# A sharp lower bound
# for locating-dominating sets in trees

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## Abstract

Let $LD(G)$ denote the minimum cardinality of a locating-dominating set for graph $G$. If $T$ is a tree of order $n$ with $l$ leaf vertices and $s$ support vertices, then a known lower bound of Blidia, Chellali, Maffray, Moncel and Semri [*Australas. J. Combin.* **39** (2007), 219–232] is $LD(T) \geq \lceil (n+1+l-s)/3 \rceil$. In this paper, we show that $LD(T) \geq \lceil (n+1+2(l-s))/3 \rceil$ and these bounds are sharp. We constructively characterize the trees achieving the lower bounds.

## 1    Introduction

For various safeguard applications that use graphical models of facilities or multiprocessor networks, various types of protection sets have been studied where the objective is to precisely locate an "intruder." Examples include identifying a malfunctioning processor in a multiprocessor network and a thief, saboteur or fire in a facility network. Being concerned with determining an intruder location in a network system, one is interested in dominating sets. For a graph $G = (V, E)$ of order $n = |V(G)|$, the open neighborhood of a vertex $v$ is $N(v) = \{x \in V(G) : vx \in E(G)\}$, and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of vertex $v$ is $deg(v) = |N(v)|$. Vertex set $D \subseteq V(G)$ is dominating if $\cup_{v \in D} N[v] = V(G)$, that is, every $x \in V(G)$ is in $D$ or is adjacent to at least one vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. If $D$ is a dominating set with $x \in D$ and $N[w] \cap D = \{x\}$, then $x$ is called a *sole dominator* of $w$ and $w$ is called a *private neighbor* of $x$. See Haynes, Hedetniemi and Slater [2, 3] for an extensive treatment of domination theory.

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As introduced in Slater [12, 13], we consider that a detection device at a vertex $v$ is able to determine if the intruder is at $v$ or if it is in $N(v)$, but which vertex in $N(v)$ cannot be determined. A locating-dominating set $L \subseteq V(G)$ (an LD-set $L$) is a dominating set with the property that for each vertex $x \in V(G) \setminus L$ the set $N(x) \cap L$ is unique. That is, any two vertices $x, y$ in $V(G) \setminus L$ are *distinguished* in the sense that there is a vertex $v \in L$ with $|N(v) \cap \{x, y\}| = 1$. The minimum size of a locating-dominating set for a graph G is the locating-dominating number of $G$, denoted $LD(G)$. A locating-dominating set $L \subseteq V(G)$ with $|L| = LD(G)$ is called an $LD(G)$-set. We interpret the set $L$ as the set of locations at which the detection devices are placed. Note that if $L$ is an $LD$-set with $x \in L$, then $x$ can be its own private neighbor and $x$ can be the sole dominator of at most one vertex in $N(x)$.

As an example, for the path $P_n : v_1, v_2, \ldots, v_n$ a $\gamma(P_n)$-set is $D = \{v_2, v_5, v_8, \ldots\}$ and $\gamma(P_n) = \lceil n/3 \rceil$. Note, for example, that $v_4, v_6 \in V(P_n) \setminus D$ and $N(v_4) \cap D = \{v_5\} = N(v_6) \cap D$. In fact, as in Slater [13], $LD(P_n) = \lceil 2n/5 \rceil$, and an $LD(P_n)$-set using two fifths of the vertices is $L = \{v_2, v_4, v_7, v_9, v_{12}, v_{14}, \ldots\}$.

There are trees $T$ with $LD(T)$-sets containing less than 40% of the vertices. For tree $T_n$ of order $n = 3k + 3$ in Figure 1, we have $LD(T_n) = k + 2$ with $LD(T_n)$-set $\{x_1, x_2, \ldots, x_k, v, w\}$. Also, $\{x_1, x_2, \ldots, x_k, v\}$ is an LD-set for both $T_n - w$ and $T_n - \{w, u\}$, with $LD(T_n - w) = LD(T_n - \{w, u\}) = k + 1$. These examples show the lower bound in the next theorem is sharp, where it is shown that any locating-dominating set for any tree must contain more than one-third of the vertices.

**Theorem 1** (Slater [12]). *For any tree $T$ of order $n \geq 2$, $\lceil (n + 1)/3 \rceil \leq LD(T) \leq n - 1$.*                                                                                                                      □

We note that other types of distinguishing sets include identifying codes as introduced in Karpovsky, Chakrabarty, and Levitin [6] and open-locating-dominating sets as described for the $n$-cubes in Honkala, Laihonen and Ranto [5] and defined for general graphs in Seo and Slater [10, 11]. See also, for example, [4, 8, 9, 10, 14]. A bibliography for papers about distinguishing sets, currently with over 250 citations, is maintained by Lobstein [7].

In Section 2, we demonstrate that for any $n$ and any $k$ within the bounds of Theorem 1 (that is, $\lceil (n + 1)/3 \rceil \leq k \leq n - 1$) there is a tree $T$ of order $n$ satisfying $LD(T) = k$ and give an explicit family of interpolating trees based on this result. We also improve the lower bounds for $LD(T)$ of Blidia, Chellali, Maffray, Moncel and Semri [1], stated in Theorem 3, and show our bound, stated in Theorem 5, to be sharp. For this lower bound, consider tree $T_{3k+2} = T_{3k+3} - w$ in Figure 1. Note that $v$ and $x_1$ adequately locate $u, v, y_1, x_1$ and $z_1$. Thereafter, each $\{y_t, x_t, z_t\}$ forms a $P_3$ attached to $v$. For $n = 3k + 1$, the lower bound is achieved for tree $T_{3k+1} = T_{3k+2} - u$. Blidia, et al. [1], present another family of trees achieving the absolute lower bound of $\lceil (n + 1)/3 \rceil$, namely, the caterpillars in Figure 2. Both families of trees illustrate the following observation.
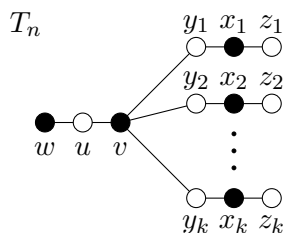
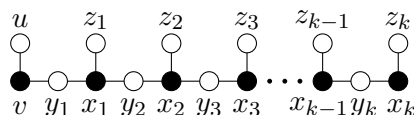Figure 1: Tree $T_n$ with $|V(T_n)| = n$ achieves the lower bound of $LD(T_n) \geq \lceil (n+1)/3 \rceil$.



Figure 2: Caterpillar $C_{3k+2}$.

**Observation 2** *If $G$ is a graph with $LD(G) = t$ containing an $LD(G)$-set $L$ with $v \in L$ and $G'$ is the graph obtained from $G$ by adding a $P_3 : y, x, z$ and the edge $yv$, then $LD(G') = LD(G) + 1$ and $L \cup \{x\}$ is an $LD(G')$-set.* □

In Sections 3 and 4 we use operations of this type to constructively characterize the trees satisfying the improved lower bounds in Theorem 5.

## 2   Bounds for $LD(T)$

The bounds for $LD(T)$ in Theorem 1 can be improved if, for example, one considers the leaf/support structure of the tree. In a tree $T$, a vertex of degree one is called an *endpoint* or a *leaf*; each vertex adjacent to a leaf vertex is called a *support vertex*; and a support vertex adjacent to at least two leaves is a *strong support vertex*. As in Blidia, et al. [1], we let $l(T)$ and $s(T)$, respectively, denote the number of leaf vertices and support vertices.

**Theorem 3** (Blidia et al. [1]). *For a tree $T$ of order $n \geq 3$,*

$$\lceil (n + 1 + l(T) - s(T))/3 \rceil \leq LD(T) \leq (n + l(T) - s(T))/2.$$

□

Further, Blidia et al. [1] constructively characterize the trees satisfying the upper bound $LD(T) = (n + l(T) - s(T))/2$ for a tree $T$ of order $n$. Note that the trees

in Figures 1 and 2 satisfy $l(T) = s(T)$, that is, they have the same number of leaf and support vertices. The lower bound of Theorem 3 is sharp when $l(T) = s(T)$, but it can be improved for $l(T) > s(T)$, that is, when there is at least one strong support vertex $v$. Specifically, assume $v$ is a support vertex adjacent to leaves $w$ and $x$. Clearly, any $LD$-set $L$ must contain at least one of $x$ and $w$. We can say something much stronger for any graph with two leaves having a common neighbor.

**Lemma 4** *Assume $v$ is a (support) vertex adjacent to leaves $w$ and $x$ in graph $G$. Then $LD(G) = LD(G - w) + 1$.*

*Proof.* Let $L_1$ be an $LD(G - w)$-set. Let $L_2 = L_1 \cup \{w\}$. Given any two vertices $y$ and $z$ in $V(G) - L_2 = V(G - w) - L_1$, some vertex $s \in L_1$ has $|N(s) \cap \{y, z\}| = 1$. Trivially, $L_2$ is an $LD$-set for $G$, and $LD(G) \leq LD(G - w) + 1$. Now, start with an $LD(G)$-set $L_2$. Without loss of generality, we can assume $w \in L_2$. If $v \in L_2$, then each vertex $y \in V(G) - L_2$ has $N(y) \cap L_2 = N(y) \cap (L_2 - w)$. It follows that $L_2 - w$ is an $LD$-set for $G - w$. If $v$ is not in $L_2$, then likewise given any two vertices $y$ and $z$ in $V(G) - (L_2 \cup \{v\})$, some vertex $s \in L_2 - w$ has $|N(s) \cap \{y, z\}| = 1$. Also, $v$ is not in $L_2$ implies $x \in L_2$ in order for $x$ to be dominated, and then $x$ distinguishes $v$ from any other vertex $y$ not in $L_2$. Again, it follows that $L_2 - w$ is an $LD$-set for $G - w$. Hence, $LD(G - w) \leq LD(G) - 1$. $\square$

Now given a tree $T$ of order $n$, we can successively remove a leaf that has a common neighbor with another leaf and do this $l(T) - s(T)$ times to produce a tree $T'$ of order $n - (l(T) - s(T))$ with $l(T') = s(T')$. By Lemma 4, we have $LD(T) = LD(T') + l(T) - s(T)$, and by Theorem 1 we have $LD(T') \geq \lceil (|V(T')| + 1)/3 \rceil$. Hence, $LD(T) = LD(T') + l(T) - s(T) \geq (n - (l(T) - s(T)) + 1)/3 + l(T) - s(T) = (n + 2(l(T) - s(T)) + 1)/3$, proving the following theorem that strengthens Theorem 3.

**Theorem 5** *If $T$ is a tree of order $n \geq 3$, then*

$$LD(T) \geq \lceil (n + 2(l(T) - s(T)) + 1)/3 \rceil.$$

$\square$

Showing this lower bound is, indeed, best possible, in Sections 3 and 4 we present a constructive characterization of the trees $T$ of order $n$ achieving $LD(T) = (n + 1 + 2(l(T) - s(T)))/3$. First, we observe that any value of $LD(T)$ within the bounds $\lceil (n + 1)/3 \rceil \leq LD(T) \leq n - 1$ is achievable by some tree of order $n$. To see this, consider any graph $G$ of order $n$ with leaf $v$. If $uv \in E(G)$ (that is, $u$ is the support vertex of $v$) and $w \in V(G) \setminus \{u, v\}$, then the graph $G'$ derived from $G$ by removing the edge $uv$ and adding the edge $vw$ corresponds to moving a leaf of $G$ from one support vertex, $u$, to any vertex $w$. The following lemma bounds the amount $LD(G)$ can differ from $LD(G')$.

**Lemma 6** *If $G$ is a graph of order $n$ with leaf $v$, then $G'$, as defined above, satisfies $|LD(G) - LD(G')| \leq 1$.*

*Proof.* Let $L \subseteq V(G)$ be an $LD(G)$-set.

**Case 1.** Suppose $v \notin L$. With respect to both $G$ and $G'$, $L$ locates all vertices in $V(G) \setminus \{v\} = V(G') \setminus \{v\}$. Hence, $L \cup \{v\}$ locates all vertices with respect to $G'$; whence, $LD(G') \leq LD(G) + 1$.

**Case 2.** Suppose $v \in L$. If $u \notin L$, define $L' = (L \setminus \{v\}) \cup \{u\}$. If $u \in L$, then $u$ must be the sole dominator of a vertex $x \in V(G) \setminus L$. Otherwise, $L \setminus \{v\}$ is a locating-dominating set of $G$ which contradicts that $L$ is an $LD(G)$-set. In this case, define $L' = (L \setminus \{v\}) \cup \{x\}$. $L'$ is now a locating-dominating set of $G$. $LD(G') \leq LD(G) + 1$ follows by using $L'$ with case 1.

Since the roles of $G$ and $G'$ may be interchanged, we have the desired result. □

Consider the tree $T_n$ achieving the lower bound $LD(T_n) = \lceil (n+1)/3 \rceil$ in Figure 1. Now iteratively apply Lemma 6 to $T_n$, moving each leaf $z_i$, $x_i$, one at a time, to the common support vertex. The final result is a star, which achieves the upper bound $LD(K_{1,n-1}) = n - 1$. We use this method to prove the following theorem.

**Theorem 7** *For $n \geq 2$, if $\lceil (n+1)/3 \rceil \leq k \leq n - 1$, then there exists a tree $T$ of order $n$ with $LD(T) = k$.*

*Proof.* For $n = 2, 3$ the result is immediate. For larger $n$, consider $T_{3k+3}$ from Figure 1. Let $T_n = T_{3k+3}$, $T_{3k+3} - w$, or $T_{3k+3} - w - u$ as appropriate. Define $T^0 = T_n$, and iteratively form $T^{i+1}$ from $T^i$ as follows. Select a vertex $t \in V(T^i)$ with $\deg(t) = 1$ adjacent to vertex $s \neq v$. Define $T^{i+1} = T^i - st + vt$. Note that $T^i$ is well defined for $i = 0, 1, \ldots, I$, where $I = n - |N_{T_n}[v]|$. For $n = 3k + 3$ we have $I = 2k + 1$, and for $n = 3k + 2, 3k + 1$ we have $I = 2k$. For all $n$, the difference between upper and lower bounds for $LD(T_n)$ is $(n - 1) - \lceil (n+1)/3 \rceil \leq I$. Since, by Lemma 6, $|LD(T^i) - LD(T^{i+1})| \leq 1$, this shows that for each $k$ from $\lceil (n+1)/3 \rceil$ to $n - 1$, there is some $i$ from 0 to $I$ satisfying $LD(T^i) = k$. □

Based on this above procedure, we now show an explicit family of trees that interpolate the locating-dominating bounds for trees of order $n$. Let $n \geq 2$ and $k$ be between $\lceil (n+1)/3 \rceil$ and $n - 1$. We will construct tree $T$ of order $n$ and $LD(T)$-set $L$ with $LD(T) = k$. Begin with vertices $u, v \in V(T)$ and edge $uv \in E(T)$. For each $i$ from 1 to $\lfloor (n - k - 1)/2 \rfloor$, add $P_3 = w_i - x_i - y_i$ and edge $uw_i$. If 2 does not divide $n - k - 1$, then also add vertices $w_0, x_0$ and edges $uw_0, w_0x_0$. For each $j$ from 1 to $k - \lfloor (n - k - 1)/2 \rfloor - 1$, add vertex $z_j$ and edge $uz_j$. Finally, $LD(T)$-set $L$ consists of vertices $u$, the $x_i$'s (including $x_0$ if it is in $V(T)$) and the $z_j$'s. Figure 3(a) shows the general form of this construction, where $I = \lfloor (n - k - 1)/2 \rfloor$, $J = k - \lfloor (n - k - 1)/2 \rfloor - 1$, vertices $w_0$ and $x_0$ are included if and only if 2 does not divide $n - k - 1$, and $L$ is indicated by darkened vertices. For $n = 8$, this construction yields the trees in Figure 3(b)–(f) with $k$ ranging from $\lceil (n+1)/3 \rceil = 3$ to $n - 1 = 7$, respectively.
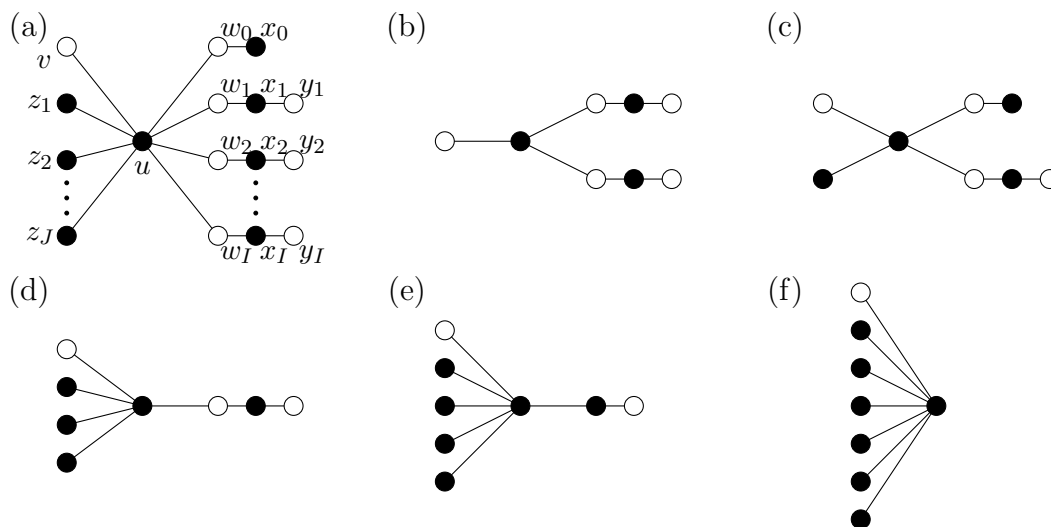
Figure 3: (a) Tree $T_n^{(k)}$ of order $n$ with $LD(T_n^{(k)}) = k$. (b)–(f) Trees $T_8^{(3)}$ through $T_8^{(7)}$, respectively.

## 3   Trees Achieving $LD(T) = \lceil (n+1)/3 \rceil$

In this section, we are interested in both the structure of trees that achieve the lower bound of Theorem 1 and the locating-dominating sets of these trees. Defining $LDT(n) = \lceil (n+1)/3 \rceil$, we say the ordered pair $(T, L)$ achieves the Theorem 1 lower bound when $T$ is a tree of order $n$ and $L$ is an $LD(T)$-set with $|L| = LDT(n)$. First, we examine some special restrictions on the structure of such ordered pairs. Then, we present five extension operations (including the one from Observation 2) that each preserve the property of achieving the Theorem 1 lower bound. That is, applying an extension operation to an ordered pair $(T, L)$ that achieves the Theorem 1 lower bound produces an ordered pair $(T^\sharp, L^\sharp)$ that achieves the same Theorem 1 lower bound with $T$ a strict subgraph of $T^\sharp$ and $L$ a subset of $L^\sharp$. In Theorem 10, we show these operations are exhaustive. That is, given the tree $T_1$ consisting of the single vertex $v$, any ordered pair $(T, L)$ that achieves the Theorem 1 lower bound can be constructed from $(T_1, \{v\})$ via successive applications of the extension operations. Using Lemma 4, this work is extended in Section 4 to a constructive characterization of trees achieving the Theorem 5 bound $LD(T) = \lceil (n + 1 + 2(l(T) - s(T)))/3 \rceil$ with no restriction on the relation of $l(T)$ to $s(T)$.

Consider an ordered pair $(T, L)$ that achieves the Theorem 1 lower bound where $T$ is of order $n = 3k+2$. In this case, $LDT(n+1) = LDT(n)+1$. Thus, given any tree $T^\sharp$ formed from $T$ by adding a single vertex $v$, more than $|L|$ vertices are required for an $LD(T^\sharp)$-set $L^\sharp$. If $u \in L$ has no private neighbor, then tree $T^\sharp$ formed from $T$ by adding vertex $v$ and edge $uv$ is locating-dominated by $L^\sharp = L$. This contradiction leads to the following proposition.

**Proposition 8** *If $T$ is a tree of order $n = 3k + 2$ and $L \subseteq V(T)$ is an $LD(T)$-set of size $\lceil (n+1)/3 \rceil$, then every vertex in $L$ has a private neighbor in $V(T) \setminus L$.* $\square$

For pair $(T, L)$, above, notice that $n \equiv 2 \pmod 3$. If $l(T) > s(T)$, then $LD(T) = LDT(n) < \lceil (n + 2(l(T) - s(T)) + 1)/3 \rceil$. By Theorem 5 this implies $l(T) = s(T)$. Also, for pair $(T^\sharp, L^\sharp)$, above, the Theorem 5 bound is the same for $l(T^\sharp) = s(T^\sharp)$ as for $l(T^\sharp) = s(T^\sharp) + 1$. That is, $\lceil ((n+1) + 2 \cdot 0 + 1)/3 \rceil = \lceil ((n+1) + 2 \cdot 1 + 1)/3 \rceil$. Indeed, if vertex $u$ is a support vertex of $T$, then it is a strong support vertex of $T^\sharp$.

The following lemma is used in the proof of Theorem 10 and specifies when a Theorem 1 lower-bound-achieving tree can have a strong support vertex. We show that no such tree can have more than one strong support vertex. To see a family of examples of this type, let $n \geq 3$ with $n \equiv 0 \pmod 3$ and $k = \lceil (n+1)/3 \rceil$. Then the tree $T_n$ constructed according to Figure 3(a) is a Theorem 1 lower-bound-achieving tree with a strong support vertex.

**Lemma 9** *If $T$ is a tree of order $n$ and $L \subseteq V(T)$ is an $LD(T)$-set of size $\lceil (n+1)/3 \rceil$, then $l(T) - s(T) \leq 1$. Moreover, $l(T) - s(T) = 1$ implies $n \equiv 0 \pmod 3$.*

*Proof.* First, we show $T$ has at most one strong support vertex. Suppose distinct vertices $v_1, v_2 \in V(T)$ are strong support vertices with leaves $w_1$ and $x_1$ adjacent to $v_1$ and leaves $w_2$ and $x_2$ adjacent to $v_2$. Let $i$ be 1 or 2. We cannot have $v_i$, $w_i$, and $x_i$ in $L$. Otherwise, we can form tree $T'$ from $T$ by removing edge $v_i x_i$ and adding edge $w_i x_i$. Then $L' = L \setminus \{x_i\}$ is an LD-set of $T'$ with $|L'| < |L| = LDT(n)$, a contradiction. If $v_i \notin L$, then $(L \setminus \{w_i\}) \cup \{v_i\}$ is an LD-set of $T$. Thus, we can assume $v_1, x_1, v_2, x_2 \in L$ and $w_1, w_2 \notin L$.

Since $L$ is an LD-set and $v_1$ sole dominates $w_1$, we cannot have $v_1$ sole dominate any other vertex in $N(v_1)$ – and likewise for $v_2$, $w_2$, and $x_2$. Form tree $T'$ from $T$ by removing edge $v_2 w_2$ and adding edge $x_1 w_2$. Then $T'$ also has order $n$ and $L' = L$ is an LD-set of $T'$. Notice in $T'$ vertex $x_1$ sole dominates $w_2$ and $v_2$, which sole dominates $w_2$ in $T$, now does not have a private neighbor. Thus, $L'' = L' \setminus \{x_2\}$ is also an LD-set of $T'$. But $|L''| < |L'| = |L| = LDT(n)$, a contradiction.

Now we use a similar technique to show a strong support vertex of $T$, if it exists, has at most two adjacent leaves. Suppose $v \in V(T)$ is a strong support vertex with adjacent leaves $w, x, y \in V(T)$. As above, we can assume, without loss of generality, that $v, w, x \in L$ and $y \notin L$. Form tree $T'$ from $T$ by removing edges $vw$ and $vx$ and adding edges $yw$ and $wx$. Then $T'$ also has order $n$ and $L' = L \setminus \{x\}$ is an LD-set of $T'$. But $|L'| < |L| = \lceil (n+1)/3 \rceil = LDT(n)$, a contradiction.

To see that $l(T) - s(T) = 1$ implies $n \equiv 0 \pmod 3$, suppose $v \in V(T)$ is a strong support vertex with adjacent leaves $w, x \in V(T)$. Without loss of generality, assume $v, w \in L$ and $x \notin L$. Since $L$ is an LD-set and $v$ sole dominates $x$, it follows that $v$ cannot sole dominate any other vertex in $N(v)$. Form tree $T'$ from $T$ by removing $w$. Then $T'$ has order $n - 1$ and $L' = L \setminus \{w\}$ is an LD-set of $T'$ of size $\lceil (n+1)/3 \rceil - 1$. If $n \not\equiv 0 \pmod 3$, then $|L'| = \lceil (n+1)/3 \rceil - 1 = \lceil ((n-1) + 1)/3 \rceil - 1 = LDT(n-1) - 1$,

a contradiction. If, however, $n \equiv 0 \pmod 3$, then there is no contradiction since $|L'| = \lceil(n+1)/3\rceil - 1 = \lceil((n-1)+1)/3\rceil = LDT(n-1)$.                    $\square$

As noted, Figures 1 and 2 demonstrate infinite families of trees achieving the lower bound $LD(T) = \lceil(n+1)/3\rceil$ by iteratively adding $P_3$'s per Observation 2. This can be applied more generally as a family of lower bound preserving extension operations. This notion will be made precise in Theorem 10, below. First, we present the following extension operations. Note that Operation O5 corresponds to adding a $P_3$ as discussed above.

Let $T$ be a tree of order $n = |V(T)|$ with $LD(T)$-set $L$ of size $|L| = LDT(n) = \lceil(n+1)/3\rceil$; that is, $(T, L)$ is a Theorem 1 lower-bound-achieving ordered pair. The five operations below are presented with the conditions under which they can be applied and the justification that the operation is bound preserving. In each case, $(T^\sharp, L^\sharp)$ is the Theorem 1 lower-bound-achieving ordered pair that results from extending $(T, L)$.

**O1**   **Condition.** $n \equiv 2 \pmod 3$ and $u \in V(T)$.
   **Extension.** $V(T^\sharp) = V(T) + v$. $E(T^\sharp) = E(T) + uv$. $L^\sharp = L + v$.
   **Justification.** Since $n \equiv 2 \pmod 3$, any increase in $n$ also increases $\lceil(n+1)/3\rceil$. So, introducing $v$ as a vertex of $T^\sharp$ permits including $v$ in $L^\sharp$ while maintaining the lower bound. That is, $|L^\sharp| = |L| + 1 = LDT(n) + 1 = LDT(n+1)$. Also, $L$ distinguishes $V(T^\sharp) \setminus \{v\}$ in $T^\sharp$. Thus, $L^\sharp$ distinguishes $V(T^\sharp)$.

**O2**   **Condition.** $n \not\equiv 2 \pmod 3$, $u \in L$, and $u$ does not have a private neighbor in $V(T) \setminus L$.
   **Extension.** $V(T^\sharp) = V(T) + v$. $E(T^\sharp) = E(T) + uv$. $L^\sharp = L$.
   **Justification.** Strictly speaking, the condition $n \not\equiv 2 \pmod 3$ is superfluous, since Proposition 8 implies this is a consequence of the remaining conditions. Because $u$ does not have a private neighbor in $V(T) \setminus L$, $L$ is itself a locating-dominating set of $T^\sharp$. Also, since $n \not\equiv 2 \pmod 3$, we have $|L^\sharp| = |L| = LDT(n) = LDT(n+1)$. This shows $L^\sharp$ is an $LD(T^\sharp)$-set.

**O3**   **Condition.** $n \not\equiv 0 \pmod 3$ and $u \in V(T)$.
   **Extension.** $V(T^\sharp) = V(T) + v + w$. $E(T^\sharp) = E(T) + uv + vw$. $L^\sharp = L + v$ or $L^\sharp = L + w$.
   **Justification.** Since $n \not\equiv 0 \pmod 3$, an increase in $n$ by 2 increases $\lceil(n+1)/3\rceil$ by 1. So, introducing $v$ and $w$ as vertices of $T^\sharp$ permits including $v$ or $w$ in $L^\sharp$ while maintaining the lower bound. That is, $|L^\sharp| = |L| + 1 = LDT(n) + 1 = LDT(n+2)$. To see that $L^\sharp$ is an $LD(T^\sharp)$-set, first note that $L$ distinguishes $V(T^\sharp) \setminus \{v, w\}$ in $T^\sharp$. Suppose $v$ is added to $L^\sharp$. Then $v$ distinguishes $w$ from all other vertices in $V(T^\sharp) \setminus L^\sharp$ except, perhaps, $u$. If $u \notin L$, then $\exists x \in N(u) \cap L$. In this case, $x$ distinguishes $w$ from $u$. If $w$ is added to $L^\sharp$ instead of $v$, the argument is similar.

**O4**   **Condition.** $v \in L$, $N(v) = \{u\}$.
   **Extension.** $V(T^\sharp) = V(T) + w + x + y$. $E(T^\sharp) = E(T) + vw + ux + xy$.

$L^\sharp = L + x$.

**Justification.** An increase in $n$ by 3 increases $\lceil (n+1)/3 \rceil$ by 1. So, introducing $w$, $x$, and $y$ as vertices of $T^\sharp$ permits including $x$ in $L^\sharp$ while maintaining the lower bound. That is, $\left| L^\sharp \right| = |L| + 1 = LDT(n) + 1 = LDT(n+3)$. To see that $L^\sharp$ is an $LD(T^\sharp)$-set, first note that $L$ distinguishes all vertices in $V(T^\sharp)$ except $w$, $x$, $y$, and perhaps $u$. If $u$ is not a private neighbor of $v$ in $T$, then $L$ distinguishes $u$ in $T^\sharp$. If $u$ is a private neighbor of $v$ in $T$, then $u$ is not distinguished by $L$ in $T^\sharp$ because $N(u) \cap L = \{v\} = N(w) \cap L$. However, $N(u) \cap L^\sharp = \{v, x\}$, which is unique. This leaves $w$ as the private neighbor of $v$ and $y$ as the private neighbor of $x$. Since $x \in L^\sharp$, this shows $L^\sharp$ is a locating-dominating set of $T^\sharp$.

**O5**  **Condition.** $u \in L$.
  **Extension.** $V(T^\sharp) = V(T) + v + w + x$. $E(T^\sharp) = E(T) + uv + vw + wx$. $L^\sharp = L + w$.
  **Justification.** Following the justification of O4, $\left| L^\sharp \right| = |L| + 1 = LDT(n) + 1 = LDT(n+3)$. Since $N(v) \cap L^\sharp = \{u, w\}$ is unique and $x$ is a private neighbor of $w$, we have $L^\sharp$ is an $LD(T^\sharp)$-set.

Notice that Operations O1 and O2 extend $T$ by one vertex, Operation O3 extends $T$ by two vertices, and Operations O4 and O5 extend $T$ by three vertices. As an example of using these operations, Figure 4 shows two constructions of the same bound achieving ordered pair $(T, L)$.

We now precisely state and prove that the five extension operations O1 through O5 are exhaustive when starting from the Theorem 1 lower-bound-achieving ordered pair $(T_1, \{v\})$, where $T_1$ is the tree consisting of the single vertex $v$.

**Theorem 10** *Let $T$ be a tree of order $n$ and $L \subset V(T)$ an $LD(T)$-set. Then $L$ is an $LD(T)$-set of order $\lceil (n+1)/3 \rceil$ if and only if the ordered pair $(T, L)$ can be constructed from $(T_1, \{v\})$ using a sequence of extensions, where each member of the sequence is one of the five extensions O1 through O5.*

*Proof.*  Let $T$ be a tree of order $n$ with $LD(T) = LDT(n) = \lceil (n+1)/3 \rceil$; let $L \subset V(T)$ be an $LD(T)$-set; and let $z \in V(T)$ be a leaf of a diametric path $P$ of $T$ with $N(z) = \{y\}$. In what follows, we describe an iterative decomposition approach. Each iteration describes how $T$ can be reduced to a subtree $T^\flat \subset T$ and $L$ can be reduced to a subset $L^\flat \subseteq L$, where $L^\flat$ is an $LD(T^\flat)$-set with $LD(T^\flat) = LDT(\left| V(T^\flat) \right|)$. Also, the original tree $T$ and $LD(T)$-set $L$ can be recovered from $T^\flat$ and $L^\flat$ with one of the operations O1 through O5. The reduction chosen depends on the structure of the tree near leaf $z$. In the cases below, we consider these structures one at a time. In general, though, we do not have to apply the chosen reduction (or recovering extension operation) at or near a leaf of a diametric path.

**Case 1: $\deg(\mathbf{y}) = 1$.** $T$ is the path on two vertices. Without loss of generality, assume $y \in L$. Define $T^\flat = T - z$ and $L^\flat = L$. Then $T^\flat$ is the single vertex tree, and $L^\flat = V(T^\flat)$. Clearly, $T^\flat$ satisfies $LD(T^\flat) = LDT(\left| V(T^\flat) \right|) = 1$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O2.
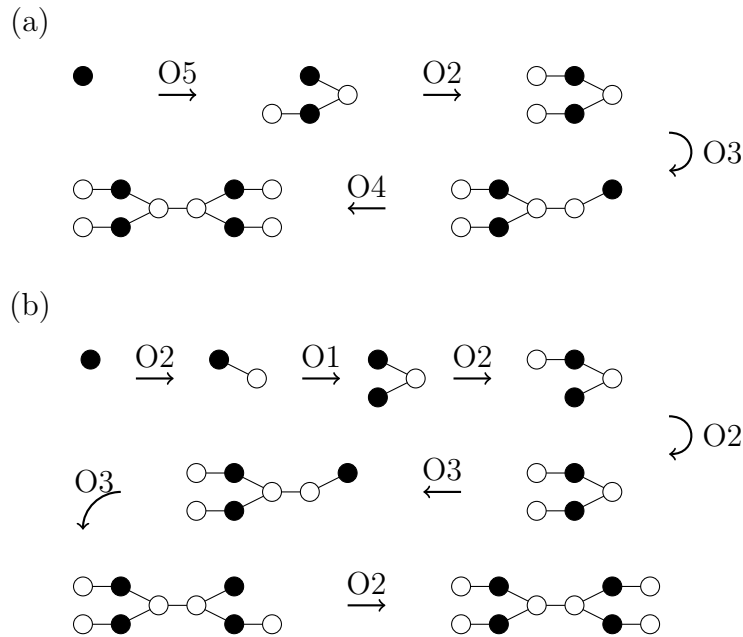
(a)



(b)



Figure 4: Two constructions for the same tree $T$ with the same locating-dominating set $L$. The transition arrows are labeled with the operation used. Construction (a) uses Operations O2 through O5 once each. Construction (b) uses Operation O1 once and Operations O2 and O3 multiple times.

**Case 2: deg(y) > 2.** Since $P$ is diametric, at most one neighbor of $y$ is not a leaf of $T$. By Lemma 9, at most two neighbors of $y$ are leaves. So, $\deg(y) = 3$ and, also by Lemma 9, $n \equiv 0 \pmod 3$. Let $z_2 \in N(y)$ be the leaf distinct from $z$. Without loss of generality, assume $z_2 \in L$. Define $T^\flat = T - z_2$ and $L^\flat = L - z_2$. Then $T^\flat$ is a tree of order $n - 1 \equiv 2 \pmod 3$ with locating-dominating set $L^\flat$. Since $\left|L^\flat\right| = LDT(n-1) = LDT(n) - 1$ we have $LD(T^\flat) = LDT(\left|V(T^\flat)\right|)$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O1.

**Case 3: deg(y) = 2 and y ∉ L.** Since $L$ dominates $z$ and $y \notin L$, $z \in L$. Define $T^\flat = T - z - y$ and $L^\flat = L - z$. Since $L^\flat$ is a locating-dominating set of $T^\flat$ and $\left|L^\flat\right| < |L|$ we cannot have $LDT(\left|V(T^\flat)\right|) = LDT(n-2) = LDT(n)$. Thus, $n \not\equiv 2 \pmod 3$ so that $n - 2 \not\equiv 0 \pmod 3$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O3.

**Case 4: deg(y) = 2, and y, z ∈ L.** Define $T^\flat = T - z$ and $L^\flat = L \setminus \{z\}$. Since $L^\flat$ is a locating-dominating set of $T^\flat$ and $\left|L^\flat\right| < |L|$ we cannot have $LDT(\left|V(T^\flat)\right|) = LDT(n-1) = LDT(n)$. Thus, $n-1 \equiv 2 \pmod 3$ so that $LDT(n) = LDT(n-1)+1$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O1.

For the remaining cases, let $x \in N(y)$ be distinct from $z$.

**Case 5: deg(y) = 2, x, y ∈ L, and z ∉ L.** Define $T^\flat = T - z - y$ and $L^\flat =$

$L - y$. Since $L^\flat$ is a locating-dominating set of $T^\flat$ and $\left|L^\flat\right| < |L|$ we cannot have $LDT(\left|V(T^\flat)\right|) = LDT(n - 2) = LDT(n)$. Thus, $n \not\equiv 2 \pmod 3$ so that $n - 2 \not\equiv 0 \pmod 3$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O3.

**Case 6: deg(y) = 2, y $\in$ L, x, z $\notin$ L, and deg(x) $\leq$ 2.** Define $T^\flat = T - z - y - x$ and $L^\flat = L - y$. Since $L^\flat$ is a locating-dominating set of $T^\flat$ with $\left|L^\flat\right| = LDT(n-3) = LDT(n) - 1$ we have $LD(T^\flat) = LDT(\left|V(T^\flat)\right|)$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O5.

**Case 7: deg(y) = 2, y $\in$ L, x, z $\notin$ L, deg(x) > 2, and x does not have a sole dominator in L $\setminus$ {y}.** Note that the last condition does not consider $y$ as a dominator of $x$. Define $T^\flat = T - z - y$ and $L^\flat = L - y$. Then, $L^\flat$ is a locating-dominating set of $T^\flat$ with $\left|L^\flat\right| < |L|$. Hence, we cannot have $LDT(\left|V(T^\flat)\right|) = LDT(n - 2) = LDT(n)$; whence, $n \not\equiv 2 \pmod 3$ so that $n - 2 \not\equiv 0 \pmod 3$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O3.

**Case 8: deg(y) = 2, y $\in$ L, x, z $\notin$ L, deg(x) > 2, and x has a sole dominator in L $\setminus$ {y}.** Note that the last condition does not consider $y$ as a dominator of $x$. Let $y_2$ be a neighbor of $x$ not on diametric path $P$. We consider three subcases based on the degree of $y_2$.

**Case 8.1: deg($y_2$) = 1.** Since $x \notin L$, we must have $y_2 \in L$ for $y_2$ to be dominated. Define $T^\flat = T - z - y$ and $L^\flat = L - y$. Since $y_2$ sole dominates only $x$ in $T^\flat$, it follows that $L^\flat$ is a locating-dominating set of $T^\flat$ with $\left|L^\flat\right| < |L|$. Hence, we cannot have $LDT(\left|V(T^\flat)\right|) = LDT(n - 2) = LDT(n)$; whence, $n \not\equiv 2 \pmod 3$ so that $n - 2 \not\equiv 0 \pmod 3$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O3.

**Case 8.2: deg($y_2$) = 2.** Let $z_2 \in N(y_2)$ be distinct from $x$. Since $P$ is diametric, $z_2$ must be a leaf of $T$. To dominate $z_2$, at least one of $z_2$ and $y_2$ must be in $L$. Suppose $y_2, z_2 \in L$. Then $L' = L - z_2$ is a locating-dominating set of $T$ with $|L'| < |L| = LDT(n)$, a contradiction.

Suppose $z_2 \in L$ and $y_2 \notin L$. Define $T^\flat = T - z_2 - y_2$ and $L^\flat = L - z_2$. Then $L^\flat$ is a locating-dominating set of $T^\flat$ with $\left|L^\flat\right| < |L|$. Hence, we cannot have $LDT(\left|V(T^\flat)\right|) = LDT(n - 2) = LDT(n)$; whence, $n \not\equiv 2 \pmod 3$ so that $n - 2 \not\equiv 0 \pmod 3$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O3.

Now suppose $y_2 \in L$ and $z_2 \notin L$. Define $T^\flat = T - z - z_2 - y_2$ and $L^\flat = L - y_2$. Then $L^\flat$ is a locating-dominating set of $T^\flat$ with $\left|L^\flat\right| = |L| - 1 = LDT(n) - 1 = LDT(n - 3) = LDT(\left|V(T^\flat)\right|)$. Now $T$ and $L$ can be recovered from $T^\flat$ and $L^\flat$ by applying Operation O4.

**Case 8.3: deg($y_2$) > 2.** Let $z_2 \in N(y_2)$ be a leaf. Define a new diametric path $P' = \ldots, x, y_2, z_2$ and apply Case 2 to $T$ and $L$ using $P'$ instead of $P$.

This completes our examination of possible structures near leaf $z$. In each case a reduction is chosen that can be reversed via one of the extension operations O1 through O5. Starting with any tree $T$ and $LD(T)$-set $L$ with $LD(T) = LDT(|V(T)|)$, iteratively apply a reduction until only the single vertex tree remains. The extensions

that correspond to the sequence of reductions can be applied in reverse order to the single vertex tree to construct the original ordered pair $(T, L)$.          □

## 4   Trees Achieving $LD(T_n) = \lceil (n + 1 + 2(l(T) - s(T)))/3 \rceil$

Theorem 10 constructively characterizes all ordered pairs $(T, L)$ where $T$ is a tree of order $n$ achieving the Theorem 1 lower bound — given by $LD(T) \geq \lceil (n + 1)/3 \rceil$ — and $L$ is an $LD(T)$-set. For $l(T) = s(T)$, this is the same as achieving the more general Theorem 5 bound given by $LD(T) \geq \lceil (n + 1 + 2(l(T) - s(T)))/3 \rceil$. In fact, these bounds are also equal when $n \equiv 0 \pmod 3$ and $l(T) - s(T) = 1$. This explains both the "moreover" part of Lemma 9 and how Theorem 10 applies to some trees with $l(T) - s(T) > 0$. In this section, we use Lemma 4 to constructively characterize ordered pairs $(T, L)$ achieving the Theorem 5 bound without restriction on $l(T) - s(T)$.

Let a $T$ be a tree of order $n$ with strong support vertex $v \in V(T)$ and leaves $w, x \in N(v)$. Let $L \subset V(T)$ be an $LD(T)$-set with $LD(T) = \lceil (n + 1 + 2(l(T) - s(T)))/3 \rceil$. Without loss of generality, we assume $w \in L$. Form tree $T^\flat$ from $T$ and vertex set $L^\flat$ from $L$ by removing vertex $w$ from each. By the proof of Lemma 4, $L^\flat$ is an $LD(T^\flat)$-set and $LD(T^\flat) = LD(T) - 1$. This reduction operation, which we call Operation RO, corresponds to removing one leaf from a strong support vertex.

In the special case that $n \equiv 0 \pmod 3$ and $l(T) - s(T) = 1$, both the original ordered pair $(T, L)$ and the ordered pair $(T^\flat, L^\flat)$, resulting from applying Operation RO to $(T, L)$, achieve the Theorem 1 lower bound. Here, $(T, L)$ can be exactly recovered by applying Extension Operation O1 to $(T^\flat, L^\flat)$. With Reduction Operation RO, we are ready to constructively characterize the ordered pairs that achieve the Theorem 5 bound without restriction on $l(T) - s(T)$.

**Theorem 11** *Let $T$ be a tree of order $n$ and $L \subset V(T)$ an $LD(T)$-set. Then $L$ is an $LD(T)$-set of size $\lceil (n + 1 + 2(l(T) - s(T)))/3 \rceil$ if and only if $l(T) - s(T)$ successive applications of Operation RO to $(T, L)$ result in an ordered pair $(T', L')$ such that $l(T') = s(T')$ and $L'$ is an $LD(T')$-set of size $\lceil (|V(T')| + 1)/3 \rceil$.*

*Proof.*    Let $T$ be a tree of order $n$ and $L \subset V(T)$ an $LD(T)$-set of size $\lceil (n + 1 + 2(l(T) - s(T)))/3 \rceil$. If $l(T) = s(T)$, then zero applications of Operation RO result in ordered pair $(T', L') = (T, L)$ and the result holds in the forward direction. Suppose $l(T) - s(T) > 0$. Since this is the only condition required to apply Operation RO, it can be applied successively $l(T) - s(T)$ times to $(T, L)$. Let ordered pair $(T', L')$ be the result of these $l(T) - s(T)$ operations. Clearly, $l(T') = s(T')$. By repeated applications of Lemma 4, one for each application of Operation RO, we see $LD(T') = LD(T) - (l(T) - s(T)) = \lceil (n + 1 + 2(l(T) - s(T)))/3 \rceil - (l(T) - s(T)) = \lceil (n + 1 - (l(T) - s(T)))/3 \rceil = \lceil (|V(T')| + 1)/3 \rceil$. By repeated applications of the proof of Lemma 4, $L'$ is an $LD(T')$-set.

Now suppose $(T', L')$ is the result of $l(T) - s(T)$ applications of Operation RO to $(T, L)$ such that $l(T') = s(T')$ and $L'$ is an $LD(T')$-set of size $\lceil (|V(T')| + 1)/3 \rceil$.

If $l(T) = s(T)$, then $(T, L) = (T', L')$ and the result holds in the reverse direction. Suppose $l(T) - s(T) > 0$. Again by repeated applications of Lemma 4 and its proof, we have $L$ is an $LD(T)$-set of size $\lceil (|V(T')| + 1)/3 \rceil + (l(T) - s(T)) = \lceil ([n - (l(T) - s(T))] + 1)/3 \rceil + (l(T) - s(T)) = \lceil (n + 1 + 2(l(T) - s(T)))/3 \rceil$. $\qquad \square$

## 5   Summary

For a tree $T$, the new lower bound of Theorem 5 takes into account the number of leaves, $l(T)$, compared to the number of support vertices, $s(T)$, and is sharp for any value of $l(T) - s(T)$. Theorems 10 and 11 constructively characterize all trees attaining the Theorem 5 bound. Moreover, these two theorems constructively characterize all locating-dominating sets for all such trees.

Further examination of the quantity $l(T) - s(T)$ is being undertaken with respect to graphical parameters related to locating-domination. In particular, the authors are investigating improved bounds on the identifying code number of a tree, $IC(T)$, along with a constructive characterization of trees achieving the bounds. Note that for the related parameter open locating-domination, $l(T) - s(T)$ must be zero. Otherwise, the tree $T$ would have two vertices, leaves in this case, that have the same open neighborhood. Such a graph does not admit an open locating-domination set.

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