ON CRITICALLY k-EXTENDABLE GRAPHS

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ABSTRACT:

Let G be a simple connected graph on 2n vertices with a perfect matching. G is **k-extendable** if for any set M of k independent edges, there exists a perfect matching in G containing all the edges of M. G is **critically k-extendable** if G is k-extendable but G + uv is not k-extendable for any non-adjacent pair of vertices u and v of G. The problem that arises is that of characterizing k-extendable and critically k-extendable graphs. This problem has been studied for k-extendable graphs and a number of results have been obtained. In particular, complete characterizations have been obtained for the case k = 1. Critically k-extendable graphs have not been studied. In this paper, we focus on the problem of characterizing critically k-extendable graphs. Complete characterizations are presented for k = 1, n - 2, n - 1 and n.

1. INTRODUCTION

All graphs considered in this paper are finite, connected,

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loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [1]. Thus G is a graph with vertex set V(G), edge set E(G) and minimum degree $\delta(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by V'. Similarly $G[E']$ denotes the subgraph induced by the edge E' of G. $N_G(u)$ denotes the neighbour set of u in G.

A **matching M** in G is a subset of E(G) in which no two edges have a vertex in common. M is a **maximum matching** if $|M| \geq |M'|$ for any other matching M' of G. A vertex v is **saturated** by M if some edge of M is incident to v; otherwise v is said to be **unsaturated**. A matching M is **perfect** if it saturates every vertex of the graph. For simplicity we let V(M) denote the vertex set of subgraph G[M] induced by M.

Let G be a simple connected graph on 2n vertices with a perfect matching. G is **k-extendable** if for any set M of k independent edges (two edges are independent if they do not have a common vertex), there exists a perfect matching in G containing all the edges of M. Clearly $1 \leq k \leq n$. We say that G is **critically k-extendable** or simply **k-critical** if it is k-extendable but G + uv is not k-extendable for any non-adjacent pair of vertices u and v of G.

Observe that the complete graph K_{2n} of order 2n and the complete bipartite graph $K_{n,n}$ with bipartitioning sets of order n are k-critical for $1 \leq k \leq n$. On the other hand, the cycle C_{2n} of order $2n \geq 6$ is 1-extendable but not 1-critical.

A number of authors have studied k-extendable graphs. An excellent survey is the paper of Plummer [6]. The problem of characterizing k-extendable graphs remains open for $k \geq 3$. k-critical graphs have not been previously investigated; the characterization problem was recently posed by Saito [7]. In this paper, we shall focus on the problem of characterizing these graphs.

For k = 1, n - 2, n - 1 and n we establish that a graph G of order 2n is k-critical if and only if $G \cong K_{n,n}$ or K_{2n}. We also characterize 2-critical graphs; for this case there exist graphs which are not complete or bipartite. We present a number of properties of k-critical graphs, including an upper bound on the minimum degree.

Section 2 contains some preliminary results that we make use of in our work. In Section 3 we prove two new properties of k-extendable that we use in establishing our main results in Section 4.

2. PRELIMINARIES

In this section, we state a number of results on k-extendable graphs which we make use of in establishing our main results. We state only results which we use; for a more detailed account we refer to the paper of Plummer [6].

We begin with an important result of Berge (see [3] p. 90). Let M be a maximum matching in a graph G. The **deficiency** def(G) of G is defined as the number of M-unsaturated vertices of G. Denoting the number of odd components in a graph H by $o(H)$ we can now state Berge's Formula :

Theorem 2.1: For any graph G

$$\mathrm{def}(G) = \max\{o(G - X) - |X| : X \subseteq V(G)\} . \qquad \square$$

As noted in the introduction 1-extendable graphs have been characterized by Grant et al [2]. The result is

Theorem 2.2: A graph G of even order is 1-extendable if and only if

(i) $o(G - S) \leq |S|$ for all $S \subset V(G)$,

and

(ii) $o(G - S) = |S|$ only if S is an independent set of vertices in G. □

Before stating a necessary condition for 2-extendable graphs we need the following definitions. A graph G is **bicritical** if $G - u - v$ has a perfect matching for every pair of vertices u and v. A graph G is **elementary** if the graph G′ induced by the edges

$$E' = \{e : e \in E(G) \text{ and } e \text{ is in some perfect matching in } G\}$$

is connected. Plummer [4] proved the following three results.

Theorem 2.3: Let G be a 2-extendable graph with $2n \geq 6$ vertices. Then G is either bicritical or elementary bipartite. □

Theorem 2.4: Let G be a k-extendable graph on 2n vertices, $1 \leq k \leq n - 1$. Then

(a) G is $(k - 1)$-extendable;

(b) G is $(k + 1)$-connected;

(c) if $d_G(u) = k + 1$, then $N_G(u)$ is independent. □

Theorem 2.5: Let G be a graph on 2n vertices and $1 \leq k \leq n - 1$. If $\delta(G) \geq n + k$, then G is k-extendable. □

For bipartite graphs, Plummer [5] proved :

Theorem 2.6: Let G be a k-extendable bipartite graph on 2n vertices, $1 \le k \le n - 1$, such that G + e is bipartite for some $e \notin E(G)$. Then G + e is also k-extendable. □

A consequence of Theorem 2.6 is the following Corollary :

Corollary: Let G be a k-extendable bipartite graph on 2n vertices, $1 \le k \le n - 1$. Then G is k-critical if and only if G is $K_{n,n}$. □

3. EXTENDABLE GRAPHS

In addition to the results mentioned in Section 2 we need, in our study of critically extendable graphs, two further results. In this section we present these results. Our first result concerns bipartite graphs.

We have noted that $K_{n,n}$ is k-extendable for all $1 \le k \le n$. Since an r-regular (connected) bipartite graph has a 1-factorization it is 1-extendable for all r. However, it need not be k-extendable, $k \ge 2$. For example, for $n \ge 2r$ it is easy to construct an r-regular bipartite graph on 2n vertices having connectivity 2; an example is given in Figure 3.1, where H and H′ are r-regular bipartite graphs on n

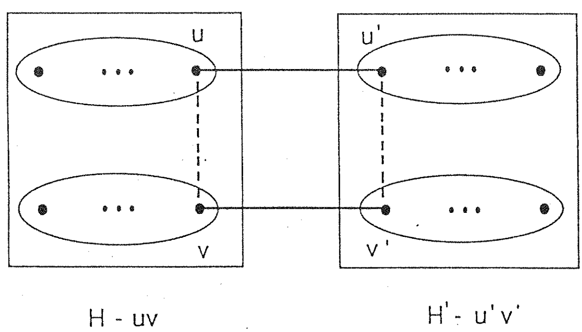

Figure 3.1.

vertices containing the edges uv and u'v', respectively. For bipartite graphs having a prescribed minimum degree we have the following result.

Theorem 3.1: Let G be a bipartite graph on 2n vertices with $\delta(G) \geq n - 1$. Then G is k-extendable for $1 \leq k \leq n - 2$.

Proof: In view of Theorem 2.4 (a), it is sufficient to prove that G is (n - 2)-extendable. Let (U,W) be the bipartition of G and let M be matching of size (n - 2) in G. Consider $G' = G - V(M)$. G' is a bipartite graph consisting of four vertices and $\delta(G') \geq 1$. If $\delta(G') = 2$, then $G' \cong K_{2,2}$ and hence has a perfect matching. If on the other hand, $\delta(G') = 1$, then G' is either 1-regular or a path of length 3. In either case it has a perfect matching. Consequently G is (n - 2)-extendable as required. □

As a Corollary we have :

Corollary: An (n - 1)-regular bipartite graph on 2n vertices is k-extendable for $1 \leq k \leq n - 2$. □

We remark that an (n - 2)-regular bipartite graph on 2n vertices need not be (n - 3)-extendable as the following graph demonstrates. Start with an (n - 5)-regular bipartite graph on 2(n - 3) vertices with bipartitioning sets X and Y. Select non-adjacent vertices $x \in X$ and $y \in Y$ and join them. Add 6 new vertices, u_1, u_2, u_3, v_1, v_2, and v_3. Join u_1 and u_2 (v_1 and v_2) to every vertex of X (Y). Join u_3 (v_3) to

u_1, u_2 and to every vertex of $Y - y$ (v_1, v_2 and to every vertex of $X - x$). Call the resultant graph G. For $n \geq 6$, G has a matching M of size $n - 3$ that saturates only the vertices of $X \cup Y$. Now $G - V(M)$ consists of 2 odd components and consequently G is not $(n - 3)$ -extendable.

In the proofs that follow we make frequent use of the following fact. If G is k-extendable, then for any vertex u, $G - u$ cannot contain a matching of size at most k that saturates $N_G(u)$.

Our next result is a generalization of Theorem 2.4 (c).

Theorem 3.2: Let G be a k-extendable graph on 2n vertices with $\delta(G) = k + t$, $1 \leq t \leq k \leq n - 1$. If $d_G(u) = \delta(G)$, then the subgraph $G[N_G(u)]$ has at most $t - 1$ independent edges.

Proof: Suppose that $d_G(u) = \delta(G)$ and $G[N_G(u)]$ has a maximum matching M of size $s \geq t$. Since G is k-extendable we must have $s \leq k - 1$. Let v be an M-unsaturated vertex of $N_G(u)$. Then $M_1 = M \cup \{uv\}$ is a matching of size $s + 1 \leq k$ in G. So M_1 can be extended to a perfect matching F of G. Let

$$F_1 = \{xy \in F : x \in N_G(u) - v \ , \ y \notin N_G(u)\} \ ,$$

$$A = V(F_1)\backslash N_G(u) \quad , \quad \text{and} \quad B = V(G) - u - N_G(u) - A.$$

Figure 3.2 depicts the situation with the edges of $M \cup F_1$ drawn in solid lines. Then

$$|A| = k + t - 2s - 1 \leq k, \quad \text{and hence}$$

$$|B| = 2n - 2k - 2t + 2s \geq 2.$$

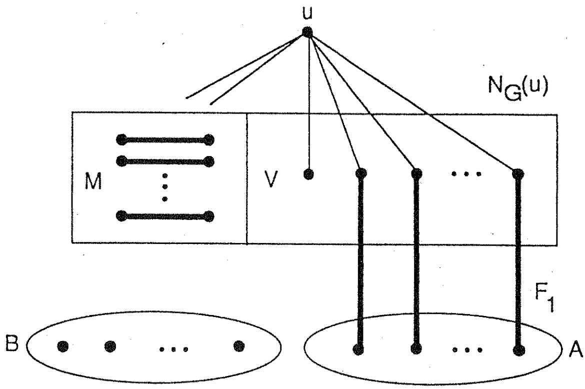

Figure 3.2.

If v is adjacent to a vertex b of B, then $M_2 = M \cup F_1 \cup \{vb\}$ is a matching in G of size $s + (k + t - 2s - 1) + 1 = k + t - s \leq k$. But then u is an isolated vertex in $G - V(M_2)$ contradicting the fact that G is k-extendable. Hence $N_G(v) \cap B = \phi$. Now for $d_G(v) \geq k + t$ the only possibility is for v to be adjacent to every vertex of $V(M) \cup A$ in which case $d_G(v) = k + t$.

If no vertex of B is adjacent to any vertex of $N_G(u)$, then $G - A$ is disconnected and hence G is at most $|A|$-connected. Since $|A| \leq k$ this contradicts Theorem 2.4(b). Let $xy \in E(G)$ with $x \in B$ and $y \in N_G(u)$. Since $y \neq v$, $y \in V(M) \cup V(F_1)$. Let $yz \in F$. Then z is in V(M) or A and so is adjacent to v. Consequently the path x, y, z, v is an F-augmenting path in G with xy and zv not in F. But then

$$M_3 = M \cup F_1 \cup \{xy, zv\} \setminus \{yz\}$$

is a matching of size $k + t - s \leq k$ that saturates the vertices of $N_G(u)$, implying that G is not k-extendable. This contradiction completes the proof of the theorem. □

As a Corollary we have :

Corollary: Let G be a k-extendable, $(k + t)$-regular graph on 2n vertices, $1 \leq t \leq k \leq n - 1$. Then $G[N_G(u)]$ contains at most $t - 1$ independent edges for every u in G. □

4. CRITICAL GRAPHS

Recall that a **k-critical** graph is one that is k-extendable, but $G + uv$ is not k-extendable for any non-adjacent pair of vertices u and v of G. Our first result provides a sufficient condition for a regular graph of diameter 2 to be k-critical.

Theorem 4.1: Let G be a k-extendable, $(k + t)$-regular graph, $1 \leq t \leq k \leq n - 1$, on 2n vertices having diameter 2. Let w be any vertex of G and u and v any pair of non-adjacent vertices of $N_G(w)$. If $G[N_G(w) - u - v]$ has exactly $t - 1$ independent edges, then G is k-critical.

Proof: Let M be a matching of size $t - 1$ in $G[N_G(w) - u - v]$. Then $M_1 = M \cup \{uw\}$ is a matching of size $t \leq k$ in G and so can be extended to a perfect matching F of G. Let

$$F_1 = \{xy \in F : x \in N_G(w) - u - v\ ,\ y \notin N_G(w)\}\ .$$

Since, by Theorem 3.2, $G[N_G(w)]$ has at most $t - 1$ independent edges, $|F_1| = k - t$. But then $M_2 = M \cup F_1 \cup \{uv\}$ is a matching in $G + uv$ of size k and $G + uv - V(M_2)$ has w as an isolated vertex. Hence G is k-critical, proving the theorem. □

We remark that the graph $G(2k,2k)$ obtained by joining two disjoint K_{2k}'s by a perfect matching satisfies the conditions in Theorem 4.1. Hence as $G(2k,2k)$ is k-extendable it is also k-critical.

Our next result provides a sufficient condition for any k-extendable graph to be k-critical. We make use of the following terminology. We call a subset S of V(G) **dependent** if G[S] has at least one edge.

Theorem 4.2: Let $G \neq K_{2n}$ be a k-extendable graph on 2n vertices, $2 \leq k \leq n-1$. If for any pair of non-adjacent vertices u and v of G there exists a dependent set S of $G - u - v$ such that $o(G - (S \cup \{u,v\})) = |S|$, then G is k-critical. Moreover, the converse is true for a non-bipartite G and $k = 2$.

Proof: Let u and v be any two non-adjacent vertices of G satisfying the hypothesis of the theorem. Then $G' = G - u - v$ contains a dependent set S such that

$$\begin{aligned} |S| &= o(G - (S \cup \{u,v\})) \\ &= o(G' - S) . \end{aligned}$$

Hence, by Theorem 2.2, G' is not 1-extendable. Consequently, G' is not $(k - 1)$-extendable and thus G is k-critical.

Suppose that G is a 2-critical non-bipartite graph. Consider the graph $G' = G - x - y$, where x and y are any two non-adjacent vertices of G. G' has a perfect matching by Theorem 2.3 but is not 1-extendable. Hence, by Theorem 2.2, there exists a dependent set S such that $o(G' - S) = |S|$. Therefore $o(G - (S \cup \{x,y\})) = |S|$, as required. This completes the proof of the theorem. □

In view of Theorem 2.6 we have the following corollary.

Corollary: Let G be a 2-extendable graph on $2n \geq 6$ vertices. G is 2-critical if and only if G is K_{2n} or $K_{n,n}$ or for any pair of non-adjacent vertices u and v of G there exists a dependent set S of G - u - v such that $o(G - (S \cup \{u,v\})) = |S|$. □

Remark 1: There exists 2-critical non-bipartite graphs which are not complete. For example, the graphs drawn in Figure 4.1.

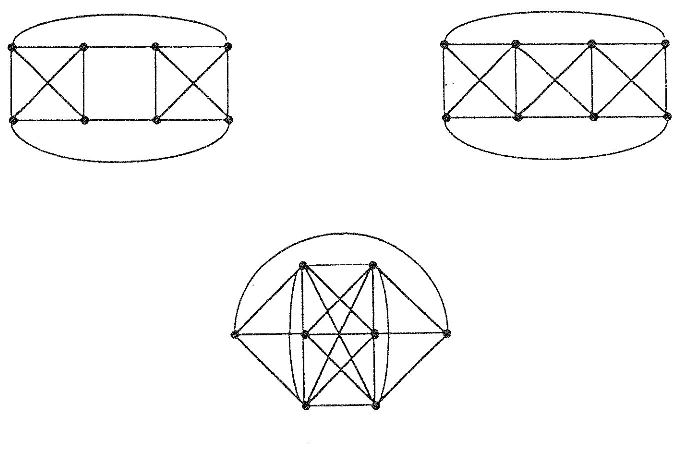

Figure 4.1.

Remark 2: None of the graphs in Figure 4.1 are 1-critical since, in each case, the deletion of any pair of non-adjacent vertices results in a graph having a perfect matching. Thus a k-critical graph need not be (k - 1)-critical.

Theorem 2.4(b) implies that a k-extendable graph G has minimum degree at least k + 1. Our next task is to establish an upper bound on the minimum degree of a k-critical graph. We start with the following lemma.

Lemma 4.1: Let $G \neq K_{2n}$ be a k-critical graph on 2n vertices, $1 \leq k \leq n - 1$, and u and v any pair of non-adjacent vertices of G. Let M be a matching of size k - 1 in G - u - v. Then the graph $G' = G - u - v - V(M)$ has a matching of size at least n - k - 1.

Proof: Suppose G′ has a maximum matching M′ of size at most n - k - 2. Then

$$\begin{aligned} \operatorname{def}(G') &= |V(G')| - 2|M'| \\ &= 2(n - k) - 2|M'| \\ &\geq 4 \end{aligned}$$

By Theorem 2.1, there exists a subset S′ of V(G′) such that

$$o(G' - S') - |S'| = \operatorname{def} G' \geq 4$$

Put $S = S' \cup \{u,v\}$ and $G_1 = G - V(M)$. Then

$$o(G_1 - S) - |S| = o(G' - S') - |S'| - 2 \geq 2 .$$

Then $\operatorname{def}(G_1) \geq 2$, implying that G is not k-extendable. This contradiction completes the proof of the Lemma. □

Lemma 4.2: Let G be a connected graph on 2n vertices with $\delta(G) \geq n - 1$ having a maximum matching M of size n - 1. Then for M-unsaturated vertices u and v of G $N_G(u) = N_G(v)$. Furthermore, no two vertices of $N_G(u)$ are joined by an edge of M, and the vertices of $V(G) - N_G(u)$ form an independent set.

Proof: Let $M = \{x_iy_i : 1 \le i \le n - 1\}$. Observe that if $x_iu \in E(G)$ then $y_iv \notin E(G)$. Let

$$M_1 = \{x_iy_i \in M : ux_i, uy_i \in E(G)\} ,$$
$$M_2 = \{x_iy_i \in M : vx_i, vy_i \in E(G)\} , \text{ and}$$
$$M_3 = M \backslash (M_1 \cup M_2) .$$

From our earlier observation it follows that $M_1 \cap M_2 = \phi$. By definition, if $x_iy_i \in M_3$ then u and v can each be joined to at most one of x_i and y_i. Consequently

$$\begin{aligned} 2(n - 1) \le d_G(u) + d_G(v) &\le 2(|M_1 \cup M_2 \cup M_3|) \\ &= 2|M| \\ &= 2(n - 1) , \end{aligned}$$

and hence each of u and v must be joined to exactly one end of each edge in M_3. In fact, $N_G(u) \cap V(M_3) = N_G(v) \cap V(M_3)$.

If $M_3 = \phi$ then, since G is connected, we have an M-augmenting path between u and v, contradicting the maximality of M. Hence $M_3 \ne \phi$. We next establish that $M_1 = \phi$.

Suppose $M_1 \ne \phi$. Let X and Y respectively denote the vertices of $V(M_3)$ adjacent and non-adjacent to u. If $ab \in E(G)$ with $a \in Y$ and

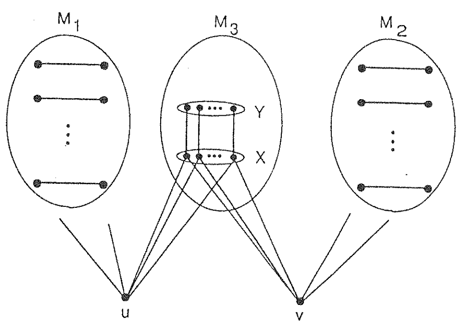

Figure 4.2.

$b \notin X$, then G contains an M-augmenting u,v path, contradicting the maximality of M. Hence Y is an independent set of vertices in G and no vertex of Y is joined to any vertex of $V(M_1) \cup V(M_2)$. Consequently for $w \in Y$ we have $d_G(w) \le |X| \le n - 2$, a contradiction. Therefore $M_1 = \phi$ and similarly $M_2 = \phi$. This proves the lemma. □

Theorem 4.3: If $G \ne K_{2n}$ is k-critical on 2n vertices, $1 \le k \le n - 1$, then

$$\delta(G) \le \begin{cases} n & , \quad n < 2k \\ n + 2\left\lfloor \frac{k-1}{2} \right\rfloor & , \quad n \ge 2k\ . \end{cases} \tag{4.1}$$

Proof: Let u and v be any pair of non-adjacent vertices of G and M a matching of size k − 1 in G − u − v. Consider the graph G′ = G − u − v − V(M). Since G is k-critical G′ has no perfect matching. Further, the subgraph $G[V(M) \cup \{u,v\}]$ has a maximum matching of size at most k-1, for otherwise G is not k-extendable. We distinguish two cases according to the value of k.

Case 1: $n < 2k$.

Suppose that $\delta(G) \ge n + 1$. Let M′ be a maximum matching in the graph G′ defined above. By Lemma 4.1, $|M'| = n - k - 1$ (note that $\nu(G') = 2n - 2k$). Let x and y be M′-unsaturated vertices of G′. Clearly x and y are not adjacent. Since $\delta(G) \ge n + 1$ and M′ is a maximum matching in G′, there must be an edge e of M such that x and y are adjacent to different end vertices of e, say a and b, respectively. Then $M' \cup \{xa, yb\}$ is a matching of size $n - k + 1 \le k$.

But

$$G - (V(M') \cup \{x,a,y,b\}) = G[(V(M) - \{a,b\}) \cup \{u,v\}]$$

has a matching of size at most $k - 2$. This contradiction proves that $\delta(G) \le n$ for $n < 2k$.

Case 2: $n \ge 2k$.

Suppose that $\delta(G) \ge n + k$. Let $G_0 = G - u - v$. Then

$$|V(G_0)| = 2(n - 1)$$

and

$$\delta(G_0) \ge \delta(G) - 2 \ge (n - 1) + (k - 1).$$

By Theorem 2.5, G_0 is $(k - 1)$-extendable contradicting the fact that G is k-critical. Hence $\delta(G) \le n + k - 1$. Thus we need only consider the case k even. For this case we will prove that $\delta(G) \le n + k - 2$.

Suppose that $\delta(G) = n + k - 1$. Now by the choice of G',

$$\delta(G') \ge \delta(G) - 2k = n - k - 1.$$

We now prove that G' is connected. Suppose that G' is disconnected. Then G' contains exactly two components as

$$\nu(G') = 2(n - k) \ge 2(\delta(G') + 1).$$

In fact, G' consists of two disjoint K_{n-k}'s. Since G' has no perfect matching, $n - k$ and hence n must be odd.

Since $\delta(G) = n + k - 1$, every vertex of G' must be adjacent, in G, to every vertex of $V(M) \cup \{u,v\}$. Let x and y be any two non-adjacent vertices of G'. Now consider the graph $\hat{G} = G + xy$. We will establish that G' is connected by showing that $\hat{G}$ is k-extendable.

Suppose $\hat{G}$ is not k-extendable. Then since G is k-extendable, there exists a set $\hat{M}$ of k independent edges, with $xy \in \hat{M}$, that does not extend to a perfect matching in $\hat{G}$. If $ab \in \hat{M}$ and $a, b \notin V(G')$, then $\hat{M}' = (\hat{M}\setminus\{xy,ab\}) \cup \{xa,yb\}$ is a matching in G of size k with $V(\hat{M}) = V(\hat{M}')$. But then G cannot be k-extendable, a contradiction. We get a similar contradiction when $ab \in \hat{M}$ with $a \in V(G')$ and $b \notin V(G')$. We conclude therefore that $V(M') \subseteq V(G')$. If $V(M) \neq V(G')$ then the graph $G'' = G - V(M) - V(\hat{M})$ consists of $\bar{K}_2 \vee (K_{2p} \cup K_{2q})$ for some p and q. Note that $V(\bar{K}_2) = \{u,v\}$. But G'' has a perfect matching implying that $\hat{M}$ is k-extendable. Hence $V(\hat{M}) = V(G')$ and so $n - k = k$ implying that n is even, a contradiction. Therefore $\hat{G}$ is k-extendable, contradicting the criticality of G. Hence G' is connected.

Now Lemma 4.1 together with the fact that G' has no perfect matching implies that G' has a maximum matching M' of size $n - k - 1$. Let u' and v' be the two M'-unsaturated vertices of G'. By Lemma 4.2 $N_{G'}(u') = N_{G'}(v')$. Let $N_{G'}(u') = \{x_1,x_2,\ldots,x_{n-k-1}\}$. Lemma 4.2 implies that no two x_i's are joined by an edge of M' and the set $V(G') - N_{G'}(u')$ is an independent set of vertices. Since $\delta(G) \geq n + k - 1$ and $G[V(M) \cup \{u,v\}]$ has a maximum matching of size at most $k - 1$, at least one of u or v, say u, is joined to a vertex, w say, of $N_{G'}(u')$. (See Figure 4.3).

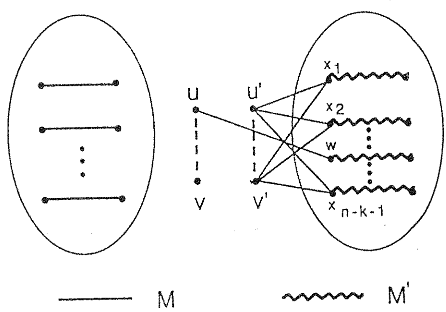

Figure 4.3.

Consider the matching $M'' = M \cup \{uw\}$. The subgraph $G''' = G - V(M'')$ contains a set $S = \{v\} \cup (N_{G'}(u')\backslash\{w\})$ such that $o(G''' - S) > |S|$. Hence G''' does not contain a perfect matching and so G is not k-extendable, a contradiction. This completes the proof of the theorem. □

Remark 3: For $n < 2k$ the graph $K_{n,n}$ achieves the bound (4.1). For $n = 2k$ the graphs H_1 and H_2 drawn in Figure 4.4 achieve the bound given in (4.1) for k odd and even, respectively. Note that in our diagrams a "double line" denotes the join. That H_1 and H_2 are k-critical is easily established. For example, in the case of H_1 if

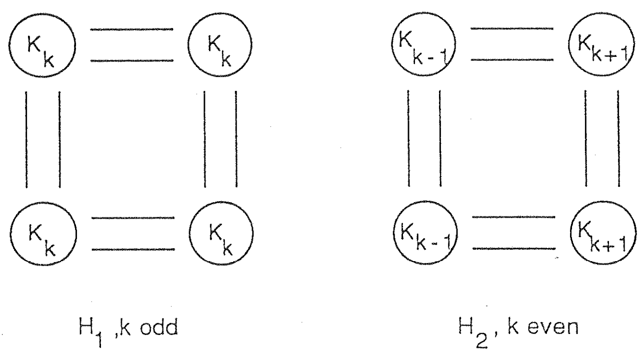

Figure 4.4.

$uv \notin E(H_1)$, then u and v are in diagonally opposite K_k's and so for odd k it is easy to find a matching M of size k, with $uv \in M$, such that $H_1 - V(M)$ consists of two odd components.

Our next lemma establishes that 1-critical graphs are regular. Observe that a graph G is 1-critical if and only if $G - u - v$ has no perfect matching for every pair of non-adjacent vertices u and v.

Lemma 4.3: If G is a 1-critical graph on 2n vertices, then G is regular.

Proof: Suppose to the contrary that G is not regular. Let $\delta(G) = r$. Since G is connected there exists adjacent vertices u and v with $d_G(u) = r$ and $d_G(v) > r$.

Let F be a perfect matching in G containing edge uv. Let

$$A = \{xy \in F | x \in N_G(u) - v,\ y \notin N_G(u)\}$$

$$B = \{xy \in F | x,y \in N_G(u)\}\ .$$

If v is adjacent to $x \in N_G(u) - v$ and $xy \in A$, then $G - u - y$ has a perfect matching, namely $(F\backslash\{uv,xy\}) \cup \{vx\}$. But this contradicts the fact that G is 1-critical. Hence v is not adjacent to any vertex of $N_G(u) \cap V(A)$. Consequently, since $|A| + 2|B| = r - 1$, v is joined to a vertex, w say, different from u that does not belong to $V(A) \cup V(B)$. Let wz be the edge of G that is in F. The choice of w implies that wz $\notin A \cup B$. Now $(F\backslash\{uv,wz\}) \cup \{vw\}$ is a perfect matching in $G - u - z$, contradicting the criticality of G. This proves the lemma. □

In the remainder of this paper, we make frequent use of the following notation. For $u \in V(G)$, we write $\bar{N}_G(u) = V(G)\backslash(N_G(u) \cup \{u\})$.

The following theorem provides a characterization of 1-critical graphs.

Theorem 4.4: A graph G on 2n vertices is 1-critical if and only if $G \cong K_{n,n}$ or K_{2n}.

Proof: The sufficiency is obvious as $K_{n,n}$ and K_{2n} are k-critical for $1 \le k \le n$. So we need to prove the necessity.

Let G be 1-critical. Then, by Lemma 4.3, G is r-regular for some $r \ge 2$. Take u, v, F, A and B as in the proof of Lemma 4.3. Then $r = |A| + 2|B| + 1$ and v is not adjacent to any vertex of $N_G(u) \cap V(A)$. We now prove that $G \cong K_{n,n}$ when $B = \phi$.

Suppose $B = \phi$. If $vw \in E(G)$, with $w \in \bar{N}_G(u)\backslash V(A)$, then

$$F' = (F\backslash\{uv, ww'\}) \cup \{vw\} ,$$

where $ww' \in F$, is a perfect matching in $G - u - w'$. But then G is not 1-critical. Hence v is not adjacent to any vertex of $\bar{N}_G(u)\backslash V(A)$. Now since v has degree r it must be joined to every vertex of $V(A) \cap \bar{N}_G(u)$. Let x be any vertex of $N_G(u) - v$. Suppose that $xy \in E(G)$ with $y \ne u$ and $y \notin \bar{N}_G(u) \cap V(A)$. Let xx′ and yy′ belong to F. Then v is adjacent to at least one of x′ or y′, say x′. Since $B = \phi$, u is not adjacent to y′. Now

$$(F\backslash\{uv, xx', yy'\}) \cup \{vx', xy\}$$

is a perfect matching in $G - u - y'$, contradicting the criticality of G. Hence $N_G(u)$ is an independent set, each vertex of which is adjacent to every vertex of $\bar{N}_G(u) \cap V(A)$. Consequently, $\bar{N}_G(u)\backslash V(A) = \phi$. Hence $r = n$ and $G \cong K_{n,n}$.

We next prove that $G \cong K_{2n}$ when $B \ne \phi$. Suppose $B \ne \phi$. Consider the edge $bb' \in B$. If $vb \notin E(G)$, then $(F\backslash\{uv, bb'\}) \cup \{ub'\}$ is a perfect matching in $G - v - b$, contradicting the criticality of G. Hence $V(B) \subseteq N_G(v)$. A similar argument establishes that any two vertices of V(B) are adjacent. Therefore the vertices u,v and V(B) form a complete subgraph in G. Now let $aa' \in A$ with $a \notin N_G(u)$. If $va \notin E(G)$, then

$(F\backslash\{uv, aa'\}) \cup \{ua'\}$ is a perfect matching in $G - v - a$, contradicting the criticality of G. Hence v is joined to every vertex of $V(A) \cap \bar{N}_G(u)$. Consider any edge $bb' \in B$. If $ab \notin E(G)$, then $(F\backslash\{aa', bb', uv\}) \cup \{ua', vb'\}$ is a perfect matching in $G - a - b$, a contradiction. Consequently each vertex of $\bar{N}_G(u) \cap V(A)$ is adjacent to every vertex of $v \cup V(B)$.

Suppose s, t are non-adjacent vertices with $s \in V(A) \cap N_G(u)$ and $t \in V(A) \cap \bar{N}_G(u)$. Let tt', $ss' \in A$. Now

$$(F\backslash\{ss', tt', uv\}) \cup \{ut', vs'\}$$

is a perfect matching in $G - s - t$, a contradiction. Hence each vertex of $V(A) \cap \bar{N}_G(u)$ is adjacent to every vertex of $V(A) \cap N_G(u)$. Consequently $N_G(u) \subseteq N_G(a)$ for every $a \in V(A) \cap \bar{N}_G(u)$. Further, since G is r-regular $N_G(u) = N_G(a)$.

Now suppose that $\bar{N}_G(u)\backslash V(A) \neq \phi$ and let $p \in \bar{N}_G(u)\backslash V(A)$. Since G is r-regular p is not adjacent to any vertex of $(V(A) \cap \bar{N}_G(u))$ or $(\{v\} \cup V(B))$. Since G is connected, $pq \in E(G)$ for some $q \in V(A) \cap N_G(u)$. Let pp', $qq' \in F$. Now

$$(F\backslash\{pp', qq', uv\}) \cup \{pq, vq'\}$$

is a perfect matching in $G - u - p'$, a contradiction. Hence $\bar{N}_G(u)\backslash V(A) = \phi$. We complete the proof by showing that $A = \phi$.

Suppose $A \neq \phi$ and let $a_1 \in V(A) \cap N_G(u)$. Since a_1 is not joined to v or any vertex of V(B), we have

$$r = |A| + 2|B| + 1 \le 2|A|$$

and hence $|A| \ge 2|B| + 1 \ge 3$. Let $a_2 \in V(A) \cap N_G(u)$ and a_1a_1', $a_2a_2' \in A$. If $a_1a_2 \in E(G)$, then $(F\backslash\{a_1a_1', a_2a_2'\}) \cup \{a_1a_2\}$ is a perfect matching in $G - a_1' - a_2'$. Since $a_1'a_2' \notin E(G)$, this contradicts the criticality of G. Hence the vertices of $N_G(u) \cap V(A)$ form an independent set. But then $d_G(a_1) \le |A| + 1 < r$, a contradiction. This proves that $A = \phi$ and hence $G \cong K_{2n}$. This completes the proof of the theorem. □

Since a graph G of order 2n is n-critical if and only if G - u - v has no perfect matching for every non-adjacent pair of vertices u and v, it follows that G is n-critical if and only if it is 1-critical. Hence we have :

Theorem 4.5: A graph G on 2n vertices is n-critical if and only if $G \cong K_{n,n}$ or K_{2n}. □

The following result gives a characterization of (n - 1) - critical graphs :

Theorem 4.6: Let G be a graph on $2n \ge 4$ vertices. Then G is (n - 1)-critical if and only if $G \cong K_{n,n}$ or K_{2n}.

Proof: We need only prove the necessity condition as $K_{n,n}$ and K_{2n} are clearly (n - 1)-critical. So suppose that G is (n - 1)-critical and $G \not\cong K_{n,n}$ and K_{2n}. We can assume that $n \ge 3$ as otherwise the result follows from Theorem 4.4. Then $n < 2(n - 1)$ and so, by theorems 2.4 (b) and 4.3, $\delta(G) = n$.

Let $d_G(u) = n$. By Theorem 2.4 (c), $N_G(u)$ is independent. Consequently every vertex in $N_G(u)$ is adjacent to every vertex in $\bar{N}_G(u)$. Consider any vertex $v \in N_G(u)$. $d_G(v) = n$ and so $N_G(v)$ is independent. Hence $\bar{N}_G(u)$ is an independent set and therefore $G \cong K_{n,n}$.

This completes the proof of the theorem. □

We now turn our attention to $(n - 2)$-critical graphs. We begin with the following lemma.

Lemma 4.4: If G is an $(n - 2)$-critical graph on $2n \geq 6$ vertices, then $\delta(G) > n - 1$.

Proof: Suppose to the contrary that $\delta(G) \leq n - 1$. Then, by Theorem 2.4(b), $\delta(G) = n - 1$. If $n = 3$, then, by Lemma 4.3 and Theorem 2.4(b), G is the cycle C_6. But C_6 is not 1-critical, and so we need only consider $n \geq 4$.

Consider a pair of adjacent vertices u and v with $d_G(u) = n - 1$. By Theorem 2.4(c) $N_G(u)$ is an independent set of vertices. Let F be a perfect matching of G containing the edge uv. Then there exists an edge xy in F such that x and y are in $\bar{N}_G(u)$. We now prove that the subgraph H induced by the vertices in $\bar{N}_G(u)$ contains only one independent edge. Suppose xy and $x'y'$ are independent edges of H. Then the graph

$$G' = G - \{x,y,x',y'\}$$

has $2n - 4$ vertices and contains $N_G(u)$ as an independent set of $n - 1$ vertices. Clearly G' cannot have a perfect matching, contradicting the fact that G is k-critical, $k \geq 2$. Hence H contains only one independent edge.

Now since H contains one independent edge, $|\bar{N}_G(u)| = n \geq 4$ and $\delta(G) = n - 1$, at least one of x or y is adjacent to a vertex of

$N_G(u)$. Suppose $xz \in E(G)$ with $z \in N_G(u)$. If $yw \in E(G)$, $w \neq z \in N_G(u)$, then the graph $G'' = G - \{x,y,z,w\}$ contains two disjoint independent sets of order $n - 1$ and $n - 3$ and hence cannot have a perfect matching. Since G is k-critical, $k \geq 2$, we must have $|N_G(y) \cap N_G(u)| \leq 1$. In fact, if $|N_G(y) \cap N_G(u)| = 1$ then $yz \in E(G)$ and so each of x,y and z have degree, in G, at least n (Theorem 2.4 (c)). Consequently, y is joined to every vertex of $\bar{N}_G(u)$. Thus H consists of a star with centre y. Therefore the graph $G''' = G - u - y$ is a bipartite graph with bipartition $(N_G(u), \bar{N}_G(u) - y)$ and $\delta(G''') \geq n - 2$. But then, by Theorem 3.1, G''' is $(n - 3)$-extendable implying that $G + uy$ is $(n - 2)$-extendable, a contradiction. This completes the proof of the lemma. □

We now characterize $(n - 2)$-critical graphs on 2n vertices which have minimum degree n.

Theorem 4.7: Let G be an $(n - 2)$-critical graph on 2n vertices with $\delta(G) = n \geq 5$. Then $G \cong K_{n,n}$.

Proof: Let $d_G(u) = n$. The main task in proving the theorem is to prove that $N_G(u)$ is an independent set. Suppose that this is not so and that v and w are adjacent vertices of $N_G(u)$. Then by Theorem 3.2, the subgraph induced by the vertices of $N_G(u)$ contains only one independent edge.

Let t be any vertex of $N_G(u) - v - w$ (since $n \geq 5$ such a t exists) and F a perfect matching of G containing the edges ut and vw. Denote the subgraph of G induced by the vertices in $\bar{N}_G(u)$ by H.

Clearly F contains an edge, xy say, of H. We claim that H contains only one independent edge. For let x′y′ and xy be a pair of independent edges in H. Then the graph $G' = G - \{x,y,x',y',v,w\}$ has $2n - 6$ vertices and contains an independent set of order $n - 2$ and hence cannot contain a perfect matching. This contradicts the fact that G is k-extendable, $k \geq 3$. Hence H contains only one independent edge. Consequently the graph $\hat{G} = G - \{v,w,x,y\}$ is bipartite with bipartitioning sets (X,Y), with $X = N_G(u)\backslash\{v,w\}$ and $Y = (\bar{N}_G(u) \cup \{u\})\backslash\{x,y\}$.

If $(N_G(x) \cup N_G(y)) \cap N_G(u) = \{v,w\}$, then every vertex of $\bar{N}_G(u)$ is joined to x and y, as otherwise $d_G(x)$ or $d_G(y)$ is less than n. But then, since $n \geq 5$, H contains a pair of independent edges. Consequently, we may assume without loss of generality that G contains the edge xz, $z \in N_G(u) - v - w$. Since $n \geq 5$, y is joined to vertices other than v, w, x and z. Let z′ be any such vertex. If $z' \notin \bar{N}_G(u)$, then $\hat{G} - z - z'$ is bipartite with bipartitioning sets of order $n - 2$ and $n - 4$ and hence does not have a perfect matching. But the subgraph $G[v,w,x,y,z,z']$ has 3 independent edges and these edges must extend to a perfect matching in G. Hence $z' \in \bar{N}_G(u)$. Consequently $|N_G(y) \cap \bar{N}_G(u)| \geq n - 3$, and $|N_G(y) \cap N_G(u)| \leq 3$.

If $|N_G(y) \cap N_G(u)| = 3$, then vw and xz are two independent edges in $G[N_G(y)]$ and so $d_G(y) \geq n + 1$ (Theorem 3.2). Consequently y is joined to every vertex of $\bar{N}_G(u)$ and $\bar{N}_G(u) - y$ is an independent set; otherwise, H contains a pair of independent edges. This establishes that $\bar{N}_G(u) - y$ is an independent set.

We claim that $N_G(u) - v$ or $N_G(u) - w$ is independent. Suppose that this is not the case. Then tv and $tw \in E(G)$ for some

$t \in N_G(u)$. Now consider any vertex $t' \in N_G(u)\setminus\{v,w,t,z\}$; t' exists since $n \ge 5$. Since $G[N_G(u)]$ contains only one independent edge, t' is not adjacent to any vertex in $N_G(u)$ and hence $N_G(t') \subseteq \bar{N}_G(u) \cup \{u\}$. From our earlier discussion we known that t' is not adjacent to y. But then $|N_G(t')| \le n - 1$, a contradiction. Thus at least one of $N_G(u) - v$ or $N_G(u) - w$ is independent. Suppose without any loss of generality that $N_G(u) - v$ is independent.

If $vy \notin E(G)$, then $d_G(y) = n$ and $N_G(y) = \{w,z\} \cup (\bar{N}_G(u)\setminus\{y\})$. Since $N_G(u) - v$ is independent, v is the only vertex of $N_G(u)$ that is adjacent to w. Therefore w is joined to at least $n - 4 \ge 1$ vertices of $\bar{N}_G(u)\setminus\{x,y\}$. Let w' be such a vertex. But now ww' and xz are two independent edges in $G[N_G(y)]$, contradicting Theorem 3.2. Hence $vy \in E(G)$.

We now show that $N_G(v) \cap \bar{N}_G(u) = \{y\}$. Suppose that this is not the case and v is adjacent to the vertex $v' \ne y$ in $\bar{N}_G(u)$. Theorem 3.2 together with the fact that uv and xy are independent edges implies that w is joined to a vertex, w' say, of $\bar{N}_G(u)$ that is different from x, y and v'. If $x \ne v'$, then vv', ww' and xz are three independent edges in G. Further, since $N_G(y) \subseteq \bar{N}_G(u) \cup \{v,w,z\}$ at least two of these independent edges are in $G[N_G(y)]$, contradicting Theorem 3.2. Hence $x = v'$. Now if $vz \in E(G)$, then applying to z the above argument used on w, we establish the existence of the edge zz' with $z' \in \bar{N}_G(u)\setminus\{x,y,w'\}$. Note that if $vz \notin E(G)$, then for $d_G(z) \ge n$ there must still exist such a vertex z'. Now the edges vx, ww' and zz' are independent and at least two are in $G[N_G(y)]$, again contradicting Theorem 3.2. This establishes that $N_G(v) \cap \bar{N}_G(u) = \{y\}$.

Now the graph $G^* = G - u - y$ is bipartite with bipartitioning sets $A = N_G(u) - v$ and $B = \{v\} \cup (\bar{N}_G(u)\backslash\{y\})$. Further $\delta(G^*) \geq n - 2$. By Theorem 3.1, G^* is $(n - 3)$-extendable. But then $G + uy$ is $(n - 2)$-extendable, contradicting the fact that G is $(n - 2)$-critical. This proves that $N_G(u)$ is an independent set. Consequently the neighbour set of every vertex of degree n is an independent set. It thus follows that $G \cong K_{n,n}$. This completes the proof of the theorem. □

Remark 4: When $n = 4$, the graphs in Figure 4.1 having 8 vertices, are 2-critical and all non-bipartite.

Our final result characterizes $(n - 2)$-critical graphs of order 2n.

Theorem 4.8: A graph G on $2n \geq 10$ vertices is $(n - 2)$-critical if and only if $G \cong K_{n,n}$ or K_{2n}.

Proof: Again we need only consider the necessity part. Suppose G is an $(n - 2)$-critical graph on $2n \geq 10$ vertices and $G \ncong K_{2n}$ and $K_{n,n}$. Then $n < 2(n - 2)$ and so, by theorems 2.4(b) and 4.3, we have $n - 1 \leq \delta(G) \leq n$. But now, by Lemma 4.4, $\delta(G) = n$ and so, by Theorem 4.7, $G \cong K_{n,n}$. This completes the proof of the theorem. □

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