - 1$. By taking $k_1 + k_2 = m$, the necessory conditions for the existance of proper set coloring is satisfied.

Let $GF^m(2)$ be a vector space over a finite Galois field $GF(2)$, and let $P(x)$ and $Q(y)$ be two primitive irreducible polynomials of degree k_1 and k_2 respectively. Then $\{x^i \bmod P(x)\}$ and $\{y^j \bmod Q(y)\}$ are two disjoint sets of non-zero polynomials of degree at the most $2^{k_1} - 1$ and $2^{k_2} - 1$, respectively, over $GF(2)$. Then label the vertices of the cycles $C_{2^{k_1}-1}$ and $C_{2^{k_2}-1}$ clockwise respectively by $x^1, x^2, \ldots, x^{2^{k_1}-1}$ and $y^1, y^2, \ldots, y^{2^{k_2}-1}$, where $x^i \neq y^j$ as explained in Mollard and Payan [6].

Analogously we assign the subsets of a nonempty set X of cardinality m to the vertices of G in place of polynomials as follows:

Let $X_1 = \{x_1, x_2, \ldots, x_{k_1-1}\}$ and $X_2 = \{y_1, y_2, \ldots, y_{k_2-1}\}$ be two disjoint nonempty subsets of X where $X_1 \cup X_2 = X$. The term x^i in each polynomial $\{x^i \bmod P(x)\}$ is replaced by an element x_i of X_1 for $i = 0, 1, 2, \ldots, k_1 - 1$ and y^j in each polynomial $\{y^j \bmod Q(y)\}$ is replaced by y_j of X_2 for $j = 1, 2, \ldots\ k_2 - 1$. Then the polynomials $\{x^i \bmod P(x)\}$ are replaced by the subsets of X_1 by replacing each term of the polynomials by the respective element in X_1. For instance if $x = x_1$ and $x^2 = x_2$ then $x + x^2 = \{x_1, x_2\}$, and so on. Similarly, the polynomials $\{y^j \bmod Q(y)\}$ are replaced by the subsets of X_2. One can easily observe that G is properly set colorable. □

For example, a proper set coloring of $C_{2^3-1} + C_{2^2-1}$ is displayed in Figure 1.

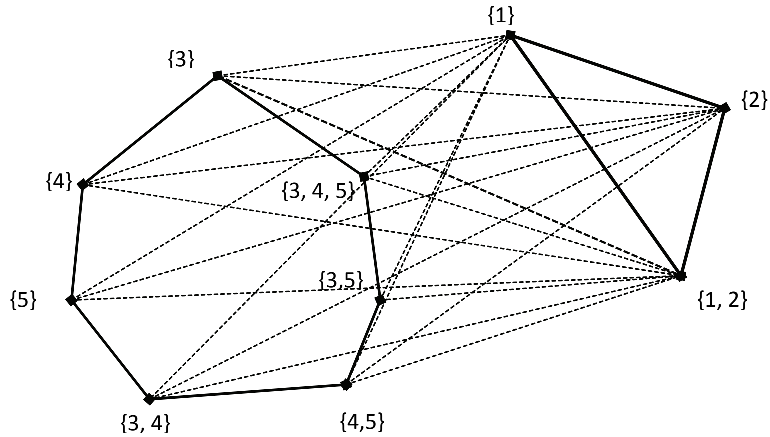

Figure 1: Properly set colored $C_7 + C_3$.

Definition 1 (Gallian [3]) Umbrella graphs are the graphs obtained by joining an edge to the center of a wheel.

From the necessary condition mentioned in Hegde [5], it is clear that no wheel is properly set colorable. The following result shows that the umbrella graphs of the form U_{2^n-1} are properly set colorable.

Corollary 4.1 *An umbrella graph of the form U_{2^n-1} is properly set colorable.*

Proof: Consider the wheel of the form W_{2^n-1}, where n is the cardinality of a set X. Let the pendant vertex w be joined to the central vertex v of the wheel.

For any $x \notin X$, let $X' = X \cup \{x\}$. Assign the set X' to the central vertex, and $\emptyset$ to the pendant vertex w. According to the method explained in Theorem 4, the vertices of the outer cycle C_{2^n-1} are assigned by the nonempty subsets of X. One can easily verify that U_{2^n-1} is properly set colorable. □

Figure 2 is an example of a proper set coloring of U_7.

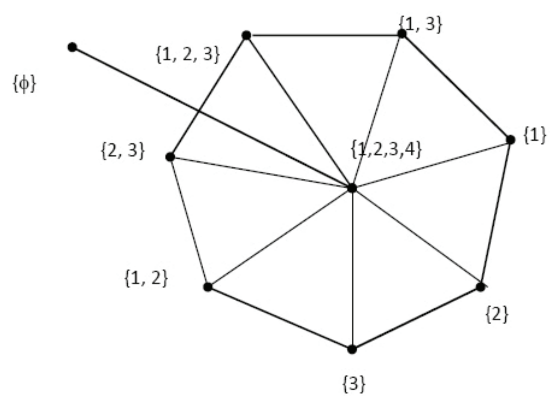

Figure 2: Properly set colored U_7.

As mentioned in Hegde [5], a necessary condition for a (p, q) graph G to be strongly set colorable is that $p + q + 1 = 2^m$ for the positive integer $m = |X|$. Given below is a relation between strongly set colorable trees and properly set colorable trees.

We call a set having odd number of elements an *odd set* and having even number of elements an *even set.*

Theorem 5 *a) Every properly set colorable tree is strongly set colorable.*
b) A strongly set colorable tree whose vertex labels are odd sets is properly set colorable.

Proof: a) Let T be a properly set colorable tree with a proper set coloring f with respect to a nonempty set X of cardinality m. Let $X' = X \cup \{x\}$. Since

T is properly set colorable, all the subsets of X are assigned to the vertices and $f(T) = \{f(v) \mid v \in V(T)\} = 2^X$ and $f^{\oplus}(T) = \{f(e) \mid e \in E(T)\} = 2^X - \emptyset$.

Define a function $F : V(T) \to 2^{X'}$ by $F(v) = f(v) \cup \{x\}$, for all $v \in V(T)$.

Since f and $f^{\oplus}$ are injective, F and $F^{\oplus}$ are also injective. Also $F(V(T)) \cap F^{\oplus}(E(T)) = \emptyset$.

Since $f(T) = 2^X$ and $f^{\oplus}(T) = 2^X - \emptyset$, we get $F(T) = 2^{X'} - 2^X$ and $F^{\oplus}(T) = 2^X - \emptyset$. That is, $f^{\oplus}(T) = F^{\oplus}(T)$. Further,

$$\begin{aligned} |F(T)| &= 2^{|X'|} - 2^{|X|} \\ &= 2^{m+1} - 2^m \\ &= 2^m(2-1) \\ &= 2^m \quad \text{and} \quad |F^{\oplus}(T)| = 2^m - 1. \end{aligned}$$

Therefore $|F(T)| + |F^{\oplus}(T)| = 2^m + 2^m - 1 = 2^{m+1} - 1 = 2^{|X'|} - 1$. This implies that F is a strong set coloring of T.

b) Let f be a strong set coloring of T with respect to the set X of n colors, such that $f(v)$ is an odd set for each $v \in V(T)$.

Define $F : V(T) \cup E(T) \longrightarrow 2^{X'}$ where $X' = X - \{w\},\ w \in X$ as

$$F(v)) = \begin{cases} f(v) - \{w\} & \text{if } w \in f(v) \\ f(v) & \text{if } w \notin f(v) \\ \emptyset & \text{if } w = f(v). \end{cases}$$

Then we have $F^{\oplus}(uv) = f^{\oplus}(uv) - \{w\}$ when $w \in f(u)$ or $w \in f(v)$, and $F^{\oplus}(uv) = f^{\oplus}(uv)$ when $w \in f(u)$ and $w \in f(v)$, or $w \notin f(u)$ and $w \notin f(v)$.

This implies that $F^{\oplus}$ is injective. Further, if $F(v) = f(v) - \{w\}$ then $F(v)$ is an even set and if $F(u) = f(u)$ then $F(u)$ is an odd set, so that $F^{\oplus}(uv)$ becomes an odd set. The number of even subsets in $f(T)$ is $2^{n-1} - 1$, out of which 2^{n-2} subsets contain w. Therefore the number of odd sets on the edges under F is 2^{n-2}. The remaining $2^{n-2} - 1$ subsets are even sets on the edges under F. Hence the total number of subsets assigned to the edges of T is $2^{n-2} + 2^{n-2} - 1 = 2^{n-1} - 1 = 2^{|X'|} - 1$. This implies that all the nonempty subsets of X' are on the edges of T. Hence F is a proper set coloring of T. □

Figure 3 illustrates that a properly set colorable tree is strongly set colorable and Figure 4 illustrates that a strongly set colorable tree with odd sets assigned to the vertices is properly set colorable.

Definition 2 (Chen [2]) An (n, k)-banana tree, denoted by $B(n, k)$ is a graph obtained by connecting one leaf of each of n copies of a k-star graph with a single root vertex that is distinct from all the stars.

Theorem 6 *For any positive integer, $r \geq 2$, a banana tree $B(k, n)$, where $k = 2^{r/2} - 1$ and $n = 2^{r/2}$, is properly set colorable.*

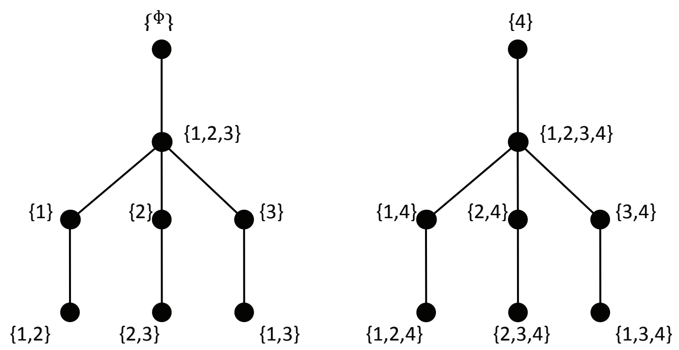

Figure 3: An illustrative example of Theorem 5(a).

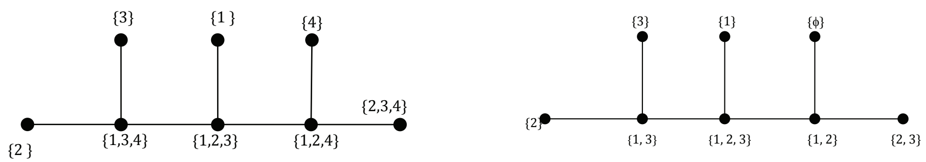

Figure 4: An illustrative example of Theorem 5(b).

Proof: Let w_0 be the central vertex, and $w_1, w_2, \ldots, w_k$ be the roots of the stars joining the central vertex. Let $v_{i,1}, v_{i,2}, \ldots, v_{i,n}$ denote the pendant vertices joining w_i where $i = 1, 2, \ldots, k$. Let X be a nonempty set with $|X| = r$ and let X' be a subset of X where $|X'| = r/2$.

We define a mapping $f : V(B) \cup E(B) \longrightarrow 2^X$ as follows:

$f(w_0) = \emptyset$, $f(w_j) = A_j$, where $A_j \subseteq X'$, $f(v_{i,1}) = B_i$, where $B_i \subseteq X - X', i = 1, 2, \ldots, k$, $f(v_{i,j}) = B_i \cup A_j$.

Since the A_j and B_i are disjoint, vertices are assigned by the distinct subsets. Therefore the mapping f is injective. Further,

$$f^{\oplus}(w_i, v_{i,1}) = A_j \oplus B_i,\ f^{\oplus}(w_0, w_j) = A_j,\ f^{\oplus}(w_j, v_{i,j}) = A_j \oplus (B_i \cup A_j) = B_i$$

which shows $f^{\oplus}$ is injective. One can see that the total number of edges is $kn + k = (2^{r/2} - 1)(2^{r/2} + 1) = 2^r - 1$, which is equal to the total number of nonempty subsets of X.

Hence $B(k, n)$ is properly set colorable, and by Theorem 5, $B(k, n)$ is also strongly set colorable. □

Figure 5 is a properly set colored banana tree $B(3, 4)$.

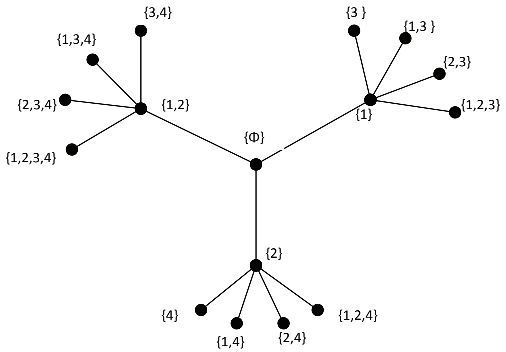

Figure 5: Properly set colored banana tree $B(3,4)$.

Construction of strongly (properly) set colorable caterpillars:

Construction of an infinite family of strongly (properly) set colorable caterpillars is given below.

Let X_1 be a nonempty set with $|X_1| = m_1$ where $m_1 \geq 2$ is a positive integer. Consider the star $K_{1,2^{m_1-1}-1} = T_0(m_1)$, say. Let u_0 be the central vertex and $v_{11}, v_{12}, \ldots, v_{1,2^{m_1-1}-1}$ be the pendant vertices of $T_0(m_1)$. We define a mapping $f_1 : V(T_0(m_1)) \to 2^{X_1}$ as follows:

$$f_1(u_0) = \{x_0\}, \text{ where } x_0 \in X_1,$$

$$f_1(v_{1,i}) = A_r, \text{ where } A_r \subset X_1 - \{x_0\},\ i = 1, 2, \ldots, 2^{m_1-1} - 2,$$

$$f_1(v_{1,2^{m_1-1}-1}) = X_1.$$

Clearly $f_1, f_1^{\oplus}$ are injectives. Let X_2 be a set of cardinality m_2 where $m_2 > m_1$. Introduce new vertices, say $u_{11}, u_{12}, u_{13} \ldots, u_{1k_1}$, where $k_1 = 2^{m_2-1} - 2^{m_1-1}$, and join each of them to $v_{1,2^{m_1-1}-1}$. Let the resulting caterpillar be denoted by $T_1(m_2)$ and define the mapping $f_2 : V(T_1(m_2)) \to 2^{X_2}$ as follows:

$$\begin{cases} f_2(u_{1,j}) = B_j \ \text{ where } \ B_j \subset X_2 - X_1, \quad j = 1, 2 \ldots, 2^{m_2-m_1}; \\ f_2(u_{1,j}) = A_i \cup B_j \ \text{ where } \ B_j \subset X_2 - X_1 \\ \qquad\qquad \text{and } \ A_i \subset X_1 \backslash \emptyset \ \text{ containing } \leq (m_1-1)/2 \text{ elements if } m_1 \text{ is odd,} \\ \qquad\qquad \leq (m_1-2)/2 \ \text{ elements if } \ m_1 \text{ is even and} \\ \qquad\qquad \text{containing } m_1/2 \text{ elements, where } A_i^c \text{ are not included;} \\ f_2(u_{1,k_1}) = X_2. \end{cases}$$

Let $f_2^{\oplus} : E(T_1(m_2)) \longrightarrow 2^{X_2}$ denote the induced edge function defined by $f_2^{\oplus}(uv) = f_2(u) \oplus f_2(v)$. Then it is not hard to verify that $f_2 \cup f_2^{\oplus}$ is the extension of strong set coloring of $f_1 \cup f_1{}^{\oplus}$ in $T_0(m_1)$ and is also strongly set colorable. Next, introduce $2^{m_3-1} - 2^{m_2-1}$ new vertices, say $v_{21}, v_{22}, \ldots, v_{2k_2}$, where $k_2 = 2^{m_3-1} - 2^{m_2-1}$, and join each of them to u_{1k_1}. Let X_3 be the set of cardinality m_3 where $X_1 \subset X_2 \subset X_3$ and $m_1 < m_2 < m_3$. Let the resulting caterpillar be $T_2(m_3)$.

Define $f_3 : T_2(m_3) \to 2^{X_3}$ by

$$\begin{cases} f_3(v_{2,i}) & = B_i \text{ where } B_i \subset X_3 - X_2; \\ & = A_k \cup B_i \text{ where } B_i \subset X_3 - X_2 \\ & \quad \text{and } A_k \subset X_2 \backslash \emptyset \text{ containing } \le (m_2-1)/2 \text{ elements if } m_2 \text{ is odd}, \\ & \quad \le (m_2-2)/2 \text{ elements if } m_2 \text{ is even, and} \\ & \quad \text{containing } m_2/2 \text{ elements, where} A_k^c \text{ are not included;} \\ f_3(v_{2,k_2}) & = X_3. \end{cases}$$

It turns out that $f_3 \cup f_3^{\oplus}$ is the extension of the strongly set colorable mapping $f_2 \cup f_2^{\oplus}$ of $T_1(m_1)$ to $T_2(m_3)$ which is also strongly set colorable. We may iterate this procedure indefinitely to obtain the strongly set colorable caterpillar at the n^{th} step, $n = 1, 2, \ldots$ where $X_1 \subset X_2 \subset \cdots \subset X_n$, and $m_1 < m_2 < \cdots < m_n$ are chosen quite arbitrarily.

An illustrative example for the construction of strongly set colorable caterpillars is given in Figure 6.

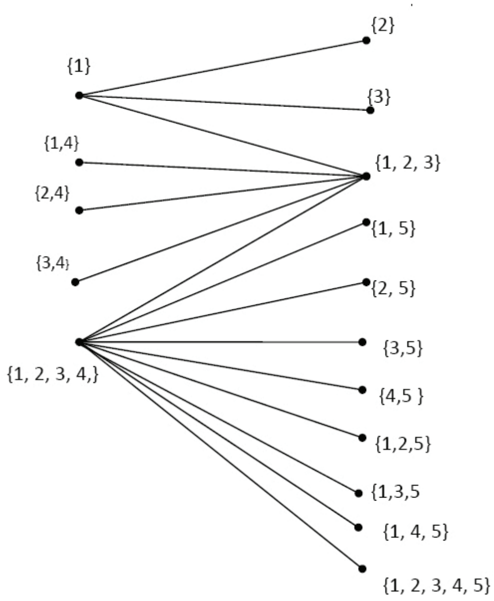

Figure 6: Strongly set colored caterpillar.

Given below is a method of constructing a properly set colorable caterpillar from the strongly set colorable caterpillar constructed above.

Remove an element from X_1, say x_0. Let the resulting set be X_1'. The mapping F_1 is defined as follows:

$F_1(u_0) = \emptyset$; $F_1(v_{1,i}) = A_r$ where $A_r \subset X_1'$, $i = 1, 2, \ldots, 2^{m_1-1}-2$; $F_1(v_{1,2^{m_1-1}-1}) = X_1'$. Clearly F_1 is injective and since $\emptyset$ is assigned to the central vertex of $T_0(m_1)$, those subsets, namely A_r, assigned to the other end vertices of the edges incident to the central vertex that is $v_{1,i}$, are also on the edges. Therefore $F^{\oplus} : E(T_0(m_1)) \rightarrow 2^{X_1'}$ is injective. The number of vertices other than the central vertex is $2^{m_1-1} - 1$ and is equal to the number of nonempty subsets of X_1'. This implies that F_1 is the proper set coloring of $T_0(m_1)$.

Let X_2' be the set obtained by deleting the same element x_0 from X_2. The mapping F_2 is defined as follows:

$$\begin{cases} F_2(u_{1,j}) & = & B_j \text{ where } B_j \subseteq X_2' - X_1'; \\ & = & A_i \cup B_j, \text{ where } A_i \text{ is the subset of } X_1' \\ & & \text{and } B_j \text{ is the subset of } X_2' - X_1'; \\ F_2(u_{1,k_1}) & = & X_2'. \end{cases}$$

It is not difficult to verify that the mapping F_2 and the induced edge mapping ${F_2}^{\oplus}$: $E(T_1(m_2)) \longrightarrow 2^{X_2'}$ are injective. Further, as explained above, $E(T_2(m_1)) = Y(X_2')$. This implies that F_2 is a proper set coloring of $T_1(m_2)$.

By proceeding in the same manner, we can show that the above constructed strongly set colored caterpillar is properly set colored.

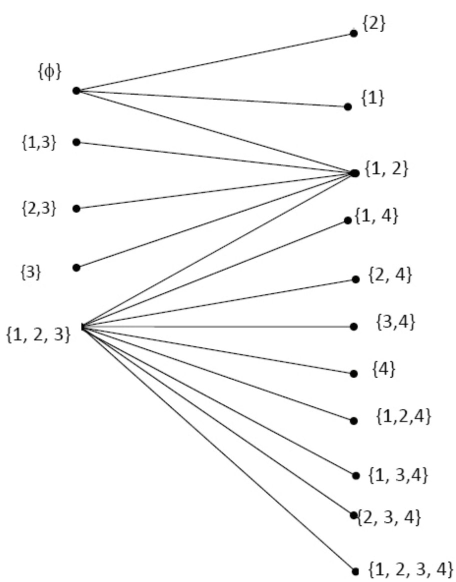

Figure 7: Properly set colored caterpillar.

Acknowledgements

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