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A characterisation of cycle-disjoint graphs with unique minimum weakly connected dominating set

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Abstract

Let G be a connected graph with vertex set $V(G)$. A set S of vertices in G is called a *weakly connected dominating set of* G if (i) S is a dominating set of G and (ii) the graph obtained from G by removing all edges joining two vertices in $V(G) \setminus S$ is connected. A weakly connected dominating set S of G is said to be minimum or a γ_w-set if $|S|$ is minimum among all weakly connected dominating sets of G. We say that G is γ_w*-unique* if it has a unique γ_w-set. Recently, a constructive characterisation of γ_w-unique trees was obtained by Lemanska and Raczek [*Czechoslovak Math. J.* 59 (134) (2009), 95–100]. A graph is said to be *cycle-disjoint* if no two cycles in G have a vertex in common. In this paper, we extend the above result on trees by establishing a constructive characterisation of γ_w-unique cycle-disjoint graphs.

1 Introduction

Let G be a (simple) graph with vertex set $V(G)$ and edge set $E(G)$. We may write V for $V(G)$ and E for $E(G)$ if there is no danger of confusion. The *order* $v(G)$ *of* G is $|V(G)|$, while the *size* $e(G)$ *of* G is $|E(G)|$. G is *non-trivial* if $v(G) \geq 2$. For any vertex $v \in V$, the *open neighbourhood* $N(v)$ *of* v is the set $\{u \in V : uv \in E\}$, while the *closed neighbourhood* $N[v]$ is $N(v) \cup \{v\}$. For $S \subseteq V$, define $N[S]$ as $\cup_{v \in S} N[v]$. We call S a *dominating set of* G if $N[S] = V$.

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Let $S \subseteq V$. The *subgraph of G weakly induced by S*, denoted by $\langle S\rangle_w$, is the graph with vertex-set $N[S]$ and edge-set $E \cap (S \times N[S])$. We call S a *weakly connected dominating set(WCDS) of G* if S is a dominating set of G and $\langle S\rangle_w$ is connected (i.e., the graph obtained from G by removing all edges joining two vertices in $V(G) \setminus S$ is connected). The *weakly connected domination number of G*, denoted by $\gamma_w(G)$, is defined by $\gamma_w(G) = min\{|S| : S$ is a WCDS of $G\}$. A WCDS S of G is called a *γ_w-set of G* if $|S| = \gamma_w(G)$. We say that G is *γ_w-unique* if G has a unique γ_w-set. The parameter $\gamma_w(G)$ was first introduced in [2]. For some existing results on $\gamma_w(G)$, see [1, 2, 3, 4, 5].

A vertex v in a graph is called an *end-vertex* if the degree $d(v) = 1$. A vertex is called a *cycle-vertex* if it is contained in a cycle. Let G be a connected graph. A vertex v in G is called a *cut-vertex* if $G - v$ is disconnected. An edge e in G is called a *bridge* if $G - e$ is disconnected.

A *unicyclic* graph is a connected graph which contains exactly one cycle. A *sunflower* is a unicyclic graph G such that $V(C)$ is a WCDS of G, where C is the only cycle in G. A graph G is said to be *cycle-disjoint* if no two cycles in G have a vertex in common.

Recently, a constructive characterization of γ_w-unique trees was given in [5]. In this paper, our aim is to characterize all γ_w-unique cycle-disjoint graphs. In Section 3, we determine all γ_w-unique sunflowers. Then, in Section 4, we shall introduce two generating functions $g_0(\mathcal{G})$ and $g(\mathcal{G})$, where $\mathcal{G}$ is a set of graphs, which produce all γ_w-unique trees, all γ_w-unique unicyclic graphs and all γ_w-unique cycle-disjoint graphs (see Theorem 4.6).

2 Preliminary Results

To begin with, we introduce in this section two elementary graph operations, called *edge-linking* and *vertex-gluing* respectively, to combine two vertex-disjoint graphs G_1 and G_2 into a new graph G. We derive some basic relations among $\gamma_w(G_1)$, $\gamma_w(G_2)$ and $\gamma_w(G)$, and study some conditions relating γ_w-uniqueness of G_1, G_2 and G.

Let G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. For $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, let $G_1(v_1) - G_2(v_2)$ denote the graph obtained from G_1 and G_2 by adding an edge joining v_1 and v_2, and let $G_1(v_1) \cdot G_2(v_2)$ denote the graph obtained from G_1 and G_2 by gluing (identifying) v_1 with v_2, as shown in Figure 1.

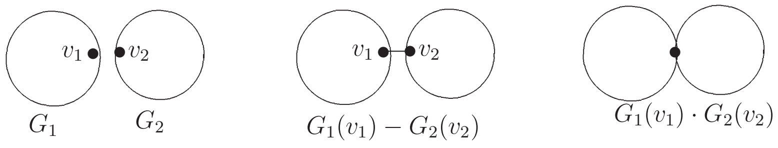


Figure 1

Lemma 2.1 ([3])**.** *Let $G = G_1(v_1) - G_2(v_2)$ be the graph defined above, $S \subseteq V(G)$ and $S_i = S \cap V(G_i)$ for each $i = 1, 2$. Assume that $v(G_i) \geq 2$ for each $i = 1, 2$. Then*

(i) *S is a WCDS of G if and only if S_i is a WCDS of G_i for each $i = 1, 2$ and $\{v_1, v_2\} \cap (S_1 \cup S_2) \neq \emptyset$.*

(ii) *If S is a γ_w-set of G, then $|S| = \gamma_w(G) \geq \gamma_w(G_1) + \gamma_w(G_2)$, where the equality holds if and only if S_i is a γ_w-set of G_i for each $i = 1, 2$ and $\{v_1, v_2\} \cap (S_1 \cup S_2) \neq \emptyset$.* □

Lemma 2.2. *Let G_1 and G_2 be γ_w-unique non-trivial graphs with γ_w-sets S_1 and S_2 respectively. Let $v_1 \in S_1$ and $v_2 \in S_2$. Then the graph $G_1(v_1) - G_2(v_2)$ is γ_w-unique with γ_w-set $S_1 \cup S_2$.*

Proof. By Lemma 2.1, $S_1 \cup S_2$ is a γ_w-set for $G_1(v_1) - G_2(v_2)$. Suppose $S'(\neq S_1 \cup S_2)$ is another γ_w-set for $G_1(v_1) - G_2(v_2)$. Then by the same lemma, $S'_1 = S' \cap V(G_1)$ and $S'_2 = S' \cap V(G_2)$ are γ_w-sets for G_1 and G_2 respectively. Since $S' \neq S_1 \cup S_2$, either $S'_1 \neq S_1$ or $S'_2 \neq S_2$, contradicting the uniqueness of S_1 and S_2. □

Lemma 2.3 ([3])**.** *Let $G = G_1(v_1) \cdot G_2(v_2)$ be the graph defined above, $S \subseteq V(G)$ and $S_i = S \cap V(G_i)$ for each $i = 1, 2$. Then*

(i) *S is a WCDS of G if and only if S_i is a WCDS of G_i for each $i = 1, 2$.*

(ii) *If S is a γ_w-set of G, then $|S| = \gamma_w(G) \geq \gamma_w(G_1) + \gamma_w(G_2) - 1$, where the equality holds if and only if S_i is a γ_w-set of G_i and each $v_i \in S_i$ for $i = 1, 2$.* □

3 Sunflowers

In this section, we shall characterize all γ_w-unique sunflowers.
We first develop some properties on γ_w-unique graphs. For any graph G and integer $i \geq 0$, let $V_i(G)$ be the set of vertices of degree i in G. For any $u \in V(G)$, let G_u denote the graph obtained from G by deleting u and all vertices in $N(u) \cap V_1(G)$.

Lemma 3.1. *Let G be a γ_w-unique graph with unique γ_w-set S. Then the following hold:*

(i) *$V_1(G) \cap S = \emptyset$;*

(ii) *$N(V_1(G)) \subseteq S$;*

(iii) *For any $u \in S$, if u is not contained in any cycle of G, then each component H_j of G_u with $|V(H_j)| \geq 2$ is γ_w-unique with the unique γ_w-set $S \cap V(H_j)$;*

(iv) *If $u \in S$, then $|N(u) \setminus S| \geq 2$ (i.e., $|N(u) \cap S| \leq d(u) - 2$).*

Proof. (i) and (ii). Let u, v be adjacent vertices with $d(v) = 1$. If $v \in S$, then $(S \setminus \{v\}) \cup \{u\}$ is a WCDS of G with size $|S|$, contradicting the uniqueness of S. Thus $v \notin S$ and so $u \in S$. Hence both (i) and (ii) hold.
(iii). Note that $|N(u) \cap V(H_j)| = 1$, as u is not contained in any cycle of G. Let $N(u) \cap V(H_j) = \{v\}$. Since S is an WCDS of G and $|V(H_j)| \geq 2$, $S \cap V(H_j)$ must be a WCDS of H_j; otherwise, v is not dominated by $S \cap V(H_j)$ and so $\langle S \rangle_w$ is not connected, a contradiction. If H_j has another WCDS U such that $|U| \leq |S \cap V(H_j)|$, then $S' = (S \setminus V(H_j)) \cup U$ is a WCDS of G such that $S' \neq S$ and $|S'| \leq |S|$, contradicting the condition that G is a γ_w-unique graph with the unique γ_w-set S. Hence (iii) holds.
(iv). If $N(u) \setminus S = \emptyset$, then $N(u) \subseteq S$ and $S \setminus \{u\}$ is also a WCDS of G. If $N(u) \setminus S = \{w\}$, then $(S \setminus \{u\}) \cup \{w\}$ is also a WCDS of G. Both cases imply that S is not the unique γ_w-set of G, a contradiction. □

For any graph H and $D \subseteq V(H)$, an edge uv in H is called a *bad edge* of D if this edge is not in the subgraph $\langle D \rangle_w$ of H. Note that uv is a bad edge of D if and only if $\{u, v\} \cap D = \emptyset$. If H' is a subgraph of H containing edge uv and $D' = D \cap V(H')$, then uv is a bad edge of D if and only if it is a bad edge of D'.
The following result can be verified easily by the definition of bad edges.

Lemma 3.2. *Let $P : u_0u_1 \ldots u_m$ be a path and D be a subset of $\{u_i : i = 0, 1, \ldots, m\}$. Then*

(i) *if $|D| \leq (m-2)/2$, then D has at least two bad edges in P;*

(ii) *if $|D| = m/2$, then D has no bad edge in P if and only if $D = \{u_{2i-1} : 1 \leq i \leq m/2\}$;*

(iii) *if $|D| = (m-1)/2$, then D has at least one bad edge in P, and D has exactly one bad edge in P if and only if D is a subset of $\{u_i : 1 \leq i \leq m-1\}$ and D is independent in P.* □

In the remainder of this section, we always assume that

(*) G is a sunflower with the only cycle $C : v_1v_2 \ldots v_mv_1$ and $S_0 = V(G) \setminus (V_1(G) \cup V_2(G)) = \{v_{j_i} : i = 1, 2, \ldots, r\}$, where $1 = j_1 < j_2 < \cdots < j_r \leq m$.

Note that r is the number of vertices in C which have end-vertex neighbours, and if $r = 0$, then G is not γ_w-unique. Thus we assume that $r \geq 1$. Let $j_{r+1} = m + 1$ and $v_{m+1} = v_1$. Let

$$S_1 = S_0 \cup \{v_{j_i+s} : 1 \leq i \leq r,\ s \text{ is even and } 2 \leq s \leq j_{i+1} - j_i - 1\} \tag{1}$$

and $S_2 = (S_1 \setminus \{v_{j_2-1}\}) \cup \{v_1\}$. Observe that

$$|S_1| = |S_0| + \sum_{i=1}^{r} \lfloor (j_{i+1} - j_i - 1)/2 \rfloor, \tag{2}$$

and if $j_2 - j_1 \in \{1\} \cup \{2k : k = 1, 2, 3, \dots\}$, then $S_2 = S_1$; otherwise, $|S_2| = |S_1| - 1$. For each $i = 1, 2, \dots, r$, let $Q_i = \{v_t : j_i < t < j_{i+1}\}$, and P_i be the subgraph of G induced by Q_i if $j_{i+1} \geq j_i + 2$. Note that P_i is a path of length $j_{i+1} - j_i - 2$.
The following result is not difficult to verify.

Lemma 3.3. *Let G be the sunflower as defined in $(*)$ with $r \geq 1$.*

(i) *If $j_{i+1} - j_i \in \{1\} \cup \{2k : k = 1, 2, 3, \dots\}$ for all i with $1 \leq i \leq r$, then S_1 is a WCDS of G;*

(ii) *If $j_2 - j_1 \geq 3$ and $j_2 - j_1$ is odd, then S_2 is a WCDS of G and $|S_2| = |S_1| - 1$.* □

Lemma 3.4. *Let G be the sunflower as defined in $(*)$ and D be any WCDS of G. Then*

(i) *D has at most one bad edge in G, and the only possible bad edge must be on C;*

(ii) *$|D \cap Q_i| \geq \lfloor (j_{i+1} - j_i - 2)/2 \rfloor$ for all i with $1 \leq i \leq r$, and $|D \cap Q_i| = (j_{i+1} - j_i - 3)/2$ holds for at most one i with $1 \leq i \leq r$.*

Proof. (i) As C is the only cycle in G, if D has more than one bad edge in G, then $\langle D \rangle_w$ is disconnected; if D has a bad edge which is not on C, then $\langle D \rangle_w$ is also disconnected. Thus the result holds.
(ii) If $|D \cap Q_i| < \lfloor (j_{i+1} - j_i - 2)/2 \rfloor$, then, by Lemma 3.2(i), D has at least two bad edges in P_i, a contradiction. If $|D \cap Q_i| = (j_{i+1} - j_i - 3)/2$, then, by Lemma 3.2(iii), D has at least one bad edge in G. As D has at most one bad edge in G, the result holds. □

Lemma 3.5. *Let G be the sunflower as defined in $(*)$ of page 180. If $j_{i+1} - j_i \in \{1\} \cup \{2k : k = 1, 2, 3, \dots\}$ for all i with $1 \leq i \leq r$, then $\gamma_w(G) = |S_1|$; otherwise, $\gamma_w(G) = |S_1| - 1$.*

Proof. We say case 1 occurs if $j_{i+1} - j_i \in \{1\} \cup \{2k : k = 1, 2, 3, \dots\}$ for all i with $1 \leq i \leq r$, and case 2 occurs otherwise. It suffices to show that $|D| \geq |S_1|$ in case 1 and $|D| \geq |S_1| - 1$ in case 2 for any WCDS D of G.
Let D be any WCDS of G. By Lemma 3.4(i), we have $|D \cap (\{v_{j_i}\} \cup (N(v_{j_i}) \cap V_1(G)))| \geq 1$. Since

$$|D| = \sum_{i=1}^{r} |D \cap (\{v_{j_i}\} \cup (N(v_{j_i}) \cap V_1(G)))| + \sum_{i=1}^{r} |D \cap Q_i|,$$

Lemma 3.4(ii) implies that $|D| \geq |S_1|$ in case 1 and $|D| \geq |S_1| - 1$ in case 2. □

Corollary 3.6. *Let G be the sunflower as defined in $(*)$ of page 180. Assume that G is γ_w-unique. Then $r \geq 1$ and*

(i) *if $j_{i+1} - j_i \in \{1\} \cup \{2k : k = 1, 2, 3, \dots\}$ for all i with $1 \leq i \leq r$, then S_1 is the γ_w-unique set of G;*

(ii) *if $j_2 - j_1 \geq 3$ and $j_2 - j_1$ is odd, then S_2 is the γ_w-unique set of G.* □

Now we are going to derive some necessary conditions on G if G is γ_w-unique, and will later prove that these conditions are also sufficient for G to be γ_w-unique.

Lemma 3.7. *Let G be the sunflower as defined in $(*)$ of page 180. Assume that G is γ_w-unique. Then the following conditions hold:*

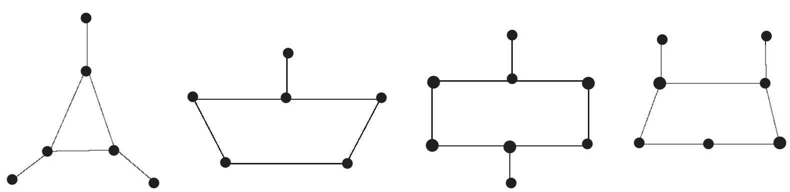
(i) $r \geq 1$;

(ii) *For $1 \leq i \leq r$, if $j_{i-1} + 1 = j_i = j_{i+1} - 1$, then $d(v_{j_i}) \geq 4$;*

(iii) *There is no i with $1 \leq i \leq r$ such that $j_{i+1} - j_i \geq 5$ and $j_{i+1} - j_i$ is odd;*

(iv) *There is at most one i with $1 \leq i \leq r$ such that $j_{i+1} - j_i = 3$; and*

(v) *If there is no i with $1 \leq i \leq r$ such that $j_{i+1} - j_i = 3$, then, for all s with $1 \leq s \leq r$, either $j_{s+1} - j_s = 2$ or $j_{s+1} - j_s = 1$ with $(d(v_{j_{s+1}}) - 3)(d(v_{j_s}) - 3) \geq 1$.*



Examples of graphs violating each of conditions (ii)–(v)
Figure 2

Proof. Let S be the γ_w-unique set of G.

(i) is obvious since every cycle is not γ_w-unique.

(ii) Note that $v_{j_{i-1}}, v_{j_i}, v_{j_{i+1}} \in S_0 \subseteq S$ by Lemma 3.1 (ii). If $d(v_{j_i}) = 3$ and $j_{i-1} + 1 = j_i = j_{i+1} - 1$, then $|N(v_{j_i}) \setminus S| \leq 1$, contradicting Lemma 3.1 (iv).

(iii) Suppose that $j_2 - j_1 \geq 5$ and $j_2 - j_1$ is odd, i.e., $j_2 \geq 6$ and j_2 is even. Let $S' = S \cap \{v_1, v_2, \ldots, v_{j_2}\}$. By Lemma 3.1 (ii), $v_1, v_{j_2} \in S'$. By Lemma 3.1 (iv), every two vertices in S' are not consecutive on the cycle C of G, implying that $|S'| \leq j_2/2$. As $\langle S \rangle_w$ is connected, Lemma 3.3 (ii) implies that $|S'| \geq j_2/2$. Thus $|S'| = j_2/2$. Observe that for any $S'' \subseteq \{v_1, v_2, \ldots, v_{j_2}\}$, if $\{v_1, v_{j_2}\} \subseteq S''$, $|S''| = j_2/2$ and S'' is independent set, then $(S \setminus S') \cup S''$ is a WCDS of G. It is clear that such S'' is not unique, for example, both $\{v_1, v_{j_2}\} \cup \{v_{2i+1} : i = 1, 2, \ldots, j_2/2 - 2\}$ and $\{v_1, v_{j_2}\} \cup \{v_{2i+2} : i = 1, 2, \ldots, j_2/2 - 2\}$ are such sets. But, as G is γ_w-unique, such S'' must be unique, and hence (iii) holds.

(iv) Suppose that $j_2 - j_1 = 3 = j_{i+1} - j_i$ for some i with $2 \leq i \leq r$. Then, $(\{v_1, v_2, v_3, v_4\} \cup \{v_{j_i}, v_{j_i+1}, v_{j_i+2}, v_{j_i+3}\}) \cap S = \{v_1, v_4\} \cup \{v_{j_i}, v_{j_{i+1}}\}$ by Lemma 3.1 (ii) and (iv). But, then $\langle S \rangle_w$ is disconnected, a contradiction.

(v) Assume that $j_{i+1} - j_i \in \{1\} \cup \{2k : k = 1, 2, \ldots\}$ for all i with $1 \leq i \leq r$. By Lemma 3.5, $\gamma_w(G) = |S_0| + \sum_{i=1}^{r} \lfloor (j_{i+1} - j_i - 1)/2 \rfloor$. By Corollary 3.6, S_1 is the γ_w-unique set of G.

If $j_2 = 2$ and $d(v_1) = 3$, then $(S_1 \setminus \{v_1\}) \cup \{u\}$, where u is the only vertex in $V_1(G) \cap N(v_1)$, is also a WCDS of G with size equal to $|S_1| = \gamma_w(G)$, a contradiction. If $j_2 - j_1$ (≥ 4) is even, then $S_1 \setminus \{v_{j_2-2}\}) \cup \{v_{j_2-1}\}$ is also a WCDS of G with size equal to $|S_1| = \gamma_w(G)$, a contradiction too. Thus (v) holds. □

Theorem 3.8. *Let G be the sunflower as defined in $(*)$ after Lemma 3.2 above. Then G is γ_w-unique if and only if G satisfies conditions (i)–(v) in Lemma 3.7.*

Proof. It suffices to prove the sufficiency. Assume that all conditions in Lemma 3.7 are satisfied. Let S' be any WCDS of G such that $|S'| = \gamma_w(G)$. We shall show that $S' = S$ and hence complete the proof.
We say case 1 occurs if $j_{i+1} - j_i \in \{1\} \cup \{2k : k = 1, 2, \dots\}$ for all i with $1 \leq i \leq r$, and case 2 occurs otherwise. By conditions in Lemma 3.7, case 2 occurs if and only if $j_{i+1} - j_i \notin \{1\} \cup \{2k : k = 1, 2, \dots\}$ for only one i with $1 \leq i \leq r$, and moreover, $j_{i+1} - j_i = 3$.

Claim 1: For any $u \in V(G)$, if $d(u) \geq 4$, then $u \in S'$.
If $u \notin S'$, then $N(u) \cap V_1(G) \subseteq S'$ and so $(S' \setminus (N(u) \cap V_1(G))) \cup \{u\}$ is a WCDS of G with size smaller than S', a contradiction. So this claim holds.

Claim 2: For any $u \in V(G)$, if $d(u) = 3$, then $u \in S'$.
Without loss of generality, suppose that $d(v_{j_2}) = 3$ and $v_{j_2} \notin S'$. By Lemma 3.7(ii), either $j_3 - j_2 \geq 2$ or $j_2 - j_1 \geq 2$. Assume that $j_3 - j_2 \geq 2$.
Let u be the only vertex in $N(v_{j_2}) \cap V_1(G)$. As $v_{j_2} \notin S'$, we have $u \in S'$. As $S' \setminus \{u\}$ is a WCDS of $G - u$, by Lemma 3.5, we have

$$|S'| - 1 = |S' \setminus \{u\}| = |S_0| - 1 - \delta' + \lfloor (j_3 - j_1 - 1)/2 \rfloor + \sum_{i=3}^{r} \lfloor (j_{i+1} - j_i - 1)/2 \rfloor, \quad (3)$$

where $\delta' \in \{0, 1\}$, and $\delta' = 0$ if and only if $j_3 - j_1$ is even and $j_{i+1} - j_i \in \{1\} \cup \{2k : k = 1, 2, \dots\}$ for all i with $3 \leq i \leq r$. As $|S'| = \gamma_w(G)$, by Lemma 3.5, (3) implies that

$$\lfloor (j_3 - j_2 - 1)/2 \rfloor + \lfloor (j_2 - j_1 - 1)/2 \rfloor - \delta = -\delta' + \lfloor (j_3 - j_1 - 1)/2 \rfloor, \quad (4)$$

where $\delta \in \{0, 1\}$, and $\delta = 0$ if and only if case 1 occurs.
If case 1 occurs, then $\delta = 0$ and condition (v) in Lemma 3.7 implies that $j_{i+1} - j_i \in \{1, 2\}$ for all i with $1 \leq i \leq r$. So we have $\lfloor (j_3 - j_2 - 1)/2 \rfloor = \lfloor (j_2 - j_1 - 1)/2 \rfloor = 0$, but $\lfloor (j_3 - j_1 - 1)/2 \rfloor = 1$, and so (4) implies that $0 = -\delta' + 1$, i.e., $\delta' = 1$. Then $j_3 - j_1$ is odd. As $j_3 - j_2 = 2$, we have $j_2 - j_1 = 1$, contradicting condition (v) in Lemma 3.7.
Thus case 2 occurs, and we have $\delta = 1$. (4) implies that

$$\lfloor (j_3 - j_2 - 1)/2 \rfloor + \lfloor (j_2 - j_1 - 1)/2 \rfloor = 1 - \delta' + \lfloor (j_3 - j_1 - 1)/2 \rfloor. \quad (5)$$

As $1 - \delta' \geq 0$ and

$$\lfloor (j_3 - j_2 - 1)/2 \rfloor + \lfloor (j_2 - j_1 - 1)/2 \rfloor \leq \lfloor (j_3 - j_1 - 1)/2 \rfloor,$$

we have $\delta' = 1$ and

$$\lfloor (j_3 - j_2 - 1)/2 \rfloor + \lfloor (j_2 - j_1 - 1)/2 \rfloor = \lfloor (j_3 - j_1 - 1)/2 \rfloor.$$

But, as $\delta' = 1$, we have $j_3 - j_1$ is odd, implying that

$$\lfloor (j_3 - j_1 - 1)/2 \rfloor = 1 + \lfloor (j_3 - j_1 - 2)/2 \rfloor \geq 1 + \lfloor (j_3 - j_2 - 1)/2 \rfloor + \lfloor (j_2 - j_1 - 1)/2 \rfloor,$$

a contradiction. Hence Claim 2 holds.

By Claims 1, 2 and condition (v) in Lemma 3.7, we have $S' = S = S_0$ in case 1. Now assume that case 2 occurs. Without loss of generality, assume that $j_2 - j_1 = 3$. Then $j_{i+1} - j_i \in \{1\} \cup \{2k : k = 1, 2, \dots\}$ for all i with $2 \leq i \leq r$. By Lemma 3.4, $|S' \cap Q_i| \geq (j_{i+1} - j_i - 2)/2$ for all i with $2 \leq i \leq r$. Thus, by the following equality:

$$|S_0| + \sum_{i=1}^{r} |S' \cap Q_i| = |S'| = |S_0| - 1 + \sum_{i=1}^{r} \lfloor (j_{i+1} - j_i - 1)/2 \rfloor = |S_0| + \sum_{i=2}^{r} \lfloor (j_{i+1} - j_i - 1)/2 \rfloor$$

we have $|S' \cap Q_1| = 0$ and $|S' \cap Q_i| = (j_{i+1} - j_i - 2)/2$ for all i with $2 \leq i \leq r$ and $j_{i+1} - j_i > 1$. As $|S' \cap Q_1| = 0$, S' has one bad edge in P_1. So S' has no bad edge in P_i for all i with $2 \leq i \leq r$ and $j_{i+1} - j_i > 1$. By Lemma 3.2(ii), if $2 \leq i \leq r$ and $j_{i+1} - j_i$ is even, then $S' \cap Q_i = \{v_{j_i+s} : 2 \leq s \leq j_{i+1} - j_i - 2, s \text{ is even}\}$. Hence $S' = S$ and the proof is complete. □

4 Families of γ_w-unique graphs generated

In this section, we shall define some operations which generate the family of γ_w-unique trees, the family of γ_w-unique unicyclic graphs and the family of γ_w-unique cycle-disjoint graphs. Let $S(G)$ be the γ_w-unique set of G if G is γ_w-unique. We first establish two results.

Lemma 4.1. *Let G be a connected graph and u be a vertex in G. If $d(u) \geq 2$ and $N(u) \setminus V_1(G) = \{v\}$, then G is γ_w-unique with γ_w-unique set S if and only if G_u is γ_w-unique with γ_w-unique set $S \setminus \{u\}$ and one of the following conditions holds:*

(i) *$d(u) \geq 3$;*

(ii) *$d(u) = 2$ and v is not contained in the γ_w-unique set $S(G_u)$.*

Proof. ($\Rightarrow$) Since $d(u) \geq 2$ and $N(u) \setminus V_1(G) = \{v\}$, we have $u \in N(V_1(G))$ and so $u \in S$. By Lemma 3.1 (iii), G_u is γ_w-unique with the γ_w-unique $S \setminus \{u\}$.
It remains to show that if $d(u) = 2$, then v is not contained in the γ_w-unique set $S(G_u)$. Let $N(u) = \{v, v'\}$. Then $v' \in V_1(G)$. If v is contained in the γ_w-unique set of G_u, then $(S \setminus \{u\}) \cup \{v'\}$ is also a WCDS of G, a contradiction.
($\Leftarrow$) Note that a star $K_{1,p}$ is γ_w-unique if $p \geq 2$, and its γ_w-unique set is $\{u\}$, where u is the only vertex with degree larger than 1. The result then follows directly from Lemmas 2.2 and 2.3. □

Lemma 4.2. *Let G be a γ_w-unique graph with γ_w-unique set S and u_1u_2 be a bridge of G such that $d(u_i) \geq 2$ for $i = 1, 2$. Let G_1, G_2 be the components of $G - u_1u_2$ with $u_i \in V(G_i)$ and G'_i be the graph obtained from G_i by adding a new vertex v_i and a new edge u_iv_i, as shown in Figure 3. Then*

(i) $S \cap \{u_1, u_2\} \neq \emptyset$;

(ii) *if $S \cap \{u_1, u_2\} = \{u_1\}$, then both G'_1 and G_2 are γ_w-unique with $S(G'_1) = S(G) \cap V(G_1)$ and $S(G_2) = S(G) \cap V(G_2)$;*

(iii) *if $S \cap \{u_1, u_2\} = \{u_1, u_2\}$, then both G_1 and G_2 are γ_w-unique with $S(G_i) = S(G) \cap V(G_i)$ for $i = 1, 2$.*

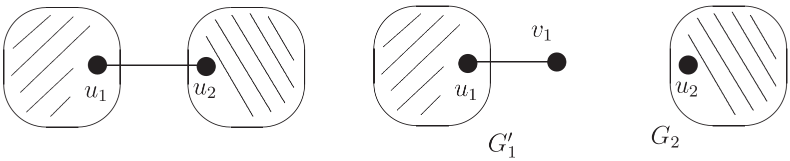


Figure 3

Proof. (i) is obvious because S is a WCDS of G.

(ii) First, it is clear that $S(G) \cap V(G_1)$ and $S(G) \cap V(G_2)$ are the WCDS of G'_1 and G_2 respectively. If S' is a WCDS of G'_1 such that $S' \neq S(G) \cap V(G_1)$ and $|S'| \leq |S(G) \cap V(G_1)|$, then $\{u_1, v_1\} \cap S' \neq \emptyset$. When $v_1 \notin S'$, $S' \cup (S(G) \cap V(G_2))$ is a WCDS of G. When $v_1 \in S'$, $(S' \setminus \{v_1\}) \cup (S(G) \cap V(G_2)) \cup \{u_2\}$ is a WCDS of G. Furthermore, in both cases, G has a different WCDS whose size is not larger than $|S(G)|$, a contradiction. If S' is a WCDS of G_2 such that $S' \neq S(G) \cap V(G_2)$ and $|S'| \leq |S(G) \cap V(G_2)|$, then $S' \cup (S(G) \cap V(G_1))$ is a WCDS of G such that its size is not larger than $|S(G)|$, a contradiction too. Thus (ii) holds.

(iii) Observe that $S(G) \cap V(G_i)$ is a WCDS of G_i for $i = 1, 2$. If S' is a WCDS of G_1 such that $S' \neq S(G) \cap V(G_1)$ and $|S'| \leq |S(G) \cap V(G_1)|$, then $S' \cup (S(G) \cap V(G_2))$ is a WCDS of G such that its size is not larger than $|S(G)|$, a contradiction. Thus (iii) holds. ☐

Let $\mathcal{G}$ be a set of non-trivial graphs G with a subset $U(G)$ of $V(G)$ for each $G \in \mathcal{G}$. Define the set $g_0(\mathcal{G})$ of graphs:

(1.1) $\mathcal{G} \subseteq g_0(\mathcal{G})$;

(1.2) for any graph G, let $G \in g_0(\mathcal{G})$ with $U(G) = U(G_u) \cup \{u\}$ if G contains a vertex u with $N(u) \setminus V_1(G) = \{v\}$ such that $G_u \in g_0(\mathcal{G})$ and either $d(u) \geq 3$ or $v \notin U(G_u)$.

By the definition of g_0, g_0 does not produce new cycles. Actually, we have the following result:

Lemma 4.3. (i) *If all graphs in $\mathcal{G}$ are trees, then all graphs in $g_0(\mathcal{G})$ are trees.*

(ii) *For any $G \in g_0(\mathcal{G})$, if G is not a tree, then there is a graph G_0 in $\mathcal{G}$ such that G_0 is a subgraph of G and every cycle of G is also a cycle of G_0.* $\square$

Let $g(\mathcal{G})$ be the set of graphs defined below:

(2.1) $\mathcal{G} \subseteq g(\mathcal{G})$;

(2.2) let $G \in g(\mathcal{G})$ if G contains a bridge u_1u_2 with $d(u_i) \geq 2$ for $i = 1, 2$ such that both G'_1 and G_2 are graphs in $g(\mathcal{G})$ with $u_2 \notin U(G_2)$, and in this case let $U(G) = U(G'_1) \cup U(G_2)$, where G_1 and G_2 are the two components of $G - u_1u_2$ with $u_i \in V(G_i)$ for $i = 1, 2$ and G'_1 is the graph obtained from G_1 by adding a new vertex v_1 and a new edge u_1v_1, as shown in Figure 3;

(2.3) let $G \in g(\mathcal{G})$ if G contains a bridge u_1u_2 with $d(u_i) \geq 2$ for $i = 1, 2$ such that both G_1 and G_2 are graphs in $g(\mathcal{G})$ and $u_i \in U(G_i)$ for $i = 1, 2$, and in this case let $U(G) = U(G_1) \cup U(G_2)$.

Note that in the definition of g_0 and g, if $\mathcal{G}$ is a set of γ_w-unique graphs, then $S(G)$ will be taken to be $U(G)$ for all $G \in \mathcal{G}$.

Lemma 4.4. *If $\mathcal{G}$ contains all stars $K_{1,p}$ where $p \geq 2$, then $g_0(\mathcal{G}) \subseteq g(\mathcal{G})$.*

Proof. Let $G \in g_0(\mathcal{G})$. If $G \in \mathcal{G}$, then it is clear that $G \in g(\mathcal{G})$. Now assume that $G \notin \mathcal{G}$. Then, by the definition of $g_0(\mathcal{G})$, G contains a vertex u with $N(u) \setminus V_1(G) = \{v\}$ such that $G_u \in g_0(\mathcal{G})$ and either $d(u) \geq 3$ or $v \notin U(G_u)$.
Assume that $G_u \in g(\mathcal{G})$. Let G_1 be the subgraph of G induced by $N[u] \setminus \{v\}$ and $G_2 = G_u$. So $G_1 \cong K_{1,p}$ and $G'_1 \cong K_{1,p+1}$, where $p = d(u) - 1$. Let $U(G_1) = U(G'_1) = \{u\}$. If $d(u) \geq 3$, then $G_1, G'_1 \in \mathcal{G}$, and so G is contained in $g(\mathcal{G})$ by (2.2) if $v \notin U(G_u)$ or by (2.3) otherwise. If $d(u) = 2$, then $G'_1 \cong K_{1,2} \in \mathcal{G}$ and $v \notin U(G_2)$, and thus $G \in g(\mathcal{G})$ by (2.2). $\square$

Lemma 4.5. *If all graphs in $\mathcal{G}$ are γ_w-unique, then all graphs in $g_0(\mathcal{G}) \cup g(\mathcal{G})$ are γ_w-unique.*

Proof. Assume that all graphs in $\mathcal{G}$ are γ_w-unique. Then $U(G) = S(G)$ for all $G \in \mathcal{G}$. By Lemma 4.1, all graphs in $g_0(\mathcal{G})$ are γ_w-unique, and by Lemmas 2.2 and 2.3, all graphs in $g(\mathcal{G})$ are γ_w-unique. $\square$

Theorem 4.6. *Let $\mathcal{G}_1$ be the set of stars $K_{1,p}$ for all $p \geq 2$ and $\mathcal{G}_2$ be the set of γ_w-unique sunflowers. Then*

(i) *$g_0(\mathcal{G}_1)$ is the set of all γ_w-unique trees;*

(ii) *$g_0(\mathcal{G}_2)$ is the set of all γ_w-unique unicyclic graphs;*

(iii) *$g(\mathcal{G}_1 \cup \mathcal{G}_2)$ is the set of all γ_w-unique cycle-disjoint graphs.*

Proof. Note that each graph in $\mathcal{G}_1 \cup \mathcal{G}_2$ is γ_w-unique. For any $G \in \mathcal{G}_1 \cup \mathcal{G}_2$, $S(G)$ is taken to be $U(G)$.

(i) By Lemma 4.3, every graph in $g_0(\mathcal{G}_1)$ is a a tree. Let T be any γ_w-unique tree. If T is a star, then $T \cong K_{1,p}$ for some $p \geq 2$, and so $T \in g_0(\mathcal{G}_1)$. Now assume that T is not a star. Then it has a vertex u such that $N(u) \setminus V_1(G) = \{v\}$ for some vertex v. By Lemma 4.1, T_u is γ_w-unique and either $d(u) \geq 3$ or $v \notin S(T_u)$. By induction, $T_u \in g_0(\mathcal{G}_1)$ and so $T \in g_0(\mathcal{G}_1)$ by the definition of g_0.

(ii) This can be proved similarly.

(iii) By Lemma 4.5, all graphs in $g(\mathcal{G}_1 \cup \mathcal{G}_2)$ are γ_w-unique. By the definition of g, all graphs in $g(\mathcal{G}_1 \cup \mathcal{G}_2)$ are cycle-disjoint.
It is clear that all graphs in $\mathcal{G}_1 \cup \mathcal{G}_2$ are γ_w-unique cycle-disjoint graphs. Now let G be any γ_w-unique cycle-disjoint graph such that $G \notin \mathcal{G}_1 \cup \mathcal{G}_2$. Then G contains a bridge u_1u_2 such that $d(u_i) \geq 2$ for $i = 1, 2$. As $S(G)$ is a WCDS of G, we have $S(G) \cap \{u_1, u_2\} \neq \emptyset$. Then either $S(G) \cap \{u_1, u_2\} = \{u_i\}$ for some $i \in \{1, 2\}$ (we may assume that $i = 1$) or $S(G) \cap \{u_1, u_2\} = \{u_1, u_2\}$. By Lemma 4.2, if $S(G) \cap \{u_1, u_2\} = \{u_1\}$, then both G'_1 and G_2 are γ_w-unique and $u_1 \in S(G'_1)$ but $u_2 \notin S(G_2)$; if $S(G) \cap \{u_1, u_2\} = \{u_1, u_2\}$, then both G_1 and G_2 are γ_w-unique and $u_i \in S(G_i)$ for $i = 1, 2$. It is clear that G'_1, G_1, G_2 are all cycle-disjoint. Thus, by induction, they are all contained in $g(\mathcal{G}_1 \cup \mathcal{G}_2)$. Then, by the definition of g, $G \in g(\mathcal{G}_1 \cup \mathcal{G}_2)$. □

Acknowledgments

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