

Perfect countably infinite Steiner triple systems

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Abstract

We use a free construction to prove the existence of perfect Steiner triple systems on a countably infinite point set. We use a specific countably infinite family of partial Steiner triple systems to start the construction, thus yielding 2^{\aleph_0} non-isomorphic perfect systems.

1 Introduction

The problem of finding perfect finite Steiner triple systems is difficult; only fourteen such systems are known [7, 8]. The question of the existence of perfect countably infinite Steiner triple systems was posed at the British Combinatorial Conference in 2009 [3]. In this paper we show that it is possible to use a free construction to obtain a perfect countably infinite Steiner triple system, and furthermore, by carefully choosing the partial Steiner triple system used to start the construction, 2^{\aleph_0} non-isomorphic perfect countably infinite Steiner triple systems can be produced. We begin with some basic definitions and an explanation of the terminology.

A *finite Steiner triple system of order v* is a pair (V, \mathcal{B}) where V is a finite set of v elements (the *points*) and \mathcal{B} is a collection of 3-element subsets (the *blocks*) of V such that each 2-element subset of V is contained in exactly one block of \mathcal{B} . It is well known that a Steiner triple system (STS) of order v exists if and only if $v \equiv 1$ or $3 \pmod{6}$; such values of v are called *admissible*. If we replace the requirement that V be finite by the requirement that V have cardinality \aleph_0 , then the resulting pair (V, \mathcal{B}) is called a *countably infinite Steiner triple system*.

A *partial Steiner triple system* (or *configuration*) is a set of 3-element subsets (blocks), taken from a point set V , which has the property that every pair of distinct elements of V occurs in at most one block. A partial Steiner triple system (partial STS) is called *finite* if it has a finite set of blocks; it is called *countably infinite* if the block set is countably infinite. If \mathcal{C} is a partial STS, we denote its set of points by $P(\mathcal{C})$. The *degree* of a point in a partial STS is the cardinality of the set of blocks that contain that point. A partial STS is *connected* if, for any pair of its points x and y , there is a sequence $x = x_0, x_1, \dots, x_n = y$ of its points x_i , such that the pair x_i and x_{i+1} are contained in together in a block, for all i .

A *sub-system* (V', \mathcal{B}') of an STS (V, \mathcal{B}) is a Steiner triple system with $V' \subseteq V$ and $\mathcal{B}' \subseteq \mathcal{B}$. A *partial sub-system* of an STS, or a partial STS, is defined analogously except that each 2-element subset of V' is contained in at most one block of \mathcal{B}' rather than in exactly one.

For any two points a and b of a (finite or countably infinite) Steiner triple system \mathcal{S} , contained in a block abc , the *cycle graph* $G(a, b)$ has vertex set $V \setminus \{a, b, c\}$, with an edge coloured a (resp., b) joining x to y if and only if axy is a block (resp., bxy is a block). In the finite case, it is well known that $G(a, b)$ is a set of disjoint cycles $\{C_{n_1}, C_{n_2}, \dots, C_{n_r}\}$, where $n_1 + n_2 + \dots + n_r = v - 3$; hence the name cycle graph. Moreover each n_i is even with $n_i \geq 4$. In the infinite case, besides finite cycles, $G(a, b)$ may have “infinite cycles”, components that we interpret as “two-way infinite paths”. We define the cycle graph, $G(a, b)$, of any two points a and b of a partial STS analogously, noting that here a and b may not be contained together in a block.

A *uniform* STS has each cycle graph $G(a, b)$ isomorphic and a *perfect* STS has each cycle graph $G(a, b)$ a single cycle (in analogy with a *perfect 1-factorisation* where the union of the 1-factors is a Hamiltonian cycle). Thus, a perfect STS is uniform, but the converse is not true. The existence of uncountably many uniform countably infinite Steiner triple systems is known [4], but none of these systems is perfect.

Note that an infinite perfect STS cannot be larger than countable since a Hamiltonian cycle cannot be larger than countable.

2 Results

Many countably infinite structures can be constructed “freely” with any special properties “built in” during the construction. Examples include Steiner systems with large values of t , and highly transitive Steiner systems [2]. Here we use such a construction to obtain a perfect countably infinite Steiner triple system by starting with a suitable partial STS and carefully “building in” the perfect property during the construction.

The following are the steps required in a free construction of a countably infinite Steiner triple system from *any* finite (or countably infinite) partial STS.

- Start with a finite (or countably infinite) *partial Steiner triple system*
- Adjoin alternatively:
 - *new blocks* incident with those pairs of points not already in a block

- *new points* so each existing block has 3 points
- After countably many steps the result is a Steiner triple system.

To obtain a perfect countably infinite Steiner triple system we must ensure that no cycle graph $G(a, b)$ in our system contains a finite cycle, and that each cycle graph $G(a, b)$ is connected. Thus we start with a partial STS with no cycle graph $G(a, b)$ containing a finite cycle. At each stage of the construction we insist that any new block contains at least one new point. This rules out any finite cycle being created in any cycle graph $G(a, b)$ because if the block axy creates a cycle in $G(a, b)$ then there exists an xy -path in $G(a, b)$ say $[x, w, v, \dots, z, y]$ where possibly $v = z$. But then we have blocks avw, bwx, byz ; so all of a, x, y are existing points, and axy cannot be a new block. In this way, all that we need to check is that the result is a countably infinite Steiner triple system and that all the cycle graphs $G(a, b)$ are connected.

Theorem 2.1 *There exists a countably infinite perfect Steiner triple system.*

Proof We start with a (finite or countably infinite) partial STS with no cycle graph $G(a, b)$ containing a finite cycle and we construct a countably infinite perfect Steiner triple system in stages.

At even-numbered stages, we list all pairs of existing points, and for each such pair ab which is not already contained in a block, we add a new point n and a block abn .

At odd-numbered stages, we list all pairs of existing points. For each such pair ab , consider the cycle graph $G(a, b)$. This will be a union of paths and isolated vertices. Suppose that x and y are either ends of paths or isolated vertices (if both are path ends, it should not be the same path). Now add new vertices and blocks as follows:

- If x is isolated or on an a -coloured edge and y is isolated or on a b -coloured edge, add blocks bxn and any , where n is a new point. Similarly if the colours are reversed.
- If x and y are on a -coloured edges, then add blocks bxn_1, an_1n_2 , and bn_2y . Similarly if they are both on b -coloured edges.

An easy check shows that we never introduce a second block containing two points already in a block. (In the first case, for example, if there were a block containing b and x , then x would not be isolated or the end of a path in $G(a, b)$.)

Now given any two points, there will be a first stage of the construction at which they have both appeared; and in the next even-numbered stage, a block containing them will be introduced. So we construct a Steiner triple system \mathcal{S} .

Given points a and b , and x and y in $V \setminus \{a, b, c\}$, where abc is a block, at the next odd-numbered stage after all these points have been introduced, x and y will be in the same connected component of the cycle graph $G(a, b)$. So the countably infinite Steiner triple system produced is perfect. \square

To construct non-isomorphic countably infinite perfect Steiner triple systems in this way we need to choose the partial STS used at the start of the construction

carefully. The infinite family of partial STS used to construct the uniform countably infinite Steiner triple systems in [4] are ‘perfect’ for our needs. We take the construction from the proof of Lemma 2.3 in that paper.

Let \mathcal{D}_1 be a partial STS with $P(\mathcal{D}_1) = \{a, b, c, d, e, f, g, h, i\}$ and the six blocks:

$$\{a, b, c\}, \{d, e, f\}, \{g, h, i\}, \{a, d, g\}, \{b, e, h\}, \{c, f, i\}.$$

It is connected and has 9 points all of degree 2.

We form an infinite sequence of partial STS called $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \dots$ starting from \mathcal{D}_1 . We construct \mathcal{D}_{i+1} from \mathcal{D}_i in the following manner. \mathcal{D}_i is a connected partial STS with $3(i + 2)$ points all of degree 2, where $i \geq 1$. There exist two points of $P(\mathcal{D}_i)$, say x, y , that do not lie together in a block of \mathcal{D}_i . Replace x by a new point x_1 in one of the two blocks containing x , and replace x by another new point x_2 in the other block containing x . Carry out a similar replacement of y by new points y_1 and y_2 . This may disconnect the configuration, but we can assume that x_1 remains in the same component as y_1 . Now introduce a new point z and two new blocks x_1y_2z and x_2y_1z . The resulting partial STS \mathcal{D}_{i+1} has $3(i + 3)$ points, all of degree 2, and is connected. Clearly no \mathcal{D}_i can contain any other \mathcal{D}_j as a (partial) sub-system.

We consider the cycle graphs $G(a, b)$ of \mathcal{D}_i ; because the degree of each point is 2, there are at most 4 edges in each such graph (when the points a and b are not contained together in a block). The shortest cycle possible in any cycle graph $G(a, b)$ is of length 4 and occurs only when the STS or partial STS contains a Pasch configuration (also called a quadrilateral) with four blocks of the form

$$\{a, x, y\}, \{b, y, z\}, \{a, z, t\}, \{b, x, t\}.$$

The way that the partial STS \mathcal{D}_i are constructed precludes this possibility, so none of the cycle graphs $G(a, b)$ of any of the \mathcal{D}_i contains a cycle.

Let \mathcal{D}_0 be the trivial partial STS comprising just a single point and no blocks. Denote by \mathcal{C} the partial STS formed from some collection of partial STS \mathcal{D}_i ($i \geq 0$); note that they need not all be distinct.

Theorem 2.2 *There exists a countably infinite perfect Steiner triple system containing the partial STS \mathcal{C} , which contains no other connected finite partial sub-system all of whose points have degree 2.*

Proof The cycle graphs $G(a, b)$ of the partial STS \mathcal{C} contain no cycles since each component of this partial STS is simply one of the \mathcal{D}_i and there are no edges in any $G(a, b)$ between these components.

We therefore follow the construction in stages, as in the proof of Theorem 2.1, starting with the partial STS \mathcal{C} . The resulting countably infinite Steiner triple system will be perfect by Theorem 2.1.

At every stage in the construction, every new block added contains a new point, and so no new finite partial sub-system can be formed all of whose points are of degree 2, since there must be at least one new point of degree 1 on every block added. □

Corollary 2.2.1 *There are 2^{\aleph_0} non-isomorphic countably infinite perfect Steiner triple systems.*

Proof If two distinct infinite subsets of $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \dots\}$ are taken to form two initial partial STS used in two applications of the construction, then the two resulting perfect countably infinite STS are non-isomorphic since they contain different numbers of the partial STS \mathcal{D}_i . There are 2^{\aleph_0} such subsets, so there are that many non-isomorphic countably infinite STS. \square

3 Concluding Remarks

The existence of these perfect countably infinite Steiner triple systems fits with the assertion that the question of the existence of countably infinite Steiner systems is in general inordinately simpler than the finite case, since there is usually “space” to fit everything in. Ad hoc methods, “free” constructions and more general constructions using Model Theoretic results, such as the Fraïssé Limit can be used to construct many countably infinite Steiner systems with specified properties.

Care has to be exercised though, as some infinite systems with specified properties do not exist. On one hand, the existence of some systems is easily ruled out by combinatorial restrictions; for example, the non-existence of a block-transitive, point-intransitive Steiner triple system [1, 9], although block-transitive, point-intransitive Steiner systems with block size greater than 3 do exist [6]. On the other hand, some countably infinite Steiner systems with specified properties do not exist and there is no obvious combinatorial reason why. It has recently been shown that *all* infinite Steiner systems with block size strictly less than v are resolvable [5], and so no non-resolvable infinite Steiner triple system exists. Note, however, that the projective plane is an example of a non-resolvable Steiner system with block size equal to v .

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