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Erdős-Ko-Rado with separation conditions

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Abstract

A family $\mathcal{A}$ of sets is said to be *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{A}$. For a family $\mathcal{F}$ of sets, let

$$\mathrm{ex}(\mathcal{F}) := \{\mathcal{A} \subseteq \mathcal{F} \colon \mathcal{A} \text{ is a largest intersecting subfamily of } \mathcal{F}\}.$$

For $n \geq 0$ and $r \geq 0$, let $[n] := \{i \in \mathbb{N} \colon i \leq n\}$ and $\binom{[n]}{r} := \{A \subseteq [n] \colon |A| = r\}$. For a sequence $\{d_i\}_{i \in \mathbb{N}}$ of non-negative integers that is *monotonically non-decreasing* (i.e. $d_i \leq d_{i+1}$ for all $i \in \mathbb{N}$), let $\mathcal{P}(\{d_i\}_{i\in\mathbb{N}}) := \{\{a_1, \dots, a_r\} \subset \mathbb{N} \colon r \in \mathbb{N},\, a_{i+1} > a_i + d_{a_i} \text{ for each } i \in [r-1]\}$. Let ${\mathcal{P}_n}^{(r)} := \mathcal{P}(\{d_i\}_{i\in\mathbb{N}}) \cap \binom{[n]}{r}$. We determine $\mathrm{ex}({\mathcal{P}_n}^{(r)})$ for $d_1 > 0$ and any r, and for $d_1 = 0$ and $r \leq \frac{1}{2}\max\{s \in [n] \colon {\mathcal{P}_n}^{(s)} \neq \emptyset\}$. We particularly have that $\{A \in {\mathcal{P}_n}^{(r)} \colon 1 \in A\} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$; Holroyd, Spencer and Talbot established this for the case where $d_1 > 0$ and $d_i = d_1$ for all $i \in \mathbb{N}$, and a part of the paper generalises a compression method that they introduced. The Erdős-Ko-Rado Theorem and the Hilton-Milner Theorem provide the solution for the case where $d_i = 0$ for all $i \in \mathbb{N}$.

1 Introduction

We start by setting up some basic notation. Unless otherwise stated, we shall use small letters such as x to denote non-negative integers or functions or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (i.e. sets whose members are sets themselves). The set of positive integers $\{1, 2, \dots\}$ is denoted by $\mathbb{N}$. For $m, n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ is denoted by $[m, n]$; we abbreviate $[1, n]$ to $[n]$, and we take $[0]$ to be the empty set $\emptyset$. For a set X, the *power set* $\{A \colon A \subseteq X\}$ of X is denoted by 2^X, and the family $\{A \subseteq X \colon |A| = r\}$ is denoted by $\binom{X}{r}$.

We next develop some notation for certain sets and families defined on a family $\mathcal{F}$. Let $\mathcal{F}^{(r)} := \{F \in \mathcal{F} \colon |F| = r\}$, $\alpha(\mathcal{F}) := \max\{|F| \colon F \in \mathcal{F}\}$, $U(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} F$.

For any $V \subseteq U(\mathcal{F})$, let

$$\mathcal{F}[V] := \{F \in \mathcal{F} \colon V \subseteq F\}, \quad \mathcal{F}\langle V\rangle := \{F \backslash V \colon F \in \mathcal{F}[V]\},$$
$$\mathcal{F}(V) := \{F \in \mathcal{F} \colon F \cap V \neq \emptyset\}, \quad \mathcal{F}(\overline{V}) := \{F \in \mathcal{F} \colon F \cap V = \emptyset\}.$$

For $u \in U(\mathcal{F})$, we abbreviate $\mathcal{F}(\{u\})$ $(= \mathcal{F}[\{u\}])$, $\mathcal{F}\langle\{u\}\rangle$ and $\mathcal{F}(\overline{\{u\}})$ to $\mathcal{F}(u)$, $\mathcal{F}\langle u\rangle$ and $\mathcal{F}(\overline{u})$, respectively.

Let $\mathcal{A}$ be a family. $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{A}$. $\mathcal{A}$ is said to be *centred* if $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$ (i.e. $\mathcal{A} = \mathcal{A}(a) \neq \emptyset$ for some $a \in U(\mathcal{A})$), and *non-centred* otherwise. Note that a centred family is trivially intersecting.

For a family $\mathcal{F}$, we define

$$\mathrm{L}(\mathcal{F}) := \{\mathcal{A} \subseteq \mathcal{F} \colon \mathcal{A} \text{ is a largest centred subfamily of } \mathcal{F}\},$$
$$\mathrm{ex}(\mathcal{F}) := \{\mathcal{A} \subseteq \mathcal{F} \colon \mathcal{A} \text{ is an } \textit{extremal} \text{ (i.e. largest) intersecting subfamily of } \mathcal{F}\}.$$

One of the most popular endeavours in extremal set theory is that of determining the size or structure of a largest intersecting subfamily of a given family $\mathcal{F}$. This originated in [10], which features a classical result that says that if $r \leq n/2$, then the size of a largest intersecting subfamily of $\binom{[n]}{r}$ is the size $\binom{n-1}{r-1}$ of the centred subfamily $\{A \in \binom{[n]}{r} \colon 1 \in A\}$. Thus, if $r \leq n/2$, then $\mathrm{L}(\binom{[n]}{r}) \subseteq \mathrm{ex}(\binom{[n]}{r})$; note that $\mathrm{L}(\binom{[n]}{r}) = \{\{A \in \binom{[n]}{r} \colon i \in A\} \colon i \in [n]\}$. This result is known as the Erdős-Ko-Rado (EKR) Theorem. Note that if $n/2 < r \leq n$, then we trivially have that $\binom{[n]}{r}$ itself is intersecting. There are various proofs of the EKR Theorem, two of which are particularly short and beautiful: Katona's [21], introducing the elegant cycle method, and Daykin's [8], using the fundamental Kruskal-Katona Theorem [22, 23]. If $r < n/2$, then, by the Hilton-Milner (HM) Theorem [15], $\mathrm{L}(\binom{[n]}{r}) = \mathrm{ex}(\binom{[n]}{r})$. The EKR Theorem inspired a wealth of results that establish how large a system of sets can be under certain intersection conditions; see [9, 12, 11, 4].

For a *monotonically non-decreasing (mnd) sequence* $\{d_i\}_{i \in \mathbb{N}}$ of non-negative integers (i.e. $0 \leq d_1 \leq d_2 \leq \cdots$) and a set $X \subset \mathbb{N}$, we define

$$\mathcal{P}(\{d_i\}_{i \in \mathbb{N}}) := \{\{a_1, \dots, a_r\} \subset \mathbb{N} \colon r \in \mathbb{N},\, a_{i+1} > a_i + d_{a_i} \text{ for each } i \in [r-1]\},$$
$$\mathcal{P}_X(\{d_i\}_{i \in \mathbb{N}}) := \mathcal{P}(\{d_i\}_{i \in \mathbb{N}}) \cap 2^X.$$

If $X = [n]$, then we may write $\mathcal{P}_n(\{d_i\}_{i \in \mathbb{N}})$ instead of $\mathcal{P}_X(\{d_i\}_{i \in \mathbb{N}})$. For convenience and neatness of notation, we assume that $\{d_i\}_{i \in \mathbb{N}}$ is some fixed mnd sequence, and we drop the argument '$(\{d_i\}_{i \in \mathbb{N}})$' from any of the notation for the families defined above unless we consider a different sequence.

In this paper we are concerned with the EKR problem for the family ${\mathcal{P}_n}^{(r)}$. Since the nature of the problem for the case $d_1 > 0$ is fundamentally different from that for the case $d_1 = 0$, we will treat the two cases separately. One difference has to do with the extremal structures. As we will show, if $d_1 > 0$, then ${\mathcal{P}_n}^{(r)}(1) \in \mathrm{L}({\mathcal{P}_n}^{(r)}) \subseteq \mathrm{ex}({\mathcal{P}_n}^{(r)})$ for all r. Now suppose $d_1 = 0$. If $r = \alpha(\mathcal{P}_n)$, then $1 \in A$ for all $A \in {\mathcal{P}_n}^{(r)}$, and hence $\mathrm{ex}({\mathcal{P}_n}^{(r)}) = \{{\mathcal{P}_n}^{(r)}(1)\}$. We will show that ${\mathcal{P}_n}^{(r)}(1) \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$ also if

$r \le \alpha(\mathcal{P}_n)/2$. However, if $\alpha(\mathcal{P}_n)/2 < r < \alpha(\mathcal{P}_n)$, then $\mathrm{L}({\mathcal{P}_n}^{(r)}) \subseteq \mathrm{ex}({\mathcal{P}_n}^{(r)})$ is not guaranteed to hold. For example, if $d_i = 0$ for all $i \in \mathbb{N}$, then ${\mathcal{P}_n}^{(r)} = \binom{[n]}{r}$, and hence, for $n/2 < r < n$ $(= \alpha(\mathcal{P}_n))$, ${\mathcal{P}_n}^{(r)}$ is non-centred, intersecting, and of course larger than ${\mathcal{P}_n}^{(r)}(1)$ $(\in \mathrm{L}({\mathcal{P}_n}^{(r)}))$; other examples with $0 = d_1 < d_{n-1}$ can easily be constructed.

For the case $d_1 > 0$, we determine every single extremal structure and exactly when it arises (i.e. for which sequences $\{d_i\}_{i\in\mathbb{N}}$ it is extremal); the proof is self-contained.

For the case $d_1 = 0$, we restrict ourselves to $r \le \alpha(\mathcal{P}_n)/2$, and in addition to showing that

$$\begin{aligned} \mathrm{L}({\mathcal{P}_n}^{(r)}) &= \{\{A \in {\mathcal{P}_n}^{(r)} \colon i \in A\} \colon i \in [n], \text{ either } d_i = 0 \text{ or } i = n \text{ and } d_{n-1} = 0\} \\ &\subseteq \mathrm{ex}({\mathcal{P}_n}^{(r)}), \end{aligned}$$

we determine precisely when $\mathrm{L}({\mathcal{P}_n}^{(r)}) = \mathrm{ex}({\mathcal{P}_n}^{(r)})$. The proof makes use of the EKR Theorem, the HM Theorem and a slight extension [2] of a cross-intersection theorem of Hilton [13]. For the case where $\mathrm{L}({\mathcal{P}_n}^{(r)}) \neq \mathrm{ex}({\mathcal{P}_n}^{(r)})$, we give a characterisation of the families in $\mathrm{ex}({\mathcal{P}_n}^{(r)})$, similar to that in Theorem 1.1(ii) below, in terms of necessary and sufficient conditions that their sets must satisfy.

Our proofs are based on the compression technique (see Section 4), which was introduced in the original proof of the EKR Theorem [10]. We remark that although they may appear somewhat lengthy, an effort has been made to make them as comprehensible and as detailed as possible.

Solutions to our problem for the case $d_1 = \cdots = d_{n-1} = d$ already exist. If $d = 0$, then ${\mathcal{P}_n}^{(r)} = \binom{[n]}{r}$, and hence, by the EKR Theorem and the HM Theorem, we have the following complete solution.

Theorem 1.1 ([10, 15]) *Suppose $d_i = 0$ for each $i \in [n-1]$.*
(i) If $1 \le r < n/2$, then $\mathrm{ex}({\mathcal{P}_n}^{(r)}) = \{{\mathcal{P}_n}^{(r)}(j) \colon j \in [n]\}$.
(ii) If $r = n/2$, then $\mathrm{ex}({\mathcal{P}_n}^{(r)}) = \{\mathcal{A} \subset {\mathcal{P}_n}^{(r)} \colon |\{A, [n]\backslash A\} \cap \mathcal{A}| = 1$ for each $A \in {\mathcal{P}_n}^{(r)}\} \supset \{{\mathcal{P}_n}^{(r)}(j) \colon j \in [n]\}$.
(iii) If $n/2 < r \le n$, then $\mathrm{ex}({\mathcal{P}_n}^{(r)}) = \{{\mathcal{P}_n}^{(r)}\}$.

For $d > 0$, Holroyd, Spencer and Talbot showed that ${\mathcal{P}_n}^{(r)}(1) \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$ (and hence $\mathrm{L}({\mathcal{P}_n}^{(r)}) \subseteq \mathrm{ex}({\mathcal{P}_n}^{(r)})$), but they left the problem of determining $\mathrm{ex}({\mathcal{P}_n}^{(r)})$ open (this is solved here).

Theorem 1.2 ([19]) *If $d \in \mathbb{N}$ and $d_i = d$ for each $i \in [n-1]$, then ${\mathcal{P}_n}^{(r)}(1) \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$ for each $r \in [\alpha(\mathcal{P}_n)]$.*

Before stating our main results, for which we need to develop some further notation and definitions, we describe our problem using a graph-theoretical formulation that makes it easy for us to relate the work in this paper to certain results that were of a high degree of inspiration and motivation behind it.

2 A graph-theoretical interpretation and a brief review of some motivating results

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a set and $E(G) \subseteq \binom{V(G)}{2}$. An element of $V(G)$ is called a *vertex of* G, and an element of $E(G)$ is called an *edge of* G. A subset I of $V(G)$ is said to be an *independent set of* G if $\{v, w\} \notin E(G)$ for every $v, w \in I$ with $v \neq w$. The size of a largest independent set of G is called the *independence number of* G. We denote the family of all independent sets of G by $\mathcal{I}_G$. We may abbreviate $\alpha(\mathcal{I}_G)$ to $\alpha(G)$; so $\alpha(G)$ denotes the independence number of G. Let $\mu(G)$ denote the size of a smallest maximal independent set of G, where by 'maximal' we mean that it is not a subset of another independent set of G.

A family $\mathcal{F}$ is said to have the *EKR property* if $\mathrm{L}(\mathcal{F}) \subseteq \mathrm{ex}(\mathcal{F})$, and to have the *strict EKR property* if $\mathrm{L}(\mathcal{F}) = \mathrm{ex}(\mathcal{F})$; we may abbreviate by saying that $\mathcal{F}$ is *EKR* if $\mathrm{L}(\mathcal{F}) \subseteq \mathrm{ex}(\mathcal{F})$, and *strictly EKR* if $\mathrm{L}(\mathcal{F}) = \mathrm{ex}(\mathcal{F})$. It is interesting that many EKR-type results can be expressed in terms of the EKR or strict EKR property of ${\mathcal{I}_G}^{(r)}$ for some graph G and $r \in X \subseteq [\alpha(G)]$, the simplest case being Theorem 1.1, which completely solves the problem for the empty graph $([n], \emptyset)$. This observation surfaced in [19, 20], and more examples are given in [7]. The work in this paper can also be expressed in such graph-theoretical terms; indeed, $\mathcal{P}_n = \mathcal{I}_{M_n}$ for M_n as defined below.

Definition 2.1 *For an mnd sequence $\{d_i\}_{i\in\mathbb{N}}$ of non-negative integers, let $M := M(\{d_i\}_{i\in\mathbb{N}})$ be the graph such that $V(M) = \mathbb{N}$ and, for every $a, b \in V(M)$ with $a < b$, $\{a, b\} \in E(M)$ if and only if $b \leq a + d_a$. Let $M_n := M_n(\{d_i\}_{i\in\mathbb{N}})$ be the subgraph induced from M by the subset $[n]$ of $V(M)$, i.e. $M_n = \left([n], E(M) \cap \binom{[n]}{2}\right)$. We refer to M_n as an mnd graph.*

Suppose $M_n = M_n(\{d_i = d\}_{i\in\mathbb{N}})$, $d \in \mathbb{N}$, and G is a copy of M_n. Then G is called a *d'th power of a path*, and if $d = 1$, then G is also simply called a *path*.

Note that Theorem 1.2 establishes the EKR property of ${\mathcal{I}_{M_n}}^{(r)}$ for the special case where M_n is a power of a path.

For $1 \leq k \leq n$, let C_n^k be the graph with $V(C_n^k) = [n]$ and $E(C_n^k) = \{\{a, b\} \in \binom{[n]}{2} \colon |(a - b) \bmod n| \leq k\}$. C_n^k is called a *k'th power of a cycle*. A conjecture of Holroyd [18] about intersecting families of *separated sets* was proved by Talbot [25] and can be stated as follows.

Theorem 2.2 ([25]) *If $1 \leq r \leq \alpha(C_n^k)$, then ${\mathcal{I}_{C_n^k}}^{(r)}$ is EKR, and strictly so unless $k = 1$ and $n = 2r + 2$.*

The study of the general EKR problem for families of independent sets of graphs (i.e. for families $\mathcal{F}$ of the form ${\mathcal{I}_G}^{(r)}$) was initially motivated by Holroyd's conjecture. This study gave rise to a number of results, which we outline below.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph H is *connected* if for every $v, w \in V(H)$ there exists a subgraph P of H such that P is a path and $v, w \in V(P)$. If G is a disjoint union of connected graphs $G_1, \dots, G_q$ ('disjoint' here means that $V(G_1), \dots, V(G_q)$ are pairwise disjoint), then we say that G_j ($j \in [q]$) is a *component* of G. We denote the set of components of G by $\mathrm{C}(G)$.

Theorem 2.2 motivated the investigation of the EKR problem for the more general case where G is a disjoint union of powers of cycles. The EKR property of ${\mathcal{I}_G}^{(r)}$ for certain graphs G of this kind is established in [16, 17, 14].

A graph H is said to be *complete* if $E(H) = \binom{V(H)}{2}$. A *singleton* is a component consisting of just one vertex.

Theorem 2.3 ([19]) *Let G be a graph such that $\mathrm{C}(G)$ consists of complete graphs, paths, cycles, and at least one singleton. If $1 \leq r \leq |\mathrm{C}(G)|/2$, then ${\mathcal{I}_G}^{(r)}$ is EKR.*

Holroyd and Talbot [20] gave examples of graphs G and values r for which ${\mathcal{I}_G}^{(r)}$ is EKR or strictly EKR, and others for which ${\mathcal{I}_G}^{(r)}$ is not EKR. Their investigation led them to the following interesting conjecture.

Conjecture 2.4 ([20]) *Let G be a graph. If $1 \leq r \leq \mu(G)/2$, then ${\mathcal{I}_G}^{(r)}$ is EKR, and strictly so if $r < \mu(G)/2$.*

A special case of the main result in [3] is that this conjecture is true when $\mu(G) \geq \frac{3}{2}(r-1)^2(3r-4) + r$. Proving the full conjecture seems very difficult; however, restricting the problem to some classes of graphs containing singletons makes it tractable. When an arbitrary number of singletons are allowed in G, ${\mathcal{I}_G}^{(r)}$ may not be EKR for $r > \mu(G)/2$; in fact, if G consists solely of singletons, then, by Theorem 1.1(iii), ${\mathcal{I}_G}^{(r)}$ is not EKR for $\mu(G)/2 < r < \alpha(G)$. On the other hand, Theorem 2.2 demonstrates graphs G for which ${\mathcal{I}_G}^{(r)}$ is EKR for all $r \leq \alpha(G)$ (and not just $r \leq \mu(G)/2$).

Note that Theorem 2.3 does not live up to Conjecture 2.4 because $\mu(G)$ is at least as large as $|\mathrm{C}(G)|$ and there is no bound as to how much larger it can be. As pointed out in [7], the consideration of mnd graphs, introduced in this paper[1], led to a proof [7] of the conjecture for a class of graphs that is significantly wider than the one captured by Theorem 2.3; the trick is to apply a stronger induction hypothesis by allowing G to have copies of mnd graphs as components.

Conjecture 2.4 has been verified for other classes of graphs containing singletons; see [6, 24, 26].

3 Main results

For a finite set $A \subset \mathbb{N}$, let

$$l(A) := \min\{a \in A\}, \quad u(A) := \max\{a \in A\}.$$

[1]For the sake of precision, we remark that the work in this paper is actually part of the author's Ph.D. thesis [5].

For $i, r \in \mathbb{N}$, define $P_{i,r} := \{p_1, \dots, p_r\} \in \mathcal{P}$ by $p_1 := i$ and $p_{j+1} := p_j + d_{p_j} + 1$, $j = 1, \dots, r-1$ (if $r > 1$). We need to define $P_{i,0} := \emptyset$.

For $1 \leq r \leq \alpha(\mathcal{P}_n)$, let

$$k_{n,r} := \max\{i \in [n] \colon u(P_{i,r}) \leq n\}.$$

Let $e_1 := 0$, and for $i \geq 2$, let

$$E_i := \{a \in [i-1] \colon a + d_a \geq i\}, \quad e_i := |E_i|.$$

Clearly, since $\{d_i\}_{i \in \mathbb{N}}$ is mnd,

$$\text{either } E_i = \emptyset \text{ or } E_i = [j, i-1] \text{ for some } j \in [i-1].$$

For any $z \in \mathbb{Z} := \{0\} \cup \mathbb{N} \cup \{-n \colon n \in \mathbb{N}\}$, let $s_z \colon \mathcal{P} \to 2^{\mathbb{N}}$ be defined by

$$s_z(A) := \{a + z \colon a \in A\}.$$

We will often use the fact that

$$A \in \mathcal{P},\ l(A) \geq 2,\ x \in [l(A) - 1] \quad \Rightarrow \quad s_{-x}(A) \in \mathcal{P},$$

which is again a consequence of $\{d_i\}_{i \in \mathbb{N}}$ being mnd.

We say that $\mathcal{P}_{[x,y]}$ is *symmetric* if $\mathcal{P}_{[x,y]} = \mathcal{P}_{[x,y]}(\{d_i^* = d\}_{i \in \mathbb{N}})$ for some $d \in \mathbb{N} \cup \{0\}$; otherwise, we say that $\mathcal{P}_{[x,y]}$ is *asymmetric*. Note that if $\alpha(\mathcal{P}_{[x,y]}) > 1$, then $\mathcal{P}_{[x,y]}$ is symmetric if and only if $e_y = d_x$.

Suppose $d_1 = d_3 = 1$, $y \in P_{3,r} = s_1(P_{2,r})$, $r \geq 2$, and for

$$m := \begin{cases} \max\{a \in [y] \colon d_a = 1\} & \text{if } \mathcal{P}_y \text{ is asymmetric;} \\ y & \text{if } \mathcal{P}_y \text{ is symmetric,} \end{cases}$$

$m = 2t + 1$ for some $t \in [r]$. Then we say that ${\mathcal{P}_y}^{(r)}$ is *type I*, and we say that a subfamily $\mathcal{A}$ of ${\mathcal{P}_y}^{(r)}$ is *special* if $\mathcal{A} = \{A_1, \dots, A_q\} \cup ({\mathcal{P}_y}^{(r)}(1) \backslash \{B_1, \dots, B_q\})$ for some $q \in [t]$, where

$$A_1 := P_{3,r} = P_{3,t} \cup P_{m+2,r-t}, \quad B_t := P_{1,t} \cup P_{m+1,r-t},$$

and for each $i \in [t-1]$ (if $t > 1$),

$$A_{i+1} := \{2j \colon j \in [i]\} \cup \{2j+1 \colon j \in [i+1, t]\} \cup P_{m+2,r-t}, \quad B_i := s_{-1}(A_{i+1}).$$

If ${\mathcal{P}_y}^{(r)}$ is type I and $\mathcal{P}_y$ is symmetric, then, since $d_1 = 1$ and $m = y \in P_{3,r}$, we have $t = r$. Note that a special family $\mathcal{A}$ as above is ${\mathcal{P}_y}^{(r)}(y)$ if and only if $\mathcal{P}_y$ is symmetric and $q = t = r$ (otherwise, $P_{1,r} \in \mathcal{A} \backslash {\mathcal{P}_y}^{(r)}(y)$). Also note that

$$\text{if } {\mathcal{P}_y}^{(r)} \text{ is type I, } \mathcal{A} \subset {\mathcal{P}_y}^{(r)}, \mathcal{A} \text{ is special, and either } \mathcal{P}_y \text{ is asymmetric or } \mathcal{A} \neq {\mathcal{P}_y}^{(r)}(y),$$
$$\text{then } {\mathcal{P}_y}^{(r)}(1)(y) \cup \{P_{1,r}, P_{3,r}\} \subseteq \mathcal{A}. \tag{1}$$

That a special family is intersecting is not difficult to check; however, for the sake of completeness, this is proved in Section 6 (Lemma 6.4).

If $\mathcal{P}_y$ is asymmetric, $k := k_{y,r} \leq d_1 + 1$ and $y \in P_{k,r} = s_{k-1}(P_{1,r})$, then we say that ${\mathcal{P}_y}^{(r)}$ is *type II*. Note that $P_{k,r} = s_{k-1}(P_{1,r})$ implies that $P_{i,r} = s_1(P_{i-1,r})$ for any $i \in [2, k]$. An example of a type II family is ${\mathcal{P}_{10}}^{(3)}(\{d_i^*\}_{i\in\mathbb{N}})$ with $d_1^* = d_2^* = d_3^* = 2$ and $d_4^* = d_5^* = d_6^* = 3$.

A family ${\mathcal{P}_y}^{(r)}$ cannot be both type I and type II, because if ${\mathcal{P}_y}^{(r)}$ is type I, then $k_{y,r} = 3 > d_1 + 1$.

This brings us to our first and main result.

Theorem 3.1 *Suppose $d_1 > 0$ and $2 \leq r \leq \alpha(\mathcal{P}_n)$.*

(i) If ${\mathcal{P}_n}^{(r)}$ is type I, then $\mathrm{ex}({\mathcal{P}_n}^{(r)}) = \{{\mathcal{P}_n}^{(r)}(1)\} \cup \{\mathcal{A} \subset {\mathcal{P}_n}^{(r)} \colon \mathcal{A}$ is special$\}$.

(ii) If ${\mathcal{P}_n}^{(r)}$ is not type I, and either ${\mathcal{P}_n}^{(r)}$ is type II or $\mathcal{P}_n$ is symmetric, then $\mathrm{ex}({\mathcal{P}_n}^{(r)}) = \{{\mathcal{P}_n}^{(r)}(1), {\mathcal{P}_n}^{(r)}(n)\}$.

(iii) In any other case, $\mathrm{ex}({\mathcal{P}_n}^{(r)}) = \{{\mathcal{P}_n}^{(r)}(1)\}$.

For $r = 1$, we trivially have $\mathrm{ex}({\mathcal{P}_n}^{(r)}) = \{\{\{j\}\} \colon j \in [n]\} = \mathrm{L}({\mathcal{P}_n}^{(r)})$.

Corollary 3.2 *If $d_1 > 0$ and $1 \leq r \leq \alpha(\mathcal{P}_n)$, then ${\mathcal{P}_n}^{(r)}$ is EKR, and strictly so unless ${\mathcal{P}_n}^{(r)}$ is type I.*

Before stating our result for $d_1 = 0$, we recall that families $\mathcal{A}_1, \dots, \mathcal{A}_k$ of sets are said to be *cross-intersecting* if for every $i, j \in [k]$ with $i \neq j$, $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}_i$ and every $B \in \mathcal{A}_j$.

Theorem 3.3 *Suppose $d_1 = 0 < d_{n-1}$ and $1 \leq r \leq \alpha(\mathcal{P}_n)/2$. Let $m := \min\{i \in [n] \colon d_i \neq 0\}$. Let $\mathcal{A} \subseteq {\mathcal{P}_n}^{(r)}$.*

(i) If $n \in P_{1,2r}$ and $m = 2r - 1$, then $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$ if and only if

(a) $\mathcal{A}(\overline{[2r-1, n]}) = \binom{[2r-2]}{r} \backslash \{[2r-2] \backslash A \colon A \in \mathcal{A}\langle 2r-1\rangle\langle n\rangle\}$, $\mathcal{A}\langle 2r-1\rangle\langle n\rangle$ is intersecting,

(b) $\mathcal{A}\langle i\rangle \cap \binom{[2r-2]}{r-1} = \mathcal{A}\langle n\rangle \cap \binom{[2r-2]}{r-1} \in \mathrm{ex}(\binom{[2r-2]}{r-1})$, $i = 2r-1, \dots, n-1$, and

(c) $\mathcal{A}\langle n\rangle \cap \binom{[2r-2]}{r-1}$ and $\mathcal{A}\langle 2r-1\rangle\langle n\rangle$ are cross-intersecting.

(ii) If $n \in P_{1,2r}$ and $r + 2 \leq m \leq 2r - 2$, then $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$ if and only if for some $j \in [m-1]$ and some $\mathcal{H}_0 \subseteq \binom{[m-1]\backslash\{j\}}{r}$, $\mathcal{A} = \mathcal{H}_0 \cup ({\mathcal{P}_n}^{(r)}(j) \backslash \{P_{1,2r} \backslash A \colon A \in \mathcal{H}_0\})$.

(iii) If either $n \notin P_{1,2r}$ or $m \leq r + 1$, then $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$ if and only if $\mathcal{A} = {\mathcal{P}_n}^{(r)}(j)$ for some $j \in [m-1]$.

The reason for imposing the condition $d_{n-1} > 0$ is that if we instead have $d_{n-1} = 0$, then $\mathcal{P}_n$ is as in Theorem 1.1. It is easy to see that the above result and Theorem 1.1 together yield the following.

Corollary 3.4 *If $d_1 = 0$ and $1 \le r \le \alpha(\mathcal{P}_n)/2$, then ${\mathcal{P}_n}^{(r)}$ is EKR, and strictly so unless $n \in P_{1,2r}$ and* $\max\{i \in [2r-1] \colon d_i = 0\} \ge r+1$.

The 'non-strict' part was proved in [7] as a special case of a more general result.

4 Some general-purpose compression tools

One of the most powerful techniques in extremal set theory is that of *compression*, also known as *shifting*. The survey paper [12] gives an account of many applications of this technique.

A *compression operation*, or simply a *compression*, is a function that maps a family to another family while retaining some important properties of the original family, such as its size or the non-empty intersection of pairs of sets belonging to it. The idea is that a family resulting from a compression, or from a sequence of compressions, has key structural properties that the original family might not have.

Various forms of compression have been invented for specific problems. For example, the paper [19] features a compression that is defined in a graph-theoretical setting and yet is widely applicable. We now present a form of compression that is intended for a general purpose and generalises the one defined in [19].

For a family $\mathcal{F}$ and $u, v \in U(\mathcal{F})$, $u \neq v$, let $\Delta_{u,v} \colon 2^{\mathcal{F}} \to 2^{\mathcal{F}}$ be defined by

$$\Delta_{u,v}(\mathcal{A}) := \{\delta_{u,v}(A) \colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \delta_{u,v}(A) \in \mathcal{A}\},$$

where $\delta_{u,v} \colon \mathcal{F} \to \mathcal{F}$ is defined by

$$\delta_{u,v}(F) := \begin{cases} (F\backslash\{v\}) \cup \{u\} & \text{if } u \notin F,\ v \in F,\ (F\backslash\{v\}) \cup \{u\} \in \mathcal{F}; \\ F & \text{otherwise.} \end{cases}$$

The function $\Delta_{u,v}$ is a compression operation. The very first thing to be noted is that

$$|\Delta_{u,v}(\mathcal{A})| = |\mathcal{A}|.$$

We say that $\mathcal{F}$ is (u, v)*-compressed* if for every $F \in \mathcal{F}(\overline{u})(v)$, $(F\backslash\{v\}) \cup \{u\} \in \mathcal{F}$.

We now prove a number of properties, given by the proposition below, of the compression operation defined above. The proofs of the main results will be a manifestation of their usefulness and hence the efficacy of this technique. They arise as generalisations of properties, discovered in [19], of compressions on intersecting families of independent sets of graphs. We particularly emphasize the importance of part (iv), which is an observation of a fairly new kind in the literature and has its roots in the proof of Theorem 1.2 (see [19, First Proof of Theorem 7]); we will use it repeatedly in the proof of Theorem 3.1.

Proposition 4.1 *Let $\mathcal{F}$ be a family, and let $u, v \in U(\mathcal{F})$, $u \neq v$. Let $\mathcal{A}^*$ be an intersecting subfamily of $\mathcal{F}$, and let $\mathcal{A} := \Delta_{u,v}(\mathcal{A}^*)$.*
(i) $\mathcal{A}(\overline{v})$ is intersecting.
(ii) If $\mathcal{F}$ is (u, v)-compressed, then $\mathcal{A}$ is intersecting.
(iii) If $\mathcal{F}[\{u, v\}] = \emptyset$ and $\mathcal{F}$ is (u, v)-compressed, then $\mathcal{A}\langle v\rangle \cup \mathcal{A}(\overline{v})$ is intersecting.
(iv) If $\mathcal{F}[\{u, v\}] = \emptyset$ and there exists $w \in U(\mathcal{F})\backslash\{u, v\}$ such that $\mathcal{F}(\overline{w})$ is (u, v)-compressed, then $\mathcal{A}\langle v\rangle$ is intersecting.

Proof. Let $B_1, B_2 \in \mathcal{A}$. Then, for each $p \in [2]$, there exists $A_p \in \mathcal{A}^*$ such that either $B_p = A_p$ or $B_p = \delta_{u,v}(A_p)$. Since $\mathcal{A}^*$ is intersecting, $A_1 \cap A_2 \neq \emptyset$.

Trivially, $\mathcal{A}(\overline{v})(u)$ is intersecting. Now we clearly have that $\mathcal{A}(\overline{v})(\overline{u}) = \mathcal{A}^*(\overline{v})(\overline{u})$, and hence $A \cap A' \neq \emptyset$ for any $A \in \mathcal{A}(\overline{v})(\overline{u})$ and any $A' \in \mathcal{A}$. Hence (i).

Suppose $\mathcal{F}$ is (u, v)-compressed. It is straightforward that $B_1 \cap B_2 \neq \emptyset$ if either $B_p = A_p$, $p = 1, 2$, or $B_p = \delta_{u,v}(A_p) \neq A_p$, $p = 1, 2$. Suppose $B_1 = A_1$, $B_2 = \delta_{u,v}(A_2) \neq A_2$ (so $u \notin A_2$) and $B_1 \cap B_2 = \emptyset$. Then $A_1 \cap A_2 = \{v\}$, $u \notin A_1$, and hence, since $\mathcal{F}$ is (u, v)-compressed, $A_1 \neq \delta_{u,v}(A_1) \in \mathcal{A}$. Since $A_1, \delta_{u,v}(A_1) \in \mathcal{A}$, $A_1, \delta_{u,v}(A_1) \in \mathcal{A}^*$. But $\delta_{u,v}(A_1) \cap A_2 = \emptyset$, a contradiction. Similarly, we cannot have $B_2 = A_2$, $B_1 = \delta_{u,v}(A_1) \neq A_1$ and $B_1 \cap B_2 = \emptyset$. Hence (ii).

Suppose $\mathcal{F}[\{u, v\}] = \emptyset$ and $\mathcal{F}$ is (u, v)-compressed. By (i), $\mathcal{A}(\overline{v})$ is intersecting. By (ii), $A \cap B \neq \emptyset$ for any $A \in \mathcal{A}(\overline{v})$ and any $B \in \mathcal{A}\langle v\rangle$. So (iii) follows if we show that $\mathcal{A}\langle v\rangle$ is intersecting. So suppose $B_1, B_2 \in \mathcal{A}(v)$. Then, for each $p \in [2]$, $B_p \in \mathcal{A}^*(v) \subseteq \mathcal{F}(v)$, and $u \notin B_p$ since $\mathcal{F}[\{u, v\}] = \emptyset$. Since $\mathcal{F}$ is (u, v)-compressed, we have $B_p \neq \delta_{u,v}(B_p) \in \mathcal{A}$, which implies that $\delta_{u,v}(B_p) \in \mathcal{A}^*$ (since $B_p \in \mathcal{A}$). So $(B_1 \cap B_2)\backslash\{v\} = B_1 \cap \delta_{u,v}(B_2) \neq \emptyset$ since $u \notin B_1$, $B_1, \delta_{u,v}(B_2) \in \mathcal{A}^*$ and $\mathcal{A}^*$ is intersecting. Hence (iii).

Suppose $\mathcal{F}[\{u, v\}] = \emptyset$ and there exists $w \in U(\mathcal{F})\backslash\{u, v\}$ such that $\mathcal{F}(\overline{w})$ is (u, v)-compressed. Suppose $B_1, B_2 \in \mathcal{A}(v)$. Then, for each $p \in [2]$, $B_p \in \mathcal{A}^*(v) \subseteq \mathcal{F}(v)$, and $u \notin B_p$ since $\mathcal{F}[\{u, v\}] = \emptyset$. Thus, if $w \notin B_p$ for some $p \in [2]$, then, since $\mathcal{F}(\overline{w})$ is (u, v)-compressed, we have $B_p \neq \delta_{u,v}(B_p) \in \mathcal{A}$, which implies that $\delta_{u,v}(B_p) \in \mathcal{A}^*$ (since $B_p \in \mathcal{A}$) and hence $(B_1 \cap B_2)\backslash\{v\} = B_{3-p} \cap \delta_{u,v}(B_p) \neq \emptyset$ (since $u \notin B_{3-p}$, $B_{3-p}, \delta_{u,v}(B_p) \in \mathcal{A}^*$ and $\mathcal{A}^*$ is intersecting). If on the contrary $w \in B_p$ for each $p \in [2]$, then trivially $w \in (B_1 \cap B_2)\backslash\{v\}$. Hence (iv). $\Box$

5 The key fact and the compression lemma for the main results

An interesting key fact is that the 'forward' mnd separations d_i induce 'backward' mnd separations e_i with the following additional property.

Proposition 5.1 *For any $i \in \mathbb{N}$, $e_i \leq e_{i+1} \leq e_i + 1$.*

Proof. It is trivial that $e_i \leq e_{i+1} \leq e_i + 1$ for $i = 1$, so we fix $i \in \mathbb{N}\backslash\{1\}$.

If either $E_i = [i-1]$ or $E_i = \emptyset$, then $e_{i+1} \leq e_i + 1$ trivially. Suppose $E_i \neq [i-1]$ and $E_i \neq \emptyset$. Then $E_i = [j, i-1]$ for some $j \in [2, i-1]$. So $(j-1) + d_{j-1} < i$, and hence $E_{i+1} \subseteq E_i \cup \{i\}$. Therefore, $e_{i+1} \leq e_i + 1$.

If $E_i = \emptyset$, then $e_i \leq e_{i+1}$ trivially. Suppose $E_i \neq \emptyset$. Then $E_i = [j, i-1]$ for some $j \in [i-1]$. Since $d_{j+1} \geq d_j$, we thus have $(j+1) + d_{j+1} \geq j + d_j + 1 \geq i + 1$. So $[j+1, i] \subseteq E_{i+1}$, and hence $|E_i| \leq |E_{i+1}|$. Therefore, $e_i \leq e_{i+1}$. □

The above result makes Proposition 4.1 wholly applicable to our problem. For $p, q \in \mathbb{N}$, let $\Delta_{p,q} \colon 2^{\mathcal{P}} \to 2^{\mathcal{P}}$ be defined as in the preceding section.

Lemma 5.2 *Let $\mathcal{A}^*$ be an intersecting subfamily of $\mathcal{P}$. Let $p, q \in \mathbb{N}$ such that $d_p > 0$ and $d_q > 0$. Let $\mathcal{A} := \Delta_{p,q}(\mathcal{A}^*)$.*
(i) If $p = q-1$, then $\mathcal{A}\langle q\rangle$ and $\mathcal{A}(\overline{q})$ are intersecting.
(ii) If $p = q-1$ and $e_p < e_q$, then $\mathcal{A}\langle q\rangle \cup \mathcal{A}(\overline{q})$ is intersecting.
(iii) If $p = q+1$ and $d_p = d_q$, then $\mathcal{A}\langle q\rangle$ and $\mathcal{A}(\overline{q})$ are intersecting.

Proof. By Proposition 4.1(i), $\mathcal{A}(\overline{q})$ is intersecting.

Note that if either $p = q-1$ or $p = q+1$, then, since $d_p > 0$ and $d_q > 0$, $\mathcal{P}[\{p, q\}] = \emptyset$.

Suppose $p = q-1$ and $e_p < e_q$. Let $P \in \mathcal{P}(q)$. Then $P \cap [\max\{1, q-e_q\}, p] = \emptyset$. Since $q - e_q = p + 1 - e_q \leq p - e_p$, we thus have $P \cap [\max\{1, p-e_p\}, p] = \emptyset$, implying that $(P \backslash \{q\}) \cup \{p\} \in \mathcal{P}$. So $\mathcal{P}$ is (p,q)-compressed. By Proposition 4.1(iii), (ii) follows.

Suppose $p = q-1$ and $e_p \geq e_q$. By Proposition 5.1, $e_p = e_q$. Since $d_{q-1} = d_p > 0$, $e_q \geq 1$. So $e_p \geq 1$, and hence $p > 1$. Let $w := \max\{1, q-e_q-1\}$ and let $P \in \mathcal{P}(\overline{w})(q)$. Then $P \cap [w, p] = \emptyset$ and $w = \max\{1, p-e_p\} \notin \{p, q\}$. So $(P \backslash \{q\}) \cup \{p\} \in \mathcal{P}(\overline{w})$. So $\mathcal{P}(\overline{w})$ is (p,q)-compressed. By Proposition 4.1(iv), $\mathcal{A}\langle q\rangle$ is intersecting. Together with (ii), this gives us (i).

Suppose $p = q+1$ and $d_p = d_q$. Let $w := q + d_q + 1$ and let $P \in \mathcal{P}(\overline{w})(q)$. Then $P \cap [p, p+d_p] = P \cap [p, w] = P \cap [p, w-1] = P \cap [q+1, q+d_q] = \emptyset$, and hence $(P \backslash \{q\}) \cup \{p\} \in \mathcal{P}$. So $\mathcal{P}(\overline{w})$ is (p,q)-compressed. By Proposition 4.1(iv), (iii) follows. □

6 The case $d_1 > 0$

This section is dedicated to the proof of Theorem 3.1. Throughout the section, we assume that $d_1 > 0$ and $\alpha(\mathcal{P}_n) \geq 2$. We set

$$n' := n - e_n - 1.$$

Note that $n' \geq 1$ since $\alpha(\mathcal{P}_n) \geq 2$. So $n' + d_{n'} < n$, and hence

$$d_{n'} \leq e_n.$$

Lemma 6.1 $k_{n,r} = k_{n',r-1}$.

Proof. Let $k := k_{n,r}$ and $k' := k_{n',r-1}$. So $u(P_{k,r}) \leq n < u(P_{k+1,r})$ and $u(P_{k',r-1}) \leq n' < u(P_{k'+1,r-1})$. Thus, since $u(P_{k',r-1}) + d_{u(P_{k',r-1})} + 1 \leq n' + d_{n'} + 1 \leq n' + e_n + 1 = n$, we have $u(P_{k',r}) \leq n$, and hence $k' \leq k$. Now,

$$u(P_{k,r-1}) = u(P_{k,r}) - e_{u(P_{k,r})} - 1 \leq u(P_{k,r}) - (e_n - (n - u(P_{k,r}))) - 1 = n',$$

where the first inequality follows by $n - u(P_{k,r})$ applications of Lemma 5.1. So $k \leq k'$. Since $k' \leq k$, the result follows. □

Lemma 6.2 *Suppose* $1 \leq q \leq \alpha(\mathcal{P}_{y-1})$ *and either* $\mathcal{P}_y$ *is symmetric or* ${\mathcal{P}_y}^{(q)}$ *is type II. Then* $s_1(A) \in {\mathcal{P}_y}^{(q)}$ *for any* $A \in {\mathcal{P}_{y-1}}^{(q)}$.

Proof. If either $q = 1$ or $\mathcal{P}_y$ is symmetric, then the result is straightforward. So consider $q \geq 2$ and ${\mathcal{P}_y}^{(q)}$ type II. Setting $k := k_{y,q}$, we then have $y \in P_{k,q} = s_{k-1}(P_{1,q})$ and $k \leq d_1 + 1$. For each $i \in [d_1 + 1]$, let $p_{i,1} < \cdots < p_{i,q}$ such that $P_{i,q} = \{p_{i,1}, \ldots, p_{i,q}\}$. By definition of $P_{i,q}$, $p_{i,j} = p_{i,j-1} + d_{p_{i,j-1}} + 1$ for each $j = [2, q]$. Since $P_{k,q} = s_{k-1}(P_{1,q})$, $p_{k,j} = p_{1,j} + k - 1$ for each $j \in [q]$. Thus, for each $j \in [2, q]$, $p_{k,j-1} + d_{p_{k,j-1}} + 1 = p_{k,j} = (p_{1,j-1} + d_{p_{1,j-1}} + 1) + k - 1$, and hence $d_{p_{k,j-1}} = d_{p_{1,j-1}} + p_{1,j-1} + k - 1 - p_{k,j-1} = d_{p_{1,j-1}}$. Therefore, for each $j \in [q-1]$, $d_{p_{k,j}} = d_{p_{1,j}}$, and hence, for each $i \in [k]$, $d_{p_{i,j}} = d_{p_{1,j}}$ (as $d_{p_{1,j}} \leq d_{p_{i,j}} \leq d_{p_{k,j}} = d_{p_{1,j}}$).

Now let $A \in {\mathcal{P}_{y-1}}^{(q)}$, and let $a_1 < \cdots < a_q \leq y - 1$ such that $A = \{a_1, \ldots, a_q\}$. Let $h \in [q]$ and let $A_h := \{a_{q-h+1}, \ldots, a_q\}$; so $|A_h| = h$. Since $y \in P_{k,q}$ and $k = k_{y,q}$, we have $P_{k_{y,h},h} = \{p_{k,q-h+1}, \ldots, p_{k,q}\}$ and $p_{k,q} = y$. Since $a_q \leq y - 1 = p_{k,q} - 1$ and $\{d_i\}_{i \in \mathbb{N}}$ is mnd, it follows that $a_{q-h+1} \leq p_{k,q-h+1} - 1$. So $a_j \leq p_{k,j} - 1$ for all $j \in [q]$. It is straightforward that we also have $p_{1,j} \leq a_j$ for all $j \in [q]$. So $p_{1,j} \leq a_j \leq p_{k,j} - 1$ for all $j \in [q]$. Since we established that $d_{p_{i,j}} = d_{p_{1,j}}$ for any $i \in [k]$ and any $j \in [q-1]$, the result follows. □

Lemma 6.3 *Suppose* $\mathcal{P}_n$ *is asymmetric,* $\mathcal{P}_n\langle n \rangle$ $(= \mathcal{P}_{n'})$ *is symmetric and either* $\mathcal{P}_n\langle 1 \rangle$ $(= \mathcal{P}_{[d_1+2,n]})$ *is symmetric or* $d_2 > d_1$*. Then* $\alpha(\mathcal{P}_n) \leq 3$.

Proof. Since $\mathcal{P}_n$ is asymmetric, we have $d_1 < e_n$, and hence $d_1 = \cdots = d_p < d_{p+1}$ for some $p \in [n']$. Since $\mathcal{P}_{n'}$ is symmetric, it follows that $(p+1) + d_{p+1} \geq n'$. Let $p_1 < p_2 < p_3 < p_4$ such that $P_{1,4} = \{p_1, p_2, p_3, p_4\}$. So $p_1 = 1$, $p_2 = d_1 + 2$, $p_3 = p_2 + d_{p_2} + 1$, $p_4 = p_3 + d_{p_3} + 1$.

Suppose $p \leq d_1 + 1$. Then $p + 1 \leq p_2$, and hence $p_3 \geq (p+1) + d_{p+1} + 1 \geq n' + 1$ and $p_4 \geq (n'+1) + d_{n'+1} + 1 \geq n + 1$. So $u(P_{1,4}) > n$, and hence $\alpha(\mathcal{P}_n) \leq 3$.

Now suppose $p \geq d_1 + 2$. So $d_2 = d_1$ as $d_1 \leq d_2 \leq d_p = d_1$. Thus, by the conditions of the lemma, $\mathcal{P}_{[d_1+2,n]}$ is symmetric. Since $d_1 + 2 \leq p$ and $d_1 = \cdots = d_p$, $d_{d_1+2} = d_1$. So $d_{d_1+2} < e_n$, but this is a contradiction since $\mathcal{P}_{[d_1+2,n]}$ is symmetric. □

Lemma 6.4 *Let* ${\mathcal{P}_y}^{(r)}$ *be a type I family, and let* $\mathcal{A} \subset {\mathcal{P}_y}^{(r)}$ *be a special family as defined in Section 3. Then* $\mathcal{A}$ *is intersecting.*

Proof. We need to show that for each $q \in [t]$, the sets in ${\mathcal{P}_y}^{(r)}$ that do not intersect A_q are members of $\{B_1, \dots, B_q\}$. Recall that $d_i = 1$ for all $i \in [m]$ $(m = 2t + 1)$.

Consider first $q = 1$. So $A_q = P_{3,r}$. Let $B \in {\mathcal{P}_y}^{(r)}(1)$ such that $B \cap A_q = \emptyset$, and let $B' := B \backslash \{1\}$. Since $B \cap P_{3,r} = \emptyset$ and $d_1 = 1$, $l(B') \geq 4$. So $B'' := B' \cup \{2\} \in {\mathcal{P}_{[2,y]}}^{(r)}$ as $d_2 = 1$. Now, given that $y \in P_{3,r} = s_1(P_{2,r})$, $P_{2,r}$ is the only set in ${\mathcal{P}_{[2,y-1]}}^{(r)}$, and hence, since $B \cap P_{3,r} = \emptyset$ implies that $y \notin B''$, we have $B'' = P_{2,r}$. So $B = (P_{2,r} \backslash \{2\}) \cup \{1\} = B_1$, and hence $\mathcal{A}$ is intersecting.

Now consider $q > 1$. So $A_q = \{2j : j \in [q-1]\} \cup (\{2j+1 : j \in [q,t]\} \cup P_{m+2,r-t}) = P_{2,q-1} \cup P_{2q+1,r-q+1}$. Now $P_{2q+1,r-q+1} = P_{3,r} \backslash P_{3,q-1}$. Since $y \in P_{3,r} = s_1(P_{2,r})$, we have $y \in P_{2q+1,r-q+1} = s_1(P_{2q,r-q+1})$, and hence $C := P_{2q,r-q+1}$ is the only set in ${\mathcal{P}_{[2q,y-1]}}^{(r-q+1)}$. Note that $C \cap A_q = \emptyset$. Let D be a set in ${\mathcal{P}_{[2q-1,y]}}^{(r-q+1)} \backslash \{C\}$ such that $D \cap A_q = \emptyset$. Then $y \notin D$ (since $y \in A_q$) and $2q - 1 \in D$ (otherwise $D \in {\mathcal{P}_{[2q,y-1]}}^{(r-q+1)}$, which leads to the contradiction that $D = C$). Now $d_{2q} = 1$ and, since $2q + 1 \in A_q$, $2q + 1 \notin D$. So $E := (D \backslash \{2q - 1\}) \cup \{2q\} \in {\mathcal{P}_{[2q,y-1]}}^{(r-q+1)}$, and hence $E = C$. So $D = (C \backslash \{2q\}) \cup \{2q-1\}$. Since $P_{2,q-1} \subset A_q$, $P_{1,q-1}$ is the only set in ${\mathcal{P}_{[2q-2]}}^{(q-1)}$ that does not intersect A_q. So $F_1 := P_{1,q-1} \cup C$ and $F_2 := P_{1,q-1} \cup D$ are the only sets in ${\mathcal{P}_y}^{(r)}$ that do not intersect A_q. It is clear from the above that $F_1 = B_{q-1}$ and $F_2 = B_q$. Hence the result. $\square$

Lemma 6.5 *If either $\mathcal{P}_y$ is symmetric or ${\mathcal{P}_y}^{(r)}$ is a type II family, then $|{\mathcal{P}_y}^{(r)}(y)| = |{\mathcal{P}_y}^{(r)}(1)|$.*

Proof. If $r = 1$, then the result is trivial. We now consider $r > 1$ and proceed by induction on r. If $\mathcal{P}_y$ is symmetric, then the result follows immediately by symmetry. Suppose ${\mathcal{P}_y}^{(r)}$ is a type II family. Clearly, $y \geq u(P_{1,r})$. If $y = u(P_{1,r})$, then ${\mathcal{P}_y}^{(r)} = \{P_{1,r}\} = {\mathcal{P}_y}^{(r)}(1) = {\mathcal{P}_y}^{(r)}(y)$. We now consider $y > u(P_{1,r})$ and proceed by induction on y. Since ${\mathcal{P}_y}^{(r)}$ is type II, we have $y \in P_{k_{y,r},r} = s_{k_{y,r}-1}(P_{1,r})$ and $k_{y,r} \leq d_1 + 1$; note that this implies that $y \in P_{k_{y,r},r} \backslash \{k_{y,r}\} = s_{k_{y,r}-1}(P_{1,r} \backslash \{1\})$ and $d_1 + 2 = l(P_{1,r} \backslash \{1\}) \leq l(P_{k_{y,r},r} \backslash \{k_{y,r}\}) \leq (d_1 + 1) + d_{d_1+1} + 1 \leq d_{d_1+2} + (d_1 + 2)$. Since $\mathcal{P}_y\langle 1 \rangle = \mathcal{P}_{[d_1+2,y]}$, it follows that either $\mathcal{P}_y\langle 1 \rangle$ is symmetric or $\mathcal{P}_y\langle 1 \rangle^{(r-1)}$ is isomorphic to a type II family in the obvious way. Also, it is fairly straightforward that either $\mathcal{P}_y(\overline{1})$ $(= \mathcal{P}_{[2,y]})$ is symmetric or $\mathcal{P}_y(\overline{1})^{(r)}$ is isomorphic to a type II family in the obvious way. Therefore, by the induction hypotheses, we have $|\mathcal{P}_y\langle 1 \rangle^{(r-1)}(y)| = |\mathcal{P}_y\langle 1 \rangle^{(r-1)}(d_1+2)|$ and $|\mathcal{P}_y(\overline{1})^{(r)}(y)| = |\mathcal{P}_y(\overline{1})^{(r)}(2)|$. So $|{\mathcal{P}_y}^{(r)}(y)| = |\mathcal{P}_y\langle 1 \rangle^{(r-1)}(d_1+2)| + |\mathcal{P}_y(\overline{1})^{(r)}(2)| = |{\mathcal{P}_y}^{(r)}(1)(d_1 + 2)| + |{\mathcal{P}_y}^{(r)}(1)(\overline{[2, d_2 + 2]})|$, and hence the result follows if $d_2 = d_1$. Since $u(P_{1,r}) < y \in P_{k_{y,r},r}$, $k_{y,r} > 1$. Thus, as we showed in the proof of Lemma 6.2, $d_2 = d_1$ indeed. $\square$

We now come to the proof of Theorem 3.1. Recall from Section 3 that $s_{-x}(A) \in \mathcal{P}$ if $A \in \mathcal{P}$, $l(A) \geq 2$ and $x \in [l(A) - 1]$; this tool will be used often in the proof. For $p, q \in \mathbb{N}$, let $\Delta_{p,q} \colon 2^{\mathcal{P}} \to 2^{\mathcal{P}}$ be defined as in Section 4.

Proof of Theorem 3.1. Let $\mathcal{J} := {\mathcal{P}_n}^{(r)}(1)$. If ${\mathcal{P}_n}^{(r)}$ is type I and $\mathcal{A}^* \subset {\mathcal{P}_n}^{(r)}$ is special, then trivially $|\mathcal{A}^*| = |\mathcal{J}|$, and Lemma 6.4 tells us that $\mathcal{A}^*$ is intersecting.

Lemma 6.5 tells us that $|{\mathcal{P}_n}^{(r)}(n)| = |\mathcal{J}|$ if either $\mathcal{P}_n$ is symmetric or ${\mathcal{P}_n}^{(r)}$ is type II. Thus, taking

$$\mathcal{A}^* \in \mathrm{ex}({\mathcal{P}_n}^{(r)}), \tag{2}$$

the result follows if we show that $|\mathcal{A}^*| = |\mathcal{J}|$ and that if $\mathcal{A}^* \neq \mathcal{J}$, then one of the following holds:

- ${\mathcal{P}_n}^{(r)}$ is type I and $\mathcal{A}^*$ is special;
- ${\mathcal{P}_n}^{(r)}$ is type II and $\mathcal{A}^* = {\mathcal{P}_n}^{(r)}(n)$;
- $\mathcal{P}_n$ is symmetric and $\mathcal{A}^* = {\mathcal{P}_n}^{(r)}(n)$.

Given that $r \leq \alpha(\mathcal{P}_n)$, we have ${\mathcal{P}_n}^{(r)} \neq \emptyset$ and hence $\mathcal{A}^* \neq \emptyset$.

Suppose $r = 2$ and $\mathcal{A}^*$ is centred. Then $\mathcal{A}^* = {\mathcal{P}_n}^{(2)}(i)$ for some $i \in [n]$. If $e_i < d_1$, then, since $\{d_i\}_{i \in \mathbb{N}}$ is mnd, we clearly must have $i \leq d_1$, in which case $n > i + d_i$ as $\mathcal{A}^* \neq \emptyset$. So

$$\begin{aligned} |\mathcal{A}^*| &= i - 1 - e_i + \max\{0, n - (i + d_i)\} \\ &= \begin{cases} i - 1 - e_i & \text{if } e_i \geq d_1,\ n \leq i + d_i; \\ n - i - d_i & \text{if } e_i < d_1,\ n > i + d_i; \\ n - 1 - d_i - e_i & \text{if } e_i \geq d_1,\ n > i + d_i. \end{cases} \end{aligned}$$

So $|\mathcal{A}^*| \leq n - 1 - d_1 = |\mathcal{J}|$, and equality holds if and only if either $i = 1$ or $i = n$ and $e_n = d_1$. Thus, by (2), either $\mathcal{A}^* = \mathcal{J}$ or $\mathcal{A}^* = {\mathcal{P}_n}^{(2)}(n)$ and $\mathcal{P}_n$ is symmetric.

Next, suppose $r = 2$ and $\mathcal{A}^*$ is non-centred. Then $\mathcal{A}^*$ can only be of the form $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$, where $1 \leq a_1 < a_2 < a_3 \leq n$. If $a_3 > a_2 + 2$, then $|{\mathcal{P}_n}^{(2)}(a_1)| \geq |\{\{a_1, a_h\} \colon h \in [a_2, a_3]\}| \geq 4 > |\mathcal{A}^*|$, which contradicts (2). So $a_3 \leq a_2 + 2$, and hence $d_{a_2} \leq 1$. Since $1 \leq d_1 \leq d_{a_2}$, $d_{a_2} = d_1 = 1$. So $|\mathcal{J}| = n - 2$, and hence, since $|\mathcal{A}^*| = 3$, $n \leq 5$ by (2). Also, $n \geq a_3 \geq a_2 + 2 \geq (a_1 + 2) + 2 \geq 5$. So $n = 5$, and hence $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $d_1 = d_3 = 1$. Together with the above, this clearly settles the result for $r = 2$.

We now consider $r \geq 3$ and proceed by induction on r. Since $n \geq u(P_{1,\alpha(\mathcal{P}_n)})$ and $r \leq \alpha(\mathcal{P}_n)$, $n \geq u(P_{1,r})$. If $n = u(P_{1,r})$, then the result is trivial since we get $\mathcal{A}^* = {\mathcal{P}_n}^{(r)} = \{P_{1,r}\}$. We now consider $n > u(P_{1,r})$ and proceed by induction on n.

Let $\mathcal{A} := \Delta_{n-1,n}(\mathcal{A}^*)$. Since $\mathcal{A}(n) \subseteq \mathcal{A}^*(n)$, we have

$$\Delta_{n-1,n}(\mathcal{A}(n)) \subseteq \mathcal{A}^*, \tag{3}$$

and since $\mathcal{A}^*$ is intersecting, the following holds:

$$A \in \mathcal{A}(\overline{n}),\ A \cap B = \emptyset \text{ for some } B \in \mathcal{A}\langle n \rangle \quad \Rightarrow \quad n - 1 \in A \notin \mathcal{A}^*,\ \delta_{n,n-1}(A) \in \mathcal{A}^*. \tag{4}$$

Note that $\mathcal{P}_n\langle n \rangle = \mathcal{P}_{n'}$. Since we are considering $3 \leq r \leq \alpha(\mathcal{P}_n)$ and $n > u(P_{1,r})$, we clearly have $2 \leq r - 1 \leq \alpha(\mathcal{P}_{n'})$ and $3 \leq r \leq \alpha(\mathcal{P}_{n-1})$. So $\mathcal{A}\langle n \rangle \subset {\mathcal{P}_{n'}}^{(r-1)} \neq \emptyset$, $\mathcal{J}\langle n \rangle = {\mathcal{P}_{n'}}^{(r-1)}(1) \neq \emptyset$, $\mathcal{A}(\overline{n}) \subset {\mathcal{P}_{n-1}}^{(r)} \neq \emptyset$, $\mathcal{J}(\overline{n}) = {\mathcal{P}_{n-1}}^{(r)}(1) \neq \emptyset$. Now, by

Lemma 5.2(i), $\mathcal{A}\langle n\rangle$ and $\mathcal{A}(\overline{n})$ are intersecting. So the induction hypothesis yields $|\mathcal{A}\langle n\rangle| \leq |\mathcal{J}\langle n\rangle|$ and $|\mathcal{A}(\overline{n})| \leq |\mathcal{J}(\overline{n})|$, and hence $|\mathcal{A}| \leq |\mathcal{J}|$. Since $|\mathcal{A}| = |\mathcal{A}^*|$ and $\mathcal{A}^* \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$, we obtain $|\mathcal{A}| = |\mathcal{J}|$ and

$$\mathcal{J} \in \mathrm{ex}({\mathcal{P}_n}^{(r)}). \tag{5}$$

So $|\mathcal{A}\langle n\rangle| = |\mathcal{J}\langle n\rangle|$, $|\mathcal{A}(\overline{n})| = |\mathcal{J}(\overline{n})|$, and hence, since the induction hypothesis gives us $\mathcal{J}\langle n\rangle \in \mathrm{ex}({\mathcal{P}_{n'}}^{(r-1)})$ and $\mathcal{J}(\overline{n}) \in \mathrm{ex}({\mathcal{P}_{n-1}}^{(r)})$, we have

$$\mathcal{A}\langle n\rangle \in \mathrm{ex}({\mathcal{P}_{n'}}^{(r-1)}), \tag{6}$$

$$\mathcal{A}(\overline{n}) \in \mathrm{ex}({\mathcal{P}_{n-1}}^{(r)}). \tag{7}$$

Thus, by the induction hypothesis again, the following must hold:

$$\mathcal{A}\langle n\rangle = \mathcal{J}\langle n\rangle \text{ or } \mathcal{A}\langle n\rangle = {\mathcal{P}_{n'}}^{(r-1)}(n') \text{ or } \mathcal{A}\langle n\rangle \text{ is special;} \tag{8}$$

$$\mathcal{A}(\overline{n}) = \mathcal{J}(\overline{n}) \text{ or } \mathcal{A}(\overline{n}) = {\mathcal{P}_{n-1}}^{(r)}(n-1) \text{ or } \mathcal{A}(\overline{n}) \text{ is special.} \tag{9}$$

Suppose $\mathcal{A}\langle n\rangle = \mathcal{J}\langle n\rangle$. Then $\mathcal{J}(n-1) \subseteq \Delta_{n-1,n}(\mathcal{A}(n))$, and hence $\mathcal{J}(n-1) \subset \mathcal{A}^*$ by (3). Suppose $\mathcal{A}^*(\overline{1})(n) \neq \emptyset$. Let $A \in \mathcal{A}^*(\overline{1})(n)$ and $B := (s_{-1}(A\backslash l(A))) \cup \{1\}$. Then $B \in \mathcal{J}(n-1)$, and hence $B \in \mathcal{A}^*$. But $A \cap B = \emptyset$, a contradiction as $\mathcal{A}^*$ is intersecting. So $\mathcal{A}^*(\overline{1})(n) = \emptyset$. Next, suppose $\mathcal{A}^*(\overline{1})(\overline{n}) \neq \emptyset$. Let $C \in \mathcal{A}^*(\overline{1})(\overline{n})$ and $D := (s_{-1}(C\backslash(l(C) \cup u(C)))) \cup \{1\}$. So $D \in \mathcal{A}\langle n\rangle$, and hence $E := D \cup \{n\} \in \mathcal{A}^*$. But $C \cap E = \emptyset$, a contradiction. So $\mathcal{A}^*(\overline{1})(\overline{n}) = \emptyset$. Together with $\mathcal{A}^*(\overline{1})(n) = \emptyset$, this gives us $\mathcal{A}^*(\overline{1}) = \emptyset$. So $\mathcal{A}^* \subseteq \mathcal{J}$. By (2), $\mathcal{A}^* = \mathcal{J}$.

We now consider $\mathcal{A}\langle n\rangle \neq \mathcal{J}\langle n\rangle$. Thus, by (8), either $\mathcal{A}\langle n\rangle = {\mathcal{P}_{n'}}^{(r-1)}(n')$ or $\mathcal{A}\langle n\rangle$ is special. We also keep in mind that $\mathcal{A}(\overline{n})$ is as in (9).

Suppose $k_{n',r-1} = 1$. Then $\mathcal{A}\langle n\rangle$ is not special. So $\mathcal{A}\langle n\rangle = {\mathcal{P}_{n'}}^{(r-1)}(n')$. By (6) and the induction hypothesis, either $\mathcal{P}_{n'}$ is symmetric or ${\mathcal{P}_{n'}}^{(r-1)}$ is type II. So $u(P_{k_{n',r-1},r-1}) = n'$. Together with $k_{n',r-1} = 1$, this gives us $\mathcal{A}\langle n\rangle = \{P_{1,r-1}\} = \mathcal{J}\langle n\rangle$, a contradiction. So

$$k_{n',r-1} \geq 2. \tag{10}$$

Thus, by Lemma 6.1,

$$k_{n,r} \geq 2. \tag{11}$$

We will consider the case where $\mathcal{P}_n$ is symmetric separately from the case where $\mathcal{P}_n$ is asymmetric.

Case 1: $\mathcal{P}_n$ is symmetric. Clearly, we then have $n \in P_{k_{n,r},r}$. By (11), $k_{n,r} \geq 2$.

The case $k_{n,r} = 2$ is trivial since then ${\mathcal{P}_n}^{(r)} = {\mathcal{P}_n}^{(r)}(1)(n) \cup \{P_{1,r}, P_{2,r}\}$ and either $\mathcal{A}^* = {\mathcal{P}_n}^{(r)}\backslash\{P_{2,r}\} = \mathcal{J}$ or $\mathcal{A}^* = {\mathcal{P}_n}^{(r)}\backslash\{P_{1,r}\} = {\mathcal{P}_n}^{(r)}(n)$.

Consider next $k_{n,r} = 3$ and $d_1 = 1$. Since $\mathcal{P}_n$ is symmetric, $n = 2r+1$. Note that this is the unique case where $\mathcal{P}_n$ is symmetric and ${\mathcal{P}_n}^{(r)}$ is type I. Let $A_1 := P_{3,r}$,

$A_{r+1} := P_{2,r}$ and $A_{i+1} := \{2j : j \in [i]\} \cup \{2j+1 : j \in [i+1, r]\}$, $i = 1, \ldots, r-1$. Let $B_{r+1} := \{1\} \cup P_{5,r-1}$ and $B_i := s_{-1}(A_{i+1})$, $i = 1, \ldots, r$. Now, for each $i \in [r]$, let $\mathcal{S}_i$ be the special family $\{A_1, \ldots, A_i\} \cup (\mathcal{J} \backslash \{B_1, \ldots, B_i\})$. For each $i \in [r+1]$, $|\mathcal{A}^* \cap \{A_i, B_i\}| \leq 1$ as $A_i \cap B_i = \emptyset$. Since $|\mathcal{A}^*| = |\mathcal{J}|$ (by (2), (5)) and ${\mathcal{P}_n}^{(r)} \backslash \mathcal{J} = \{A_1, \ldots, A_{r+1}\}$, we actually have $|\mathcal{A}^* \cap \{A_i, B_i\}| = 1$ for all $i \in [r+1]$. Suppose $\mathcal{A}^* \neq \mathcal{J}$. Then $A_q \in \mathcal{A}^*$ for some $q \in [r+1]$; assume that q is the largest such integer. Suppose $q > 1$ and there exists $p \in [2, q]$ such that $A_p \in \mathcal{A}^*$ and $A_{p-1} \notin \mathcal{A}^*$; then, since $B_{p-1} \cap A_p = \emptyset$, we get the contradiction that $|\mathcal{A}^* \cap \{A_{p-1}, B_{p-1}\}| = 0$. So $A_p \in \mathcal{A}^*$ for all $p \in [q]$. Since $A_1 \cap A_{r+1} = \emptyset$, $q \leq r$. Therefore $\mathcal{A}^*$ is the special family $\mathcal{S}_q$.

Now consider any of the remaining cases. So either $d_1 = 1$ and $k_{n,r} \geq 4$ or $d_1 > 1$ and $k_{n,r} \geq 3$. By Lemma 6.1, ${\mathcal{P}_{n'}}^{(r-1)}$ is not type I, and hence $\mathcal{A}\langle n\rangle$ is not special. So $\mathcal{A}\langle n\rangle = {\mathcal{P}_{n'}}^{(r-1)}(n')$, and hence $A_1 := P_{k_{n',r-1},r-1} \cup \{n\} \in \mathcal{A}^*$, $A_2 := (A_1 \backslash l(A_1)) \cup \{l(A_1)-1\} \in \mathcal{A}^*$ (we have $l(A_2) \geq 2$ because, since $l(A_1) = k_{n',r-1}$ and $k_{n,r} \geq 3$, $l(A_1) \geq 3$ by Lemma 6.1). Let $\mathcal{A}' := \Delta_{2,1}(\mathcal{A}^*)$. By Lemma 5.2(iii), $\mathcal{A}'\langle 1\rangle$ and $\mathcal{A}'(\overline{1})$ are intersecting. By an argument similar to the one for $\mathcal{A}$ above, $\mathcal{A}'\langle 1\rangle$ and $\mathcal{A}'(\overline{1})$ must obey conditions similar to (8) and (9); in particular, $\mathcal{A}'\langle 1\rangle$ must be one of ${\mathcal{P}_{[d_1+2,n]}}^{(r-1)}(d_1+2)$ and ${\mathcal{P}_{[d_1+2,n]}}^{(r-1)}(n)$ (note that, since $\mathcal{P}_n$ is symmetric, $\mathcal{A}'\langle 1\rangle$ cannot be isomorphic to a special family, just like $\mathcal{A}\langle n\rangle$ cannot be special). Suppose $\mathcal{A}'\langle 1\rangle = {\mathcal{P}_{[d_1+2,n]}}^{(r-1)}(d_1+2)$. Taking $A_3 := s_{-1}(A_1)$, we then have $A_4 := (A_3 \backslash \{l(A_3), l(A_3 \backslash l(A_3))\}) \cup \{1, d_1+2\} \in \mathcal{A}^*$. If $l(A_1) = d_1+2$, then $A_2 \cap A_4 = \emptyset$, otherwise $A_1 \cap A_4 = \emptyset$; this is a contradiction. So $\mathcal{A}'\langle 1\rangle = {\mathcal{P}_{[d_1+2,n]}}^{(r-1)}(n)$. Since $\mathcal{P}_n$ is symmetric, we can use an argument similar to the one we applied for the case $\mathcal{A}\langle n\rangle = \mathcal{J}\langle n\rangle$ to obtain $\mathcal{A}^* = {\mathcal{P}_n}^{(r)}(n)$.

Case 2: $\mathcal{P}_n$ is asymmetric. Note that we therefore have $e_n > 1$. As we showed above, the following are the cases that must be investigated.

Sub-case 2.1: $\mathcal{A}\langle n\rangle = {\mathcal{P}_{n'}}^{(r-1)}(n')$. By (6) and the induction hypothesis, either $\mathcal{P}_{n'}$ is symmetric or ${\mathcal{P}_{n'}}^{(r-1)}$ is type II. So

$$n' \in P_{k_{n',r-1},r-1} = s_{k_{n',r-1}-1}(P_{1,r-1}). \tag{12}$$

Suppose $\mathcal{A}(\overline{n})$ is special but not ${\mathcal{P}_{n-1}}^{(r)}(n-1)$. By definition, we have $k_{n-1,r} = 3$, $d_1 = 1$, $u(P_{3,r}) = n-1$, and hence $u(P_{1,r+1}) = u(\{1\} \cup P_{3,r}) = n-1$. So $u(P_{1,r}) = (n-1) - e_{n-1} - 1 \leq n - e_n - 1 = n'$, where the inequality follows by Proposition 5.1. Since $k_{n',r-1} = k_{n,r} \geq k_{n-1,r} = 3 > d_1 + 1$ (the first equality is given by Lemma 6.1), ${\mathcal{P}_{n'}}^{(r-1)}$ is not type II. So $\mathcal{P}_{n'}$ is symmetric, and we thus have $e_{n'} = d_1 = 1$. Suppose $u(P_{1,r}) < n'$. Since $\mathcal{P}_{n'}$ is symmetric, we then have $P_{2,r} = s_1(P_{1,r})$ and $u(P_{2,r}) \leq n'$. So $A_1 := P_{2,r-2} \cup \{n'\} \in \mathcal{A}\langle n\rangle$. By (1), $P_{1,r} \in \mathcal{A}(\overline{n})$. Since $A_1 \cap P_{1,r} = \emptyset$, (4) gives us $n-1 \in P_{1,r}$, which contradicts $u(P_{3,r}) = n-1$. So $u(P_{1,r}) = n'$. Since $P_{3,r-1} = P_{1,r} \backslash \{1\}$ and $\mathcal{A}\langle n\rangle = {\mathcal{P}_{n'}}^{(r-1)}(n')$, we therefore have $P_{3,r-1} \in \mathcal{A}\langle n\rangle$, and hence $A_2 := P_{3,r-1} \cup \{n\} \in \mathcal{A}^*$. Since $P_{3,r} = \delta_{n-1,n}(A_2)$, we obtain $P_{3,r} \in \mathcal{A}^*$ by (3). Now, since $\mathcal{A}(\overline{n})$ is special, $P_{3,r} = s_1(P_{2,r})$ and, by (1), $A_3 := \{1, n-1\} \cup (P_{2,r-1} \backslash \{2\}) \in \mathcal{A}(\overline{n})$. So $A_2 \cap A_3 = \emptyset$, and hence $A_4 := \delta_{n,n-1}(A_3) \in \mathcal{A}^*$

by (4). But then $P_{3,r} \cap A_4 = \emptyset$, a contradiction. We therefore conclude that either $\mathcal{A}(\overline{n}) = \mathcal{J}(\overline{n})$ or $\mathcal{A}(\overline{n}) = {\mathcal{P}_{n-1}}^{(r)}(n-1)$.

Sub-sub-case 2.1.1: $\mathcal{A}(\overline{n}) = \mathcal{J}(\overline{n})$. Let $A_1 := (P_{k_{n',r-1}-1,r-1} \backslash \{n'-1\}) \cup \{n'\}$. Note that $n'-1 \in P_{k_{n',r-1}-1,r-1}$ by (12). Since $\mathcal{A}\langle n \rangle = {\mathcal{P}_{n'}}^{(r-1)}(n')$, we thus have $A_1 \in \mathcal{A}\langle n \rangle$. Suppose $k_{n',r-1} > 2$. Then $A_2 := (P_{k_{n',r-1}-2,r-1} \backslash \{k_{n',r-1}-2\}) \cup \{1, n-2\} \in \mathcal{A}(\overline{n})$. By (12), we have $P_{k_{n',r-1}-1,r-1} = s_1(P_{k_{n',r-1}-2,r-1})$, and hence $A_1 \cap A_2 = \emptyset$. But then (4) gives us $n-1 \in A_2$, a contradiction. So $k_{n',r-1} \leq 2$. By (10), $k_{n',r-1} = 2$. So $u(P_{2,r-1}) = n'$ by (12). Thus, $A_3 := P_{2,r-1} \in \mathcal{A}\langle n \rangle$ and, by (12), $n'-1 = u(P_{1,r-1})$. Suppose $d_{n'-1} < e_n$. Then $(n'-1) + d_{n'-1} + 1 \leq n' + e_n - 1 = n-2$, and hence $A_4 := P_{1,r-1} \cup \{n-2\} \in \mathcal{P}$ (as $n'-1 = u(P_{1,r-1})$). So $A_4 \in \mathcal{A}(\overline{n})$. Since (12) implies that $A_3 \cap A_4 = \emptyset$, (4) gives us $n-1 \in A_4$, a contradiction. So $d_{n'-1} = d_{n'} = e_n$. Thus, since $u(P_{1,r-1}) = n'-1$ and $P_{2,r-1} = s_1(P_{1,r-1})$ (by (12)), we have $P_{1,r} = P_{1,r-1} \cup \{(n'-1)+e_n+1\} = P_{1,r-1} \cup \{n-1\}$, $P_{2,r} = P_{2,r-1} \cup \{n'+e_n+1\} = P_{2,r-1} \cup \{n\}$, and hence $P_{2,r} = s_1(P_{1,r-1} \cup \{n-1\}) = s_1(P_{1,r})$. So ${\mathcal{P}_n}^{(r)}$ is type II. Now $u(P_{1,r}) = n-1$ implies that $\mathcal{A}(\overline{n}) = \{P_{1,r}\}$. Since $P_{2,r-1} \in {\mathcal{P}_{n'}}^{(r-1)}(n') = \mathcal{A}\langle n \rangle \subseteq \mathcal{A}^*\langle n \rangle$ and $P_{1,r} \cap P_{2,r-1} = \emptyset$, it follows by (4) that $\mathcal{A}^*(\overline{n}) = \emptyset$ and $\mathcal{A}^*(\overline{n'})(n) = \{(P_{1,r} \backslash \{n-1\}) \cup \{n\}\}$. So $\mathcal{A}^*(\overline{n'})(n) = {\mathcal{P}_n}^{(r)}(\overline{n'})(n)$ as $u(P_{1,r} \backslash \{n-1\}) = u(P_{1,r-1}) = n'-1$. Since $\mathcal{A}(n')(n-1) = \emptyset$, we have $\mathcal{A}^*(n')(n) = \mathcal{A}(n')(n)$, and hence $\mathcal{A}^*(n')(n) = {\mathcal{P}_n}^{(r)}(n')(n)$. Therefore $\mathcal{A}^* = {\mathcal{P}_n}^{(r)}(n)$.

Sub-sub-case 2.1.2: $\mathcal{A}(\overline{n}) = {\mathcal{P}_{n-1}}^{(r)}(n-1)$. Suppose $d_{n'} < e_n$. Then $A_1 := P_{k_{n',r-1},r-1} \cup \{n-1\} \in \mathcal{A}(\overline{n})$. Recall that $\mathcal{A}\langle n \rangle = {\mathcal{P}_{n'}}^{(r-1)}(n')$. Thus, by (12), $A_2 := P_{k_{n',r-1},r-1} \cup \{n\} \in \mathcal{A}(n)$, and hence $A_2 \in \mathcal{A}^*$. Since $A_1 = \delta_{n-1,n}(A_2)$, $A_1 \in \mathcal{A}^*$ by (3). By (10), $k_{n',r-1} - 1 \geq 1$; so let $A_3 := P_{k_{n',r-1}-1,r-1} \cup \{n-1\}$. We have $A_3 \in \mathcal{A}(\overline{n})$, and $A_2 \cap A_3 = \emptyset$ since $A_2 = s_1(A_3)$ by (12). So $A_4 := \delta_{n,n-1}(A_3) \in \mathcal{A}^*$ by (4). But $A_1 \cap A_4 = A_2 \cap A_3 = \emptyset$, a contradiction. So $d_{n'} = e_n$, which implies that $e_{n-1} \geq e_n$. By Proposition 5.1, $e_{n-1} = e_n$.

Let $A \in \mathcal{A}(\overline{n})$. Since $n-1 \in A$ and $(n-1) - e_{n-1} - 1 = n - e_n - 2 \leq n'-1$, we have $n' \notin A$ and $B := A \backslash \{n-1\} \in \mathcal{P}_{n'-1}^{(r-1)}$. Since either $\mathcal{P}_{n'}$ is symmetric or ${\mathcal{P}_{n'}}^{(r-1)}$ is type II, Lemma 6.2 gives us $s_1(B) \in {\mathcal{P}_{n'}}^{(r-1)}$. So $C := (s_1(B) \backslash u(s_1(B))) \cup \{n', n\} \in {\mathcal{P}_n}^{(r)}(n')(n)$. Since ${\mathcal{P}_n}^{(r)}(n')(n) = \mathcal{A}(n) \subseteq \mathcal{A}^*(n)$, $C \in \mathcal{A}^*$. Since $A \cap C = \emptyset$, it follows by (4) that $A \notin \mathcal{A}^*$ and $\delta_{n,n-1}(A) \in \mathcal{A}^*(\overline{n'})(n)$. We have therefore shown that $\mathcal{A}(\overline{n})(n') = \emptyset$, $\mathcal{A}^*(\overline{n}) = \emptyset$ and $\mathcal{A}^*(\overline{n'})(n) = \Delta_{n,n-1}(\mathcal{A}(\overline{n})) = \Delta_{n,n-1}({\mathcal{P}_{n-1}}^{(r)}(n-1)) = {\mathcal{P}_n}^{(r)}(\overline{n'})(n)$. Since $\mathcal{A}(\overline{n})(n') = \emptyset$ implies that $\mathcal{A}^*(n')(n) = \mathcal{A}(n')(n)$, we have $\mathcal{A}^*(n')(n) = {\mathcal{P}_n}^{(r)}(n')(n)$. So $\mathcal{A}^* = {\mathcal{P}_n}^{(r)}(n)$.

We now show that ${\mathcal{P}_n}^{(r)}$ is type II. Recall from Section 3 that ${\mathcal{P}_n}^{(r)}(n)$ is special only if $\mathcal{P}_n$ is symmetric. However, $\mathcal{P}_n$ is asymmetric. So ${\mathcal{P}_n}^{(r)}(n)$ is not special.

Since $e_{n-1} = e_n$ and $\mathcal{P}_n$ is asymmetric, $\mathcal{P}_{n-1}$ is asymmetric. So ${\mathcal{P}_{n-1}}^{(r)}(n-1)$ is not special. Since ${\mathcal{P}_{n-1}}^{(r)}(n-1) = \mathcal{A}(\overline{n}) \in \mathrm{ex}({\mathcal{P}_{n-1}}^{(r)})$, it follows by the induction hypothesis that ${\mathcal{P}_{n-1}}^{(r)}$ is type II, and hence $k_{n-1,r} \leq d_1 + 1$. By the induction hypothesis for $\mathcal{A}\langle n \rangle = {\mathcal{P}_{n'}}^{(r-1)}(n') \in \mathrm{ex}({\mathcal{P}_{n'}}^{(r-1)})$, either ${\mathcal{P}_{n'}}^{(r-1)}$ is type II or $\mathcal{P}_{n'}$ is symmetric. If ${\mathcal{P}_{n'}}^{(r-1)}$ is type II, then, by definition, $k_{n',r-1} \leq d_1 + 1$. We now show that this inequality also holds if $\mathcal{P}_{n'}$ is symmetric.

Suppose $\mathcal{P}_{n'}$ is symmetric and $k_{n',r-1} > d_1 + 1$. Then $\{1, n\} \cup P_{k_{n',r-1},r-1} \in \mathcal{P}_n^{(r+1)}$, and hence $r < \alpha(\mathcal{P}_n)$. If either $\mathcal{P}_n\langle 1\rangle$ is symmetric or $d_2 > d_1$, then Lemma 6.3 gives us $\alpha(\mathcal{P}_n) \leq 3$, and hence $r \leq 2$; however, we are considering $r \geq 3$. So $\mathcal{P}_n\langle 1\rangle$ is asymmetric and $d_2 = d_1$. Let $\mathcal{A}' := \Delta_{2,1}(\mathcal{A}^*)$. By Lemma 5.2(iii), $\mathcal{A}'\langle 1\rangle$ and $\mathcal{A}'(\overline{1})$ are intersecting. The induction hypothesis gives us $|\mathcal{A}'\langle 1\rangle| \leq |\mathcal{P}_n\langle 1\rangle^{(r-1)}(d_1 + 2)| = |{\mathcal{P}_n}^{(r)}(1)(d_1 + 2)|$, $|\mathcal{A}'(\overline{1})| \leq |\mathcal{P}_n(\overline{1})^{(r)}(2)| = |{\mathcal{P}_n}^{(r)}(1)([2, 2 + d_2])| \leq |{\mathcal{P}_n}^{(r)}(1)([2, d_1 + 2])|$, and therefore $|\mathcal{A}'| \leq |{\mathcal{P}_n}^{(r)}(1)(d_1 + 2)| + |{\mathcal{P}_n}^{(r)}(1)([2, d_1 + 2])| = |{\mathcal{P}_n}^{(r)}(1)|$. Since $|\mathcal{A}'| = |\mathcal{A}|$ and $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$, it follows that $|\mathcal{A}'\langle 1\rangle| = |\mathcal{P}_n\langle 1\rangle^{(r-1)}(d_1 + 2)|$ and $|\mathcal{A}'(\overline{1})| = |\mathcal{P}_n(\overline{1})^{(r)}(2)|$. Since the induction hypothesis gives us $\mathcal{P}_n\langle 1\rangle^{(r-1)}(d_1 + 2) \in \mathrm{ex}(\mathcal{P}_n\langle 1\rangle^{(r-1)})$, we therefore have $\mathcal{A}'\langle 1\rangle \in \mathrm{ex}(\mathcal{P}_n\langle 1\rangle^{(r-1)})$. Thus, by the induction hypothesis, one of the following holds: (a) $\mathcal{A}'\langle 1\rangle = \mathcal{P}_n\langle 1\rangle^{(r-1)}(d_1 + 2)$, (b) $\mathcal{A}'\langle 1\rangle = \mathcal{P}_n\langle 1\rangle^{(r-1)}(n)$, (c) $\mathcal{A}'\langle 1\rangle$ is isomorphic to a special family. Suppose (a) holds. Then $P_{d_1+2,r-1} \in \mathcal{A}'\langle 1\rangle$, and hence $P_{1,r} \in \mathcal{A}'(1)$. So $P_{1,r} \in \mathcal{A}^*$ as $\mathcal{A}'(1) \subset \mathcal{A}^*$; but this clearly contradicts $\mathcal{A}^* = {\mathcal{P}_n}^{(r)}(n)$ and (11). Suppose (c) holds. Since $\mathcal{P}_n\langle 1\rangle$ is asymmetric, (1) then gives us $P_{d_1+2,r-1} \in \mathcal{A}'\langle 1\rangle$; but, as we have just shown, this leads to a contradiction. So (b) holds. Since $\mathcal{P}\langle 1\rangle$ is asymmetric, it follows by the induction hypothesis that $\mathcal{P}_n\langle 1\rangle^{(r-1)}$ is isomorphic to a type II family, and hence, by definition of a type II family, we must have $r - 1 = \alpha(\mathcal{P}_n\langle 1\rangle)$; but this clearly contradicts $r < \alpha(\mathcal{P}_n)$.

Therefore, as we claimed earlier, $k_{n',r-1} \leq d_1 + 1$. So $k_{n,r} \leq d_1 + 1$ by Lemma 6.1. Now, since ${\mathcal{P}_{n-1}}^{(r)}$ is type II, $n - 1 \in P_{k_{n-1,r},r} = s_{k_{n-1,r}-1}(P_{1,r})$. Since $e_{n-1} = e_n$, it follows that $n' - 1 = (n - 1) - e_{n-1} - 1 \in P_{k_{n-1,r},r-1} = s_{k_{n-1,r}-1}(P_{1,r-1})$. Since either ${\mathcal{P}_{n'}}^{(r-1)}$ is type II or $\mathcal{P}_{n'}$ is symmetric, $n' \in P_{k_{n',r-1},r-1} = s_{k_{n',r-1}-1}(P_{1,r-1})$. So $P_{k_{n',r-1},r-1} = s_1(P_{k_{n-1,r},r-1})$ and $k_{n-1,r} = k_{n',r-1} - 1$. Since $d_{n'} = e_n$, we have $n' + d_{n'} + 1 = n$, and hence $P_{k_{n,r},r} = P_{k_{n',r-1},r-1} \cup \{n\}$ as $n' = u(P_{k_{n',r-1},r-1})$. Bringing together the relations we have just established, we get

$$\begin{aligned}P_{k_{n,r},r} &= s_1(P_{k_{n-1,r},r-1}) \cup \{n\} = s_1(P_{k_{n-1,r},r-1} \cup \{n-1\}) = s_1(P_{k_{n-1,r},r}) \\ &= s_1(s_{k_{n-1,r}-1}(P_{1,r})) = s_{k_{n-1,r}}(P_{1,r}) = s_{k_{n',r-1}-1}(P_{1,r}).\end{aligned}$$

Together with Lemma 6.1, this gives us $P_{k_{n,r},r} = s_{k_{n,r}-1}(P_{1,r})$. Since we also established that $k_{n,r} \leq d_1 + 1$ and $n \in P_{k_{n,r},r}$, ${\mathcal{P}_n}^{(r)}$ is type II.

Sub-case 2.2: $\mathcal{A}\langle n\rangle$ is special and $\mathcal{A}\langle n\rangle \neq {\mathcal{P}_{n'}}^{(r-1)}(n')$. By (6) and the induction hypothesis, ${\mathcal{P}_{n'}}^{(r-1)}$ is type I. So $n' \in P_{3,r-1} = s_1(P_{2,r-1})$ and, by (1), $P_{1,r-1}, P_{3,r-1} \in \mathcal{A}\langle n\rangle$. Taking $Q_1 := P_{1,r-1} \cup \{n\}$ and $Q_3 := P_{3,r-1} \cup \{n\}$, we then have $Q_1, Q_3 \in \mathcal{A}^*(n)$ (as $\mathcal{A}\langle n\rangle \subseteq \mathcal{A}^*\langle n\rangle$).

Suppose $\mathcal{A}(\overline{n}) = {\mathcal{P}_{n-1}}^{(r)}(n - 1)$. So $A_1 := s_{-1}(Q_3) = P_{2,r-1} \cup \{n - 1\} \in \mathcal{A}(\overline{n})$ and $A_2 := P_{1,r-1} \cup \{n - 1\} \in \mathcal{A}(\overline{n})$. Since $A_2 = \delta_{n-1,n}(Q_1)$, it follows by (3) that $A_2 \in \mathcal{A}^*$. Since $A_1 \cap Q_3 = \emptyset$, we have $A_3 := P_{2,r-1} \cup \{n\} \in \mathcal{A}^*$ by (4). Now, by (7) and the induction hypothesis, we should have that either ${\mathcal{P}_{n-1}}^{(r)}$ is type II or $\mathcal{P}_{n-1}$ is symmetric, and hence $P_{2,r} = s_1(P_{1,r})$; but then $A_2 \cap A_3 = \emptyset$, a contradiction. So $\mathcal{A}(\overline{n}) \neq {\mathcal{P}_{n-1}}^{(r)}(n - 1)$.

Next, suppose $\mathcal{A}(\overline{n})$ is special. Let $A_1 := s_1(P_{2,r})$ and $A_2 := \{1, n - 1\} \cup (P_{2,r-1}\backslash\{2\})$. Then $A_1 = P_{3,r} = P_{3,r-1} \cup \{n - 1\} \in \mathcal{A}(\overline{n})$ and, by (1), $A_2 \in \mathcal{A}(\overline{n})$.

Since $A_1 = \delta_{n-1,n}(Q_3)$, (3) gives us $A_1 \in \mathcal{A}^*$. Since $A_2 \cap Q_3 = A_2 \cap \delta_{n,n-1}(A_1) = \emptyset$, (4) gives us $A_3 := \delta_{n,n-1}(A_2) \in \mathcal{A}^*$. But $A_1 \cap A_3 = \emptyset$, a contradiction.

Therefore, $\mathcal{A}(\overline{n}) = \mathcal{J}(\overline{n})$. Suppose $d_{n'-1} < e_n$. Then $(n'-1) + d_{n'-1} + 1 \leq n' + e_n - 1 = n - 2$. Now $n' - 1 \in P_{2,r-1}$ as $n' \in P_{3,r-1} = s_1(P_{2,r-1})$. Taking $A_1 := \{1, n-2\} \cup (P_{2,r-1} \backslash \{2\})$, we thus get $A_1 \in \mathcal{A}(\overline{n}) \cap \mathcal{A}^*$. However, since $Q_3 \ni n' \leq n - 2 - d_{n'-1} \leq n - 3$ implies that $n - 2 \notin Q_3$, we also get $A_1 \cap Q_3 = \emptyset$, a contradiction. So $d_{n'-1} = e_n$, and hence $d_{n'-1} = d_{n'} = e_n$ (as $d_{n'} \leq e_n$). Thus, since $u(P_{2,r-1}) = n' - 1$ and $u(P_{3,r-1}) = n'$, $u(P_{2,r}) = (n'-1) + d_{n'-1} + 1 = n' + e_n = n - 1$ and similarly $u(P_{3,r}) = n$. So $P_{3,r} = P_{3,r-1} \cup \{n\} = s_1(P_{2,r-1} \cup \{n-1\}) = s_1(P_{2,r})$. Since ${\mathcal{P}_{n'}}^{(r-1)}$ is type I, $d_1 = d_3 = 1$. Since $\mathcal{P}_n$ is asymmetric, we therefore have $e_n > 1$, and hence, since $d_{n'-1} = e_n$, $m := \max\{a \in [n] \colon d_a = 1\} \leq n' - 2$. Thus, since ${\mathcal{P}_{n'}}^{(r-1)}$ is type I, $m = 2t+1$ for some $t \in [r-1]$ (note that if $\mathcal{P}_{n'}$ is symmetric, then $n' = 2(r-1)+1$, $d_{n'-2} = d_1$, and hence $m = n' - 2 = 2(r-2)+1$). So ${\mathcal{P}_n}^{(r)}$ is type I.

It remains to show that $\mathcal{A}^*$ is special. Let $A_1, \dots, A_t, B_1, \dots, B_t$ be as in the definition of a special family with $y = n$ in Section 3. Since $n \in P_{3,r} = P_{3,t} \cup P_{m+2,r-t} = s_1(P_{2,r}) = s_1(P_{2,t} \cup P_{m+1,r-t})$, we have $n \in P_{m+2,r-t} = s_1(P_{m+1,r-t})$, and hence, for each $i \in [t]$, $n \in A_i$ and $n - 1 \in P_{m+1,r-t} \subset B_i$. For each $i \in [t]$, let $A'_i := A_i \backslash \{n\}$, $B'_i := B_i \backslash \{n-1\}$, $B''_i := B'_i \cup \{n\}$. Since $t \in [r-1]$, $r \geq t+1$. If $r = t+1$, then $P_{m+2,r-t} = P_{m+2,1} = \{m+2\}$, and hence $n = m+2$; however, this contradicts $m \leq n' - 2 < n - 2$. So $r \geq t+2$. Thus $P_{m+2,r-t-1} \neq \emptyset$ and $P_{m+1,r-t-1} \neq \emptyset$. Clearly, for each $i \in [t]$, $A'_i = (A_i \backslash P_{m+2,r-t}) \cup P_{m+2,r-t-1}$ and $B'_i = (B_i \backslash P_{m+1,r-t}) \cup P_{m+1,r-t-1}$ (recall that $P_{m+1,r-t} \subset B_i$). Therefore, since $\mathcal{A}\langle n \rangle$ is special and $\mathcal{A}\langle n \rangle \neq {\mathcal{P}_{n'}}^{(r-1)}(n')$, $\mathcal{A}\langle n \rangle = \{A'_1, \dots, A'_q\} \cup ({\mathcal{P}_{n'}}^{(r-1)}(1) \backslash \{B'_1, \dots, B'_q\})$ for some $q \in [t]$ (note that if $\mathcal{P}_{n'}$ is symmetric, then, since ${\mathcal{P}_{n'}}^{(r-1)}$ is type I, we have $n' = 2(r-1)+1$, $t = r-2$, and hence $\mathcal{A}\langle n \rangle = \{A'_1, \dots, A'_h\} \cup ({\mathcal{P}_{n'}}^{(r-1)}(1) \backslash \{B'_1, \dots, B'_h\})$ for some $h \in [t+1] = [r-1]$, where $A'_{t+1} = \{2j \colon j \in [r-2]\} \cup \{n'\}$ and $B'_{t+1} = P_{1,r-1}$; however, if $h = t+1$, then $\mathcal{A}\langle n \rangle = {\mathcal{P}_{n'}}^{(r-1)}(n')$, a contradiction). So $\mathcal{A}^*_1 := \mathcal{A}(n) = \{A_1, \dots, A_q\} \cup ({\mathcal{P}_n}^{(r)}(1)(n) \backslash \{B''_1, \dots, B''_q\})$. Since $\mathcal{A}(n) \subseteq \mathcal{A}^*(n)$, $\mathcal{A}^*_1 \subseteq \mathcal{A}^*$. Now, we also have $\mathcal{A}(\overline{n}) = \mathcal{J}(\overline{n}) = {\mathcal{P}_{n-1}}^{(r)}(1)$. So $\mathcal{A}^*_2 := {\mathcal{P}_{n-1}}^{(r)}(1)(\overline{n-1}) = \mathcal{A}(\overline{n})(\overline{n-1}) \subset \mathcal{A}^*$. Finally, consider $A \in \mathcal{A}(\overline{n})(n-1) = {\mathcal{P}_{n-1}}^{(r)}(1)(n-1)$. If $A = B_i$ for some $i \in [q]$, then, since $A_i \cap B_i = \emptyset$ and $A_i \in \mathcal{A}^*_1$, we must have $A \notin \mathcal{A}^*$ and $(A \backslash \{n-1\}) \cup \{n\} \in \mathcal{A}^*$. If $A \notin \{B_1, \dots, B_q\}$, then $(A \backslash \{n-1\}) \cup \{n\} \in \mathcal{A}^*_1$, and hence $A \in \mathcal{A}^*$ by (3). Setting $\mathcal{A}^*_3 := \mathcal{A}^* \backslash (\mathcal{A}^*_1 \cup \mathcal{A}^*_2)$, we therefore have $\mathcal{A}^*_3 = ({\mathcal{P}_{n-1}}^{(r)}(1)(n-1) \backslash \{B_1, \dots, B_q\}) \cup \{B''_1, \dots, B''_q\}$. So $\mathcal{A}^* = \mathcal{A}^*_1 \cup \mathcal{A}^*_2 \cup \mathcal{A}^*_3 = \{A_1, \dots, A_q\} \cup ({\mathcal{P}_n}^{(r)}(1) \backslash \{B_1, \dots, B_q\})$. So $\mathcal{A}^*$ is special. □

7 The case $d_1 = 0$

This section is dedicated to the proof of Theorem 3.3. We start with a lemma concerning sets in hereditary families. A family $\mathcal{F}$ is said to be a *hereditary family* (also called an *ideal*, a *downset*, and an *abstract simplicial complex*) if for each member F of $\mathcal{F}$, all the subsets of F are members of $\mathcal{F}$. Clearly, for any $X \subseteq \mathbb{N}$, $\mathcal{P}_X$ is a hereditary family.

Lemma 7.1 *Let $\mathcal{F}$ be a hereditary family with $\alpha(\mathcal{F}) \geq 1$. If there exist $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cap F_2 = \emptyset$ and $|F_1| = |F_2| = \alpha(\mathcal{F})$, then for every non-empty $F \in \mathcal{F}$ there exists $F' \in \mathcal{F}$ such that $F \cap F' = \emptyset$ and $|F| + |F'| > \alpha(\mathcal{F})$.*

Proof. Let $F \in \mathcal{F}$, $F \neq \emptyset$. If $F \subseteq F_1$, then the result follows immediately by taking $F' \in \binom{F_2}{\alpha(\mathcal{F})+1-|F|}$. If $F \nsubseteq F_1$, then $|F_1 \backslash F| \geq |F_1| - (|F| - 1) = \alpha(F) + 1 - |F|$, and hence the result follows by taking $F' \in \binom{F_1 \backslash F}{\alpha(\mathcal{F})+1-|F|}$. □

The converse of this result is not true; to see this, take $\mathcal{F}$ to be $2^{[n+1]} \backslash \{[n+1]\}$. Also, the conditions on F_1 and F_2 are sharp; by considering $\mathcal{F} = 2^{F_1} \cup 2^{F_2}$ and $F = \{x\}$, it is easy to see that we can neither allow F_1 and F_2 to have a non-empty intersection nor allow F_1 or F_2 to be of size less than $\alpha(\mathcal{F})$.

Lemma 7.2 *If $d_1 > 0$, $\alpha(\mathcal{P}_n) \geq 3$ and $n \in P_{1,\alpha(\mathcal{P}_n)}$, then for any non-empty $A \in \mathcal{P}_n(\overline{2})$ there exists $A' \in \mathcal{P}_n(2)$ such that $A \cap A' = \emptyset$ and $|A| + |A'| \geq \alpha(\mathcal{P}_n)$.*

Proof. Let $1 = p_1 < p_2 < \cdots < p_{\alpha(\mathcal{P}_n)}$ such that $P_{1,\alpha(\mathcal{P}_n)} = \{p_1, \ldots, p_{\alpha(\mathcal{P}_n)}\}$. We have $\mathcal{P}_n\langle 2\rangle \subset 2^{[a,n]}$, where $a := 2 + d_2 + 1$. Let $a =: q_1 < \cdots < q_{\alpha(\mathcal{P}_n)-2}$ such that $P_{a,\alpha(\mathcal{P}_n)-2} = \{q_1, \ldots, q_{\alpha(\mathcal{P}_n)-2}\}$. So

$$p_2 = 1 + d_1 + 1 < 2 + d_2 + 1 = q_1 < p_2 + d_{p_2} + 1 = p_3, \tag{13}$$

and if $\alpha(\mathcal{P}_n) \geq 4$, then, proceeding inductively, we also get

$$p_i = p_{i-1} + d_{p_{i-1}} + 1 < q_{i-2} + d_{q_{i-2}} + 1 = q_{i-1} < p_i + d_{p_i} + 1 = p_{i+1}, \tag{14}$$

$i = 3, \ldots, \alpha(\mathcal{P}_n) - 1$. Let $F_1 := P_{p_3,\alpha(\mathcal{P}_n)-2} = P \backslash \{p_1, p_2\}$, $F_2 := P_{a,\alpha(\mathcal{P}_n)-2}$. By (13) and (14), $F_1, F_2 \in \mathcal{P}_n\langle 2\rangle$ and $F_1 \cap F_2 = \emptyset$. Since $|F_1| = \alpha(\mathcal{P}_n) - 2$, $\alpha(\mathcal{P}_n\langle 2\rangle) \geq \alpha(\mathcal{P}_n) - 2 \geq 1$. By definition of a, $P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)} \in \mathcal{P}_n\langle 2\rangle$ (for the same reason that $P_{1,\alpha(\mathcal{P}_n)} \in \mathcal{P}_n$, being that $\{d_i\}_{i\in\mathbb{N}}$ is mnd). So $u(P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)}) \leq n$. Now $n = p_{\alpha(\mathcal{P}_n)}$ as we are given that $n \in P_{1,\alpha(\mathcal{P}_n)}$.

Suppose $\alpha(\mathcal{P}_n\langle 2\rangle) > \alpha(\mathcal{P}_n) - 2$. Then $q_{\alpha(\mathcal{P}_n)-2} \in P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)} \backslash \{u(P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)})\}$. Together with (13) and (14), this gives us $u(P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)}) \geq q_{\alpha(\mathcal{P}_n)-2} + d_{q_{\alpha(\mathcal{P}_n)-2}} + 1 > p_{\alpha(\mathcal{P}_n)}$, contradicting $u(P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)}) \leq n = p_{\alpha(\mathcal{P}_n)}$. So $\alpha(\mathcal{P}_n\langle 2\rangle) = \alpha(\mathcal{P}_n) - 2 = |F_1| = |F_2|$.

Let $\emptyset \neq A \in \mathcal{P}_n(\overline{2})$. Suppose $A \in \mathcal{P}_n\langle 2\rangle$. By Lemma 7.1, there exists $A'' \in \mathcal{P}_n\langle 2\rangle$ such that $A \cap A'' = \emptyset$ and $|A| + |A''| \geq \alpha(\mathcal{P}_n\langle 2\rangle) + 1 = \alpha(\mathcal{P}_n) - 1$. Hence $A' := A'' \cup \{2\} \in \mathcal{P}_n(2)$, $A \cap A' = \emptyset$ and $|A| + |A'| \geq \alpha(\mathcal{P}_n)$. Now suppose $A \notin \mathcal{P}_n\langle 2\rangle$. We have $A^* := A \cap [a,n] \in \mathcal{P}_n\langle 2\rangle \cup \{\emptyset\}$. If $A^* \neq \emptyset$, then we apply the argument above to get $|A^*| + |A'| \geq \alpha(\mathcal{P}_n)$ for some $A' \in \mathcal{P}_n(2)$ such that $A^* \cap A' = \emptyset$, and clearly this yields the result. Suppose $A^* = \emptyset$. Let $A' := F_1 \cup \{2\}$. So $A \cap A' = \emptyset$ and $|A| + |A'| \geq 1 + (\alpha(\mathcal{P}_n) - 1) = \alpha(\mathcal{P}_n)$. □

In the proof of Theorem 3.3, we will use the following slightly improved version [2] of a result of Hilton [13] for cross-intersecting families.

Theorem 7.3 ([13, 2]) *Let $r \leq n/2$ and $k \geq 2$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be cross-intersecting subfamilies of $\binom{[n]}{r}$. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{r} & \text{if } k \leq \frac{n}{r}; \\ k\binom{n-1}{r-1} & \text{if } k \geq \frac{n}{r}. \end{cases}$$

Moreover, if equality holds and $(k, \frac{n}{r}) \neq (2, 2)$, then one of the following holds:
(i) $k > n/r$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_k \in \mathrm{ex}(\binom{[n]}{r})$;
(ii) $k < n/r$, $\mathcal{A}_j = \binom{[n]}{r}$ for some $j \in [k]$, and $\mathcal{A}_i = \emptyset$ for each $i \in [k]\backslash\{j\}$;
(iii) $k = n/r$ and $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are as in (i) or (ii).

For $p, q \in \mathbb{N}$, let $\Delta_{p,q} \colon 2^{\mathcal{P}} \to 2^{\mathcal{P}}$ be defined as in Section 4.

Proof of Theorem 3.3. We start with (i), for which we have $d_{2r-2} = 0$ and $d_{2r-1} = n - 2r$. Note that $r \geq 2$ since $d_1 = 0 < d_m = d_{2r-1}$. We first consider $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$ and prove the necessary condition. Let $\mathcal{B} := {\mathcal{P}_n}^{(r)}(1)$. Let $\mathcal{A}_0 = \mathcal{A} \cap \binom{[2r-2]}{r}$, $\mathcal{A}_2 := \mathcal{A}(2r-1)(n)$ and $\mathcal{A}_{1,i} := \mathcal{A}(i)\backslash\mathcal{A}_2$, $i = 2r-1, \ldots, n$. Define $\mathcal{B}_0$, $\mathcal{B}_2$, and $\mathcal{B}_{1,i}$ similarly. Note that since $(2r-1) + d_{2r-1} + 1 = n$ (and $d_i \geq d_{2r-1}$ for all $i \geq 2r$), if $A \in \mathcal{A}$ and $|A \cap [2r-1, n]| > 1$, then $A \cap [2r-1, n] = \{2r-1, n\}$. So $\mathcal{A}_0 \cup \mathcal{A}_2 \cup \bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ is a partition for $\mathcal{A}$. Let $\mathcal{A}'_2 := \mathcal{A}\langle 2r-1\rangle\langle n\rangle \subseteq \binom{[2r-2]}{r-2}$ and $\mathcal{A}'_{1,i} := \mathcal{A}\langle i\rangle \cap \binom{[2r-2]}{r-1} = \mathcal{A}_{1,i}\langle i\rangle$, $i = 2r-1, \ldots, n$. Define $\mathcal{B}'_2$ and $\mathcal{B}'_{1,i}$ $(i = 2r-1, \ldots, n)$ similarly. So

$$|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}'_2| + \sum_{i=2r-1}^{n} |\mathcal{A}'_{1,i}|, \quad |\mathcal{B}| = |\mathcal{B}_0| + |\mathcal{B}'_2| + \sum_{i=2r-1}^{n} |\mathcal{B}'_{1,i}| \tag{15}$$

Clearly, $\mathcal{A}_0$ and $\mathcal{A}'_2$ must be cross-intersecting. So

$$|\{A, [2r-2]\backslash A\} \cap (\mathcal{A}_0 \cup \mathcal{A}'_2)| \leq 1 \text{ for all } A \in \binom{[2r-2]}{r-2} \cup \binom{[2r-2]}{r}, \tag{16}$$

and hence

$$|\mathcal{A}_0| + |\mathcal{A}'_2| \leq \binom{2r-2}{r} = |\mathcal{B}_0| + |\mathcal{B}'_2|. \tag{17}$$

Let us now consider $\mathcal{A}'_{1,i}$, $i = 2r-1, \ldots, n$. These families must also be cross-intersecting. Thus, by Theorem 7.3, we have

$$\sum_{i=2r-1}^{n} |\mathcal{A}'_{1,i}| \leq (n-2r+2)\binom{2r-3}{r-2} = \sum_{i=2r-1}^{n} |\mathcal{B}'_{1,i}|. \tag{18}$$

By (15), (17) and (18), we have $|\mathcal{A}| \leq |\mathcal{B}|$. Thus, since $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$, we actually have $|\mathcal{A}| = |\mathcal{B}|$. It follows that the inequalities in (17) and (18) are actually equalities. By Theorem 7.3 and the EKR Theorem, an equality in (18) yields $\mathcal{A}'_{1,2r-1} = \cdots = \mathcal{A}'_{1,n} \in \mathrm{ex}(\binom{[2r-2]}{r-1})$; hence (b).

Since $d_{n-1} > 0$ and $2r \le \alpha(\mathcal{P}_n)$, $n \ge 2r+1$. So the sets of $\mathcal{A}_{1,2r}$ do not intersect with those of $\mathcal{A}_2$ on $[2r-1, n]$, and hence $\mathcal{A}'_{1,2r}$ and $\mathcal{A}'_2$ are cross-intersecting. By the equalities in (b), (c) follows.

Since we established equality in (17), we also have equality in (16), which implies that $\mathcal{A}_0 = \binom{[2r-2]}{r} \backslash \{[2r-2]\backslash A \colon A \in \mathcal{A}'_2\}$. Thus, to obtain (a), it remains to show that $\mathcal{A}'_2$ is intersecting. Suppose there exist $A_1, A_2 \in \mathcal{A}'_2$ such that $A_1 \cap A_2 = \emptyset$. So $[2r-2]\backslash(A_1 \cup A_2) = \{x, y\}$ for some distinct $x, y \in [2r-2]$. Let $A_3 := A_1 \cup \{x\}$ and $A_4 := A_2 \cup \{y\}$. So $A_3 \cap A_2 = \emptyset$ and $A_4 \cap A_1 = \emptyset$. Since $\mathcal{A}'_{1,2r}$ and $\mathcal{A}'_2$ are cross-intersecting (see above), we therefore get $A_3, A_4 \notin \mathcal{A}'_{1,2r}$. Since $A_4 = [2r-2]\backslash A_3$, this implies that $\mathcal{A}'_{1,2r} \notin \mathrm{ex}(\binom{[2r-2]}{r-1})$ (see Theorem 1.1(ii)), a contradiction to (b). So $\mathcal{A}'_2$ is intersecting. Hence (a).

We now prove the sufficiency condition in (i). So let $\mathcal{A}$ be a subfamily of ${\mathcal{P}_n}^{(r)}$ that obeys (a), (b) and (c). Define $\mathcal{A}_0$, $\mathcal{A}_2$ and $\mathcal{A}_{1,i}$, $i = 2r-1, \dots, n$, as above. As we showed above, $\mathcal{A}_0 \cup \mathcal{A}_2 \cup \bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ is a partition for $\mathcal{A}$. By definition, $\mathcal{A}_0$, $\mathcal{A}_2$, $\mathcal{A}_{1,2r-1}, \dots, \mathcal{A}_{1,n}$ are intersecting. By (a), $\mathcal{A}_0 \cup \mathcal{A}_2$ is intersecting. By (b) and (c), $\mathcal{A}_2 \cup \bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ is intersecting. If $A \in \bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$, then $|A \cap [2r-2]| = r-1$, and hence A intersects each set in $\binom{[2r-2]}{r}$; so $\mathcal{A}_0$ and $\bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ are cross-intersecting. Therefore, $\mathcal{A}$ is intersecting. Now, it is immediate from (a), (b) and (c) that the bounds in (17) and (18) are attained. So $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$.

We now prove (ii) and (iii) by induction on n. The case $r = 1$ is trivial, so we assume that $r \ge 2$. We first consider $\mathcal{A}^* \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$ and prove the necessary conditions for $\mathcal{A}^*$. Unlike we did in the Proof of Theorem 3.1, we do not use $\Delta_{n-1,n}$ because if $\alpha(\mathcal{P}_n)/2 = r > \alpha(\mathcal{P}_{n-1})/2$ (which is possible), then we cannot apply the induction hypothesis. Instead, we work with $\mathcal{A} := \Delta_{m,m+1}(\mathcal{A}^*)$. By Lemma 5.2(ii), $\mathcal{A}(\overline{m+1}) \cup \mathcal{A}\langle m+1\rangle$ is intersecting. We have $\mathcal{A}(\overline{m+1}) \subset {\mathcal{P}_n}^{(r)}(\overline{m+1}) = \mathcal{P}_n(\overline{m+1})^{(r)}$ and $\mathcal{A}\langle m+1\rangle \subset {\mathcal{P}_n}^{(r)}\langle m+1\rangle = \mathcal{P}_n\langle m+1\rangle^{(r'')}$, where $r'' = r-1$. Since $m, m+d_m+1 \in P_{1,\alpha(\mathcal{P}_n)} \in \mathcal{P}_n$, we have $\alpha(\mathcal{P}_n) = \alpha(\mathcal{P}_n(\overline{m+1}))$ and

$$r'' \le (\alpha(\mathcal{P}_n) - 2)/2 = \alpha(\mathcal{P}_n\langle m\rangle\langle m+d_m+1\rangle)/2 \le \alpha(\mathcal{P}_n\langle m+1\rangle)/2. \tag{19}$$

Observe that $\mathcal{P}_n(\overline{m+1})$ and $\mathcal{P}_n\langle m+1\rangle$ are isomorphic to $\mathcal{P}_{n'}(\{d'_i\}_{i\in\mathbb{N}})$ and $\mathcal{P}_{n''}(\{d''_i\}_{i\in\mathbb{N}})$, respectively, where $n' = n-1$, $n'' = \max\{m-1, n-d_{m+1}-2\}$, and $\{d'_i\}_{i\in\mathbb{N}}$ and $\{d''_i\}_{i\in\mathbb{N}}$ are mnd sequences given by

$$d'_i := \begin{cases} d_i = 0 & \text{if } i \in [m-1]; \\ d_m - 1 & \text{if } i = m; \\ d_{i+1} & \text{if } i \in \mathbb{N}\backslash[m], \end{cases} \quad \text{and } d''_i := \begin{cases} d_i = 0 & \text{if } i \in [m-1]; \\ d_{i+d_{m+1}+2} & \text{if } i \in \mathbb{N}\backslash[m-1]. \end{cases}$$

Therefore, we can apply the induction hypothesis or Theorem 1.1 to each of $\mathcal{A}(\overline{m+1})$ and $\mathcal{A}\langle m+1\rangle$ to get

$$|\mathcal{A}(\overline{m+1})| \le |{\mathcal{P}_n}^{(r)}(\overline{m+1})(1)|, \quad |\mathcal{A}\langle m+1\rangle| \le |{\mathcal{P}_n}^{(r)}\langle m+1\rangle(1)|, \tag{20}$$

and hence $|\mathcal{A}| \le |{\mathcal{P}_n}^{(r)}(1)|$. Since $|\mathcal{A}| = |\mathcal{A}^*|$ and $\mathcal{A}^* \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$, $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$. So we actually have equalities in (20), and hence

$$\mathcal{A}(\overline{m+1}) \in \mathrm{ex}({\mathcal{P}_n}^{(r)}(\overline{m+1})), \quad \mathcal{A}\langle m+1\rangle \in \mathrm{ex}({\mathcal{P}_n}^{(r)}\langle m+1\rangle). \tag{21}$$

Claim 7.4 *Suppose* $a \in [m]$ *and* $\mathcal{A}(\overline{m+1}) = {\mathcal{P}_n}^{(r)}(\overline{m+1})(a)$. *Then* $a \in [m-1]$ *and* $\mathcal{A} = {\mathcal{P}_n}^{(r)}(a)$.

Proof. Suppose $\mathcal{A}(\overline{m+1}) = {\mathcal{P}_n}^{(r)}(\overline{m+1})(a)$, $a \in [m]$. Then, since $a \in P_{1,\alpha(\mathcal{P}_n)} \in \mathcal{P}_n$ and $r \leq \alpha(\mathcal{P}_n)/2$, for any $A \in {\mathcal{P}_n}^{(r)}\langle m+1\rangle(\overline{a})$ there exists $A' \in \mathcal{A}(\overline{m+1})$ such that $A \cap A' = \emptyset$. Since $\mathcal{A}(\overline{m+1}) \cup \mathcal{A}\langle m+1\rangle$ is intersecting, it follows that $\mathcal{A}\langle m+1\rangle \subseteq {\mathcal{P}_n}^{(r)}\langle m+1\rangle(a)$. So $\mathcal{A} = {\mathcal{P}_n}^{(r)}(a)$ as $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$. If $a = m$, then $\mathcal{A}\langle m+1\rangle = \emptyset$, and hence $|\mathcal{A}| = |{\mathcal{P}_n}^{(r)}(\overline{m+1})(a)| \leq |{\mathcal{P}_n}^{(r)}(\overline{m+1})(1)| < |{\mathcal{P}_n}^{(r)}(1)|$, contradicting $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$. □

Claim 7.5 *Suppose* $n \in P_{1,2r}$ *and* $m \leq 2r-2$. *Let* $j \in [m-1]$. *Let* $A \in {\mathcal{P}_n}^{(r)}(\overline{j})(\overline{m+1})$ *such that* $A \cap [m,n] \neq \emptyset$. *Then there exists* $A' \in {\mathcal{P}_n}^{(r)}(j)(m+1)$ *such that* $A \cap A' = \emptyset$.

Proof. Let $\mathcal{Q} := \mathcal{P}_n \cap 2^{[m,n]}$. So $\mathcal{Q}$ is isomorphic to $\mathcal{P}_{n-(m-1)}(\{d_{i+m-1}\}_{i\in\mathbb{N}})$. Clearly, $n \in P_{1,2r}$ implies that $n \in P_{m,\alpha(\mathcal{Q})}$ and $\alpha(\mathcal{P}_n) = 2r$. Since $m \leq 2r-2$ and $\alpha(\mathcal{Q}) = \alpha(\mathcal{P}_n) - (m-1) = 2r-(m-1)$, $\alpha(\mathcal{Q}) \geq 2r-(2r-3) = 3$. Let $B := A \cap [m,n] \in \mathcal{Q}(\overline{m+1})$. By the given conditions on A, $B \neq \emptyset$. Thus, by Lemma 7.2, there exists $B' \in \mathcal{Q}(m+1)$ such that $B \cap B' = \emptyset$ and $|B|+|B'| = \alpha(\mathcal{Q})$. Let $A' := B' \cup ([m-1]\backslash A)$. So $|A'| = |B'|+|[m-1]\backslash A| = (\alpha(\mathcal{Q})-|B|)+((m-1)-(r-|B|)) = \alpha(\mathcal{Q})+m-1-r = r$. Since $j \notin A$, $j \in A'$. The truth of the claim is now clear. □

Note that $P_{1,2r} \in \mathcal{P}_n$ since $2r \leq \alpha(\mathcal{P}_n)$.

Consider first $n \notin P_{1,2r}$. Since $m \in P_{1,2r}$, $m+1 \notin P_{1,2r}$. So $P_{1,2r} \in \mathcal{P}_n(\overline{m+1})(\overline{n})$. By (21) and the induction hypothesis, it follows that $\mathcal{A}(\overline{m+1}) = {\mathcal{P}_n}^{(r)}(\overline{m+1})(j)$ for some $j \in [m]$ (note that if $d_m = 1$, then $d'_m = 0$ and $d'_{m+1} > 0$). By Claim 7.4, $\mathcal{A} = {\mathcal{P}_n}^{(r)}(j)$ and $j \in [m-1]$.

Now consider $n \in P_{1,2r}$ and $m \leq 2r-2$. We have $P_{1,2r} = [m-1] \cup P_{m,\alpha(\mathcal{Q})}$, where $\mathcal{Q}$ is as in the Proof of Claim 7.5, and hence $\alpha(\mathcal{Q}) \geq 3$. Let $m =: p_1 < p_2 < \cdots < p_{\alpha(\mathcal{Q})} := n$ such that $P_{m,\alpha(\mathcal{Q})} = \{p_1, \ldots, p_{\alpha(\mathcal{Q})}\}$. Let $m'' := (m+1)+d_{m+1}+1$. Let $m'' =: q_1 < \cdots < q_{\alpha(\mathcal{Q})-2}$ such that $P_{m'',\alpha(\mathcal{Q})-2} = \{q_1, \ldots, q_{\alpha(\mathcal{Q})-2}\}$. Similarly to (13) and (14), we have

$$p_2 = m + d_m + 1 < (m+1) + d_{m+1} + 1 = q_1 < p_2 + d_{p_2} + 1 = p_3, \tag{22}$$

and if $\alpha(\mathcal{Q}) \geq 4$, then

$$p_i = p_{i-1} + d_{p_{i-1}} + 1 < q_{i-2} + d_{q_{i-2}} + 1 = q_{i-1} < p_2 + d_{p_2} + 1 = p_{i+1}, \tag{23}$$

$i = 3, \ldots, \alpha(\mathcal{Q})-1$. Let $P''_{1,2r''} := \{p''_1, \ldots, p''_{2r''}\} \in \mathcal{P}(\{d''_i\}_{i\in\mathbb{N}})$, where $p''_1 := 1$ and $p''_{l+1} := p''_l + d''_{p''_l} + 1$, $l = 1, \ldots, 2r''-1$. Clearly, $p''_j = j$, $j = 1, \ldots, m-1$, and $p''_l = q_{l-m+1} - d_{m+1} - 2$, $l = m, \ldots, 2r''$. Note that $2r'' = 2r-2 = (m-1)+\alpha(\mathcal{Q})-2$ (as $P_{1,2r} = [m-1] \cup P_{m,\alpha(\mathcal{Q})}$). Now, by (22) and (23), $q_{\alpha(\mathcal{Q})-2} < p_{\alpha(\mathcal{Q})}$. So we have $p''_{2r''} = p''_{m+\alpha(\mathcal{Q})-3} = q_{\alpha(\mathcal{Q})-2} - d_{m+1} - 2 < n - d_{m+1} - 2 = n''$. By the induction hypothesis, it follows that $\mathcal{A}\langle m+1\rangle = {\mathcal{P}_n}^{(r)}\langle m+1\rangle(j)$ for some $j \in [m-1]$. Therefore,

$$\mathcal{A}(m+1) = {\mathcal{P}_n}^{(r)}(m+1)(j). \tag{24}$$

Let $\mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2$ be the partition of $\mathcal{A}(\overline{m+1})$ given by $\mathcal{H}_0 := \mathcal{A}(\overline{m+1}) \cap \binom{[m-1]}{r}$, $\mathcal{H}_1 := \{A \in \mathcal{A}(\overline{m+1})\colon P_{m,\alpha(\mathcal{Q})} \subseteq A\}$ and $\mathcal{H}_2 := \mathcal{A}(\overline{m+1}) \backslash (\mathcal{H}_0 \cup \mathcal{H}_1)$. Define a partition $\mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2$ of ${\mathcal{P}_n}^{(r)}(\overline{m+1})(j)$ similarly. Since $\mathcal{A}$ is intersecting, it follows by (24) and Claim 7.5 that

$$\mathcal{H}_1 \subseteq \mathcal{I}_1 \quad \text{and} \quad \mathcal{H}_2 \subseteq \mathcal{I}_2. \tag{25}$$

Suppose $m \leq r+1$. If $m < r+1$, then $\mathcal{H}_0 = \emptyset$. If $m = r+1$, then $\mathcal{H}_0 = \{[m-1]\} \in {\mathcal{P}_n}^{(r)}(\overline{m+1})(j)$. Together with (24) and (25), this gives us $\mathcal{A} \subseteq {\mathcal{P}_n}^{(r)}(j)$. Since $\mathcal{A} \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$, $\mathcal{A} = {\mathcal{P}_n}^{(r)}(j)$.

Now suppose $m \geq r+2$. If $A \in \mathcal{H}_0 \backslash \mathcal{I}_0$, then $P_{1,2r} \backslash A \in \mathcal{I}_1 \backslash \mathcal{H}_1$. Thus, $|\mathcal{H}_0| + |\mathcal{H}_1| \leq |\mathcal{I}_0| + |\mathcal{I}_1|$ as $\mathcal{H}_1 \subseteq \mathcal{I}_1$ (by (25)). By (20), (21) and (25), it follows that $\mathcal{H}_2 = \mathcal{I}_2$ and $|\mathcal{H}_0| + |\mathcal{H}_1| = |\mathcal{I}_0| + |\mathcal{I}_1|$. We now prove that $\mathcal{A} = ({\mathcal{P}_n}^{(r)}(j) \backslash \{P_{1,2r} \backslash A\colon A \in \mathcal{H}_0\}) \cup \mathcal{H}_0$ by showing that for any $A \in \binom{[m-1]}{r}$, $P_{1,2r} \backslash A$ is the unique set in ${\mathcal{P}_n}^{(r)}$ that does not intersect A. Indeed, let $A' \in {\mathcal{P}_n}^{(r)}$ such that $A \cap A' = \emptyset$. Then $A' = A_1' \cup A_2'$ for some $A_1' \subseteq [m-1] \backslash A$ and $A_2' \in \mathcal{Q}$. Suppose that either $A_1' \neq [m-1] \backslash A$ or $|A_2'| < \alpha(\mathcal{Q})$ $(= 2r-m+1)$; then $|A'| < (m-1-r)+(2r-m+1) = r$, a contradiction. So $A_1' = [m-1] \backslash A$ and $|A_2'| = \alpha(\mathcal{Q})$. Clearly, since $n \in P_{m,\alpha(\mathcal{Q})} = P_{1,2r} \backslash [m-1]$, $P_{m,\alpha(\mathcal{Q})}$ is the only set in $\mathcal{Q}$ of size $\alpha(\mathcal{Q})$. So $A_2' = P_{m,\alpha(\mathcal{Q})}$, and hence $A' = P_{1,2r} \backslash A$.

We conclude the proof of the necessary conditions in (ii) and (iii) by showing that $\mathcal{A}^* = \mathcal{A}$. Suppose $\mathcal{A}^* \neq \mathcal{A}$. Then there exists $A^* \in \mathcal{A}^* \backslash \mathcal{A}$ such that $A := \delta_{m,m+1}(A^*) \in \mathcal{A} \backslash \mathcal{A}^*$. Now we have shown that for some $j \in [m-1]$ and $\mathcal{H}_0 \subseteq \binom{[m-1]\backslash\{j\}}{r}$, $\mathcal{A} = ({\mathcal{P}_n}^{(r)}(j) \backslash \{P_{1,2r} \backslash A\colon A \in \mathcal{H}_0\}) \cup \mathcal{H}_0$ (where $\mathcal{H}_0 = \emptyset$ if either $n \notin P_{1,2r}$ or $m \leq r+1$). Thus, since $m \in A$, $A \in {\mathcal{P}_n}^{(r)}(j)(m)$. So $A^* \in {\mathcal{P}_n}^{(r)}(j)(m+1) \backslash \mathcal{A}$, but this is a contradiction because, since $m+1 \notin P_{1,2r}$, ${\mathcal{P}_n}^{(r)}(j)(m+1) \subset \mathcal{A}$.

It remains to prove the sufficiency conditions in (ii) and (iii). We have shown that for any intersecting family $\mathcal{A} \subset {\mathcal{P}_n}^{(r)}$, $|\mathcal{A}| \leq |{\mathcal{P}_n}^{(r)}(1)|$. This already proves the sufficiency condition in (iii) because for any $j \in [2, m-1]$, ${\mathcal{P}_n}^{(r)}(j)$ is isomorphic to ${\mathcal{P}_n}^{(r)}(1)$. Therefore, the sufficiency condition in (ii) follows from the already established fact that if $n \in P_{1,2r}$, $r+2 \leq m \leq 2r-2$ and $A \in \binom{[m-1]}{r}$, then $P_{1,r} \backslash A$ is the unique set in ${\mathcal{P}_n}^{(r)}$ that does not intersect A. □

8 The remaining problem

Since $\mathcal{P}_n = \bigcup_{r=1}^{\alpha(\mathcal{P}_n)} {\mathcal{P}_n}^{(r)}$, it is immediate from Theorem 3.1 that

$$\mathrm{ex}(\mathcal{P}_n) = \{\mathcal{P}_n(1)\} \text{ if } d_1 > 0.$$

As noted in [10], $2^{[n]}(1) \in \mathrm{ex}(2^{[n]})$ holds because for any $\mathcal{A} \in \mathrm{ex}(2^{[n]})$, $[n] \backslash A \notin \mathcal{A}$ for all $A \in \mathcal{A}$. Now $\mathcal{P}_n = 2^{[n]}$ if $d_i = 0$ for all $i \in [n-1]$. By an inductive argument similar to the one used in Theorem 3.3 for showing that ${\mathcal{P}_n}^{(r)}(1) \in \mathrm{ex}({\mathcal{P}_n}^{(r)})$, we therefore obtain that

$$\mathcal{P}_n(1) \in \mathrm{ex}(\mathcal{P}_n) \text{ if } d_1 = 0.$$

For the case $d_1 = 0$ and $\alpha(\mathcal{P}_n)/2 < r < \alpha(\mathcal{P}_n)$, investigating the nature of families in $\mathrm{ex}({\mathcal{P}_n}^{(r)})$ seems to be a daunting task; recall from Section 1 that in this case we may have ${\mathcal{P}_n}^{(r)}(1) \notin \mathrm{ex}({\mathcal{P}_n}^{(r)})$ in this case. An achievable target, though, might be to determine a list of families such that $\mathrm{ex}({\mathcal{P}_n}^{(r)})$ must contain a family in the list. An example of an accomplishment of this kind is [1, Theorem 6], and it may well be that the extremal structures for our remaining problem are similar to those in that result.

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