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A note on the generalized factorial*

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Abstract

The set of the generalized factorials $\{(x|d)_i\}_{i=0}^n$ is a basis of the vector space of all real polynomials of degree at most n, where $(x|d)_i = x(x-d)\cdots(x-id+d)$ and d is a fixed negative real number. In this note, we show that if a polynomial has only real nonpositive zeros in the standard basis, then so does the polynomial having the same coefficients with the former in the basis $\{(x|d)_i\}_{i=0}^n$. This is a generalization of a result in the work of Brenti [F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, *Mem. Amer. Math. Soc.* 81 (1989), no. 413.].

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1 Introduction

Let i be a positive integer. The monic polynomial $x(x-1)\cdots(x-i+1)$ is referred to as the *falling factorial*, which is denoted by $(x)^{\underline{i}}$ following Graham et al. [2], i.e.,

$$(x)^{\underline{i}} := x(x-1)\cdots(x-i+1),$$

with a conventional definition $(x)^{\underline{0}} := 1$. It is clear that $(i)^{\underline{i}} = i!$, where $i!$ is the usual factorial of i. Accordingly, the *rising factorial* (or *Pochhammer symbol*, which is conventional in hypergeometric theory) is defined as

$$(x)^{\overline{i}} := x(x+1)\cdots(x+i-1),$$

with $(x)^{\overline{i}} := 1$. Let d be a real number. Define the *generalized factorial* by

$$(x|d)_i := x(x-d)(x-2d)\cdots(x-(i-1)d),$$

and $(x|d)_0 := 1$ (see [4]). For $d \neq 0$, $(x|d)_i = d^i(\frac{x}{d})^{\underline{i}}$. In particular, $(x|1)_i = (x)^{\underline{i}}$ and $(x|-1)_i = (x)^{\overline{i}}$. The generalized factorial has been extensively investigated (see [3, 4, 5, 6]).

Denote by P_n the vector space of all real polynomials of degree at most n. It is easy to see that the sets $\{x^i\}_{i=0}^n$ (standard basis of P_n), $\{\frac{x^i}{i!}\}_{i=0}^n$, $\{(x)^{\underline{i}}\}_{i=0}^n$ and $\{(x)^{\overline{i}}\}_{i=0}^n$ are respectively four bases of P_n. Brenti [1] deeply studied the real zeros of the polynomials under various bases of P_n. Here we state three of his results about the four above-mentioned bases. Assume that the polynomial coefficients $\{a_i\}_{i=0}^n$ in the following results are all nonnegative.

Theorem 1. ([1, Theorem 2.4.3]) *If the polynomial $\sum_{i=0}^n a_i x^i$ has only real and nonpositive zeros, then so does the polynomial $\sum_{i=0}^n a_i (x)^{\overline{i}}$. Moreover, the zeros of the latter are simple and the distance between any two of them is at least* 1.

Theorem 2. ([1, Theorem 2.4.2]) *If the polynomial $\sum_{i=0}^n a_i (x)^{\underline{i}}$ has only real and nonpositive zeros, then so does the polynomial $\sum_{i=0}^n a_i x^i$.*

Theorem 3. ([1, Theorem 2.4.1]) *If the polynomial $\sum_{i=0}^n a_i x^i$ has only real zeros, then so does the polynomial $\sum_{i=0}^n \frac{a_i}{i!} x^i$.*

It is natural to ask whether the three results above can be generalized to the generalized factorial or not. In the next section, we will show that only Theorem 1 has this generalization.

2 Main results

Theorem 1 due to Brenti is a special case of the following result when $d = -1$.

Theorem 4. *Let d be a negative real number and $a_0, a_1, \ldots, a_n$ be nonnegative real numbers. If the polynomial $\sum_{i=0}^{n} a_i x^i$ has only real and nonpositive zeros, then so does the polynomial $\sum_{i=0}^{n} a_i (x|d)_i$. Moreover, the zeros of the latter are simple and the distance between any two of them is at least $-d$.*

Proof. The proof is modeled after the one of Theorem 2.4.3 in [1]. We will prove the theorem by induction on n. The result is clearly true if $n = 1$. Now suppose that it is true for polynomials of the degree at most n. Let $\sum_{i=0}^{n+1} b_i x^i$ be a real polynomial of degree $n + 1$. Suppose that

$$\sum_{i=0}^{n+1} b_i x^i = b_{n+1}(x + r_1)(x + r_2) \cdots (x + r_{n+1}),$$

where $r_j \geq 0$ for $j = 0, 1, \ldots, n+1$. Note that $b_i \geq 0$ since b_i is the i-th the elementary symmetric function of the r_js. Let $\sum_{i=0}^{n} c_i x^i = b_{n+1}(x+r_1)(x+r_2) \cdots (x+r_n)$. Then

$$\begin{aligned} \sum_{i=0}^{n+1} b_i x^i &= (x + r_{n+1}) \sum_{i=0}^{n} c_i x^i \\ &= \sum_{i=0}^{n} c_i x^{i+1} + r_{n+1} \sum_{i=0}^{n} c_i x^i \\ &= \sum_{i=0}^{n+1} (c_{i-1} + r_{n+1} c_i) x^i, \end{aligned}$$

where $c_{-1} = c_{n+1} = 0$. It follows that

$$b_i = c_{i-1} + r_{n+1} c_i, \qquad i = 0, 1, \ldots, n + 1.$$

Now let $P(x) = \sum_{i=0}^{n} c_i (x|d)_i$ and $Q(x) = \sum_{i=0}^{n+1} b_i (x|d)_i$. Then

$$\begin{aligned} Q(x) &= \sum_{i=0}^{n+1} (c_{i-1} + r_{n+1} c_i)(x|d)_i \\ &= \sum_{i=0}^{n+1} c_{i-1}(x|d)_i + r_{n+1} \sum_{i=0}^{n} c_i (x|d)_i \\ &= xP(x - d) + r_{n+1} P(x). \end{aligned}$$

Now using the induction hypothesis for $\sum_{i=0}^{n} c_i x^i$, we have $P(x) = b_{n+1}(x - s_1)(x - s_2) \cdots (x - s_n)$, where $s_n < s_{n-1} < \cdots < s_1 \leq 0$, and $s_i - s_{i+1} \geq -d$ for $i = 1, 2, \ldots, n-1$. Then it follows from $Q(x) = xP(x-d) + r_{n+1} P(x)$ that $Q(0) \geq 0$, $\lim_{x \to -\infty} \operatorname{sgn}(Q(x)) = (-1)^{n+1}$, and for $i = 1, 2, \ldots, n - 1$,

$$(-1)^i Q(s_i) \geq 0, \quad (-1)^i Q(s_i + d) \geq 0.$$

This implies that $Q(x)$ has a real zero in $[s_{i+1}, s_i + d]$ for $i = 1, 2, \ldots, n-1$, a real zero in $[s_1, 0]$ and a real zero in $(-\infty, s_n + d]$. Hence $Q(x)$ has only simple nonpositive zeros: $\xi_{n+1} < \xi_n < \cdots < \xi_2 < \xi_1 \leq 0$, and $s_i \leq \xi_i$, $\xi_{i+1} \leq s_i + d$ for $i = 1, 2, \ldots, n$. Therefore $\xi_i - \xi_{i+1} \geq s_i - s_i - d = -d$. The proof is completed. □

Remark 1. When d is a positive number not equal to 1, Theorem 2 cannot be generalized analogous to Theorem 1. For example, when $d = 5$, the polynomial $1 + 3(x|5) + 3(x|5)_2 + 5(x|5)_3 + 5(x|5)_4$ has only real zeros:

$$13.8249,\ 10.1888,\ 4.98602,\ 0.000284768.$$

But the zeros of $1 + 3x + 3x^2 + 5x^3 + 5x^4$ are

$$-0.796675,\ \ -0.454591,\ 0.125633 \pm 0.732432\sqrt{-1}.$$

Remark 2. When d is a real number not equal to 0, Theorem 3 does not hold for the generalized factorial. For example,

(1) the zeros of $1 + 5(x)^{\underline{1}} + 6(x)^{\underline{2}} + 45(x)^{\underline{3}} + (x)^{\underline{4}}$ are

$$-41.8659,\ 1.6718,\ 1.20591,\ \ -0.0118478.$$

But the zeros of $1 + 5\frac{(x)^{\underline{1}}}{1!} + 6\frac{(x)^{\underline{2}}}{2!} + 45\frac{(x)^{\underline{3}}}{3!} + \frac{(x)^{\underline{4}}}{4!}$ are

$$-176.601,\ \ -0.0560534,\ 1.32835 \pm 0.812378\sqrt{-1}.$$

(2) the zeros of $3 + 15(x)^{\overline{1}} + 20(x)^{\overline{2}} + 30(x)^{\overline{3}} + 61(x)^{\overline{4}} + 41(x)^{\overline{5}} + 11(x)^{\overline{6}} + (x)^{\overline{7}}$ are

$$-12.3845, -8.03645, -5.37911, -3.15662, -2.02696, -1.01546, -0.000862455.$$

But the zeros of $3 + 15\frac{(x)^{\overline{1}}}{1!} + 20\frac{(x)^{\overline{2}}}{2!} + 30\frac{(x)^{\overline{3}}}{3!} + 61\frac{(x)^{\overline{4}}}{4!} + 41\frac{(x)^{\overline{5}}}{5!} + 11\frac{(x)^{\overline{6}}}{6!} + \frac{(x)^{\overline{7}}}{7!}$ are

$$-51.5946, -25.1559, -13.5813, -3.89986, -0.0530304, -1.85761 \pm 0.834773\sqrt{-1}.$$

Remark 3. This part is due to the helpful suggestions of an anonymous referee. Brenti [1] considered the transition matrices between different bases. For example, the transition matrix from $\{x^i\}_{i=0}^n$ to $\{(x)^{\overline{i}}\}_{i=0}^n$ is the $(n+1)\times(n+1)$ unsigned Stirling number matrix of the first kind $\mathbf{S1us}_n$[7, A132393]. It is clear that for a fixed negative real number $d = -|d| < 0$, the transition matrix from $\{x^i\}_{i=0}^n$ to $\{(x|d)_i\}_{i=0}^n$ is the $(n+1)\times(n+1)$ lower triangle matrix $\mathbf{D}_n(|d|)\mathbf{S1us}_n\mathbf{D}_n^{-1}(|d|)$, where $\mathbf{D}_n(|d|)$ denotes the diagonal $(n+1)\times(n+1)$ matrix with $|d|^m$ in the m-th row $(m = 0, 1, 2, \ldots, n)$. More precisely, write $a(n, k)$ for the coefficients a_k in a polynomial $P(x)$ of degree n, and similarly for $q_d(n, k)$, the coefficients q_k of $P(x) = \sum_{k=0}^n q_k(x|d)_k$. Then we have the following transformation

$$T_{n,|d|}:\quad \vec{q}_{-|d|}(n) = \vec{a}(n)\mathbf{D}_n(|d|)\mathbf{S1us}_n\mathbf{D}_n^{-1}(|d|),$$

where $\vec{a}(n) := (a(n,0), a(n,1), \ldots, a(n,n))$ and $\vec{q}_{-|d|}(n) := (q_d(n,0), q_d(n,1), \ldots, q_d(n,n))$.

Here we will concentrate on a group property for the fixed negative real number $d = -|d| < 0$. If we iterate the above-mentioned transformation, then the powers of the matrix $\mathbf{S1us}_n$ appear. Note that there also exist negative powers for the inverse elements of this transformation group, since the inverse of $\mathbf{S1us}_n$ is known to be the signed matrix $\mathbf{S2s}_n$ with elements $\mathbf{S2s}_n(i,j) = (-1)^{i-j}\mathbf{S2}_n(i,j)$, where $\mathbf{S2}_n(i,j)$ is the usual $(n+1) \times (n+1)$ Stirling matrix of the second kind $\mathbf{S1us}_n$[7, A048993]. The linear group transformations are then

$$T^l_{n,|d|}: \quad q^{(l)}_{-|d|}(n) = \mathbf{D}_n(|d|)\mathbf{S1us}^l_n\mathbf{D}^{-1}_n(|d|)$$

for $l \in \mathbf{Z}$. The neutral group element $\mathbf{S1us}^0_n = \mathbf{1}_n$ (the $(n+1)\times(n+1)$ unit matrix).

Similarly, for a fixed positive real number $d = |d| > 0$, we only replace $\mathbf{S1us}_n$ by the $(n+1)\times(n+1)$ signed Stirling number matrix of the first kind $\mathbf{S1s}_n$. Note that the inverse matrix of $\mathbf{S1s}_n$ is the usual $(n+1)\times(n+1)$ Stirling matrix of the second kind.

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