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More trees with equal broadcast and domination numbers*

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Abstract

A broadcast on a graph G is a function $f : V(G) \to \{0, 1, \ldots, \operatorname{diam} G\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V(G)$. The broadcast number of a graph is the minimum value of $\sum_{v \in V(G)} f(v)$ among all broadcasts f with the property that each vertex of G is within distance $f(v)$ from a vertex v with $f(v) > 0$. We characterize a class of trees with equal broadcast and domination numbers.

1 Introduction

Consider a radio station wishing to transmit a broadcast across a large area. It must decide where to place the broadcast towers (and how big the towers should be) in order to minimize the number of towers while ensuring that the entire region hears the broadcast. We can model this situation with a graph G, where the vertices represent geographic regions and two vertices are adjacent if their corresponding regions are close enough that a weak broadcast from one region can be heard from the other. If the towers can only broadcast to adjacent regions, then finding the optimal layout is equivalent to finding a minimum dominating set S of G, that is, a set of vertices of G where each vertex of G is either in S or adjacent to a vertex

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in S. If the station can use stronger towers (at a higher cost) then the goal is now to minimize the total cost of the towers. Placing the towers and determining their strength is equivalent to assigning a nonnegative integer to each vertex, where the regions corresponding to vertices with a zero do not have towers, and the strength of each tower on all other regions is proportional to the integer for that vertex.

We consider the case where the graphs representing regions are trees, and investigate a class of trees, called 1-cap trees (see Section 1.1 for the definition), for which the use of arbitrarily strong transmitters does no better than using transmitters that only broadcast to adjacent regions.

After giving basic definitions, we describe our initial class $\mathcal{C}_2$ of trees in Section 1.1, and introduce some essential tools in Sections 1.2 – 1.4. In Section 2 we characterize the 1-cap trees in a subclass $\mathcal{C}_2'$ of $\mathcal{C}_2$ and in Section 3 we combine these trees to form 1-cap trees in $\mathcal{C}_2$. We close with a short list of open problems on 1-cap trees in Section 4.

1.1 Definitions and background

For undefined concepts see [5, 13]. A *broadcast* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \operatorname{diam} G\}$ such that $f(v) \le e(v)$ (the eccentricity of v) for all $v \in V$. A *broadcast vertex* is a vertex v for which $f(v) \ge 1$. The set of all broadcast vertices is denoted V_f^+. For $v \in V_f^+$, the *f-neighbourhood* $N_f[v]$ of v is the set $\{u \in V : d(u, v) \le f(v)\}$. A vertex u *hears* a broadcast from $v \in V_f^+$, and v *broadcasts to* u, if $u \in N_f[v]$.

A broadcast f is a *dominating broadcast* if every vertex hears at least one broadcast. The *cost* of a broadcast f is defined as $\operatorname{cost}(f) = \sum_{v \in V(G)} f(v)$. The broadcast number $\gamma_b(G)$ is defined by $\gamma_b(G) = \min\{\operatorname{cost}(f) : f \text{ is a dominating broadcast of } G\}$. A dominating broadcast f of a graph G for which $\operatorname{cost}(f) = \gamma_b(G)$ is called a *γ_b-broadcast*. If f is a dominating broadcast such that $f(v) = 1$ for each $v \in V_f^+$, then V_f^+ is a *dominating set* of G, and the minimum cost of such a broadcast is the usual *domination number* $\gamma(G)$. A *γ-set* is a dominating set of cardinality $\gamma(G)$. If D is a dominating set and $v \in D$, then $\operatorname{PN}(v, D)$ is the set of all vertices dominated by v and by no other vertex in D. Let $D_{\mathrm{PN}} = \{v \in D : \operatorname{PN}(v, D) = \{v\}\}$; note that if $v \in D_{\mathrm{PN}}$, then $D - \{v\}$ dominates all vertices of G except v.

Erwin [11, 12] was the first to consider the broadcast domination problem, and to observe the trivial bound $\gamma_b(G) \le \min\{\operatorname{rad} G, \gamma(G)\}$ for any graph G. A graph G is *radial* if $\gamma_b(G) = \operatorname{rad} G$. The problems of characterizing radial trees and trees T with $\gamma_b(T) = \gamma(T)$ were first addressed by Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi in [9] and also studied in [1, 10, 23]. The former problem was solved by Herke and Mynhardt [17], while a large class of trees satisfying $\gamma_b(T) = \gamma(T)$ was studied by Mynhardt and Wodlinger in [21]. In [19], a graph G such that $\gamma_b(G) = \gamma(G)$ is called a 1-*capacity graph*, abbreviated to a 1-*cap graph*, because in these graphs the cost of an arbitrary dominating broadcast is no less than that of a dominating broadcast in which each vertex broadcasts with a capacity of 1, or not

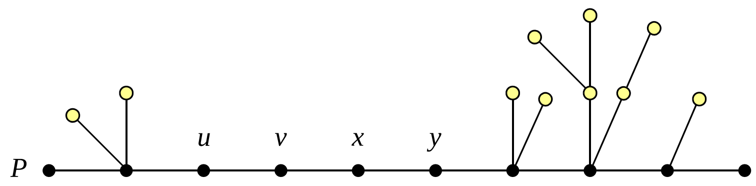

Figure 1: A tree with maximum split-sets $\{uv\}$ and $\{xy\}$

at all.

Minimum broadcast domination is solvable in polynomial time ($O(n^6)$) for any graph (Heggernes and Lokshtanov [14]) and in linear time for trees (Dabney, Dean and Hedetniemi [8]) and block graphs (Heggernes and Sæther [15]). While the problem of determining the domination number of an arbitrary graph is NP-complete, it has long been known that the domination number of a tree can be determined in linear time (see [6]). Knowing that $\gamma(T) = \gamma_b(T)$ for some tree T (or for any number of given trees) however does not adequately reveal the properties of all 1-cap trees, which merit investigation in their own right. Other work on broadcast domination includes [2, 3, 4, 20, 22].

A *diametrical path* (abbreviated d-*path*) of a tree T is a path of length $d = \operatorname{diam} T$. (Note: we use the Roman letter "d" for the abbreviation d-path, and the italic letter "d" for the length of a d-path.) We initially consider the class $\mathcal{C}_2$ of trees that can be obtained from a d-path $P = v_0, ..., v_d$ and a number of paths B_i, each of length congruent to 2 (mod 3), by identifying a leaf of B_i with a vertex v_j of P, $j = 2, ..., d-2$, in such a way that the resulting tree T has maximum degree $\Delta(T) = 3$ and $\operatorname{diam} T = d$. We explain in Section 1.3 how 1-cap trees in $\mathcal{C}_2$ are used to obtain a much larger class of 1-cap trees.

1.2 Split-sets

Let P be a d-path of a tree T. A set M of edges of P is a *split-P set* if, for each component T' of $T - M$, the path $P \cap T'$ is a d-path of T' of even positive length. A *split-set* of T is a split-P set for some d-path P of T, and a *maximum split-set* of T is a split-set of maximum cardinality. For example, the sets $\{uv\}$ and $\{xy\}$ are maximum split-P sets of the tree in Fig. 1, where P is the path of black vertices. Herke and Mynhardt [16, 17] showed that split-sets play an important role in determining the broadcast number of a tree, and Cockayne, Herke and Mynhardt [7] showed that split-sets are also relevant to the study of 1-cap trees.

Theorem 1.1 [16, 17] *For any tree T, let $m \geq 0$ be the cardinality of a maximum split-set of T. Then $\gamma_b(T) = \operatorname{rad} T - \left\lceil \frac{m}{2} \right\rceil$.*

Theorem 1.2 [7] *A tree T is 1-cap if and only if it has a maximum split-set M such that each component of $T - M$ is 1-cap.*

1.3 Shadow trees

The class of trees we consider is best described by means of so-called *shadow trees*, as first defined in [17]. Let P be a d-path of a tree T. For each $i = 0, ..., d$, let V_i be the set of all vertices of T that are connected to v_i by a (possibly trivial) path that is internally disjoint from P. Let ℓ_i be a vertex of T in V_i at maximum distance from v_i (possibly $\ell_i = v_i$), and let B_i be the $v_i - \ell_i$ path. The *shadow tree of T with respect to P*, denoted $S_{T,P}$, is the subtree of T induced by $\bigcup_{i=0}^{d} V(B_i)$. If B_i has length at least one, it is called a *branch* of $S_{T,P}$. Note that if P and P' are different d-paths of T, then it is possible that $S_{T,P} \ncong S_{T,P'}$. If the d-path P is understood or irrelevant, we abbreviate $S_{T,P}$ to S_T. Furthermore, any tree T such that $T = S_T$ is called a *shadow tree*; any shadow tree is the shadow tree of infinitely many trees. A tree and its shadow tree (in dark vertices and edges) are shown in Fig. 2. The relevance of shadow trees to the study of broadcast domination was demonstrated in [16, 17].

Theorem 1.3 [16, 17] *For any shadow tree S_T of a tree T, $\gamma_b(S_T) = \gamma_b(T)$.*

The following corollary to Theorem 1.3 demonstrates the importance of shadow trees to the class of all 1-cap trees.

Corollary 1.4 [7] *(i) If a tree T is 1-cap, then $\gamma(T) = \gamma(S_T)$ and S_T is 1-cap.*

(ii) If S_T is 1-cap and $\gamma(S_T) = \gamma(T)$, then T is 1-cap.

Due to the relatively simple structure of shadow trees, the following approach to studying 1-cap trees is useful. Let k denote a positive integer.

Step 1 Find all 1-cap shadow trees S with $\gamma_b(S) = k$.

Step 2 If S is a 1-cap shadow tree with $\gamma_b(S) = k$, use Corollary 1.4 to find all 1-cap trees T with $\gamma_b(T) = k$ that have S as shadow tree.

In order to perform Step 2 we need to know the conditions for a tree T and a subtree T' to have equal domination numbers. Let $W_1, ..., W_t$ be the components of

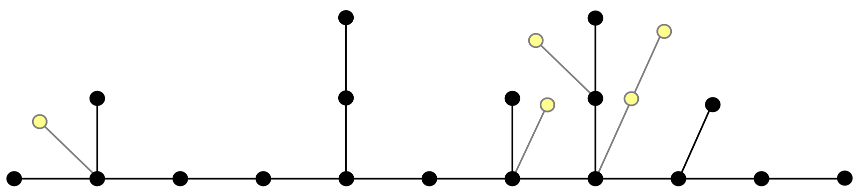

Figure 2: A tree and its shadow tree (dark vertices and edges)

$T - E(T')$. For $i = 1, ..., t$, let u_i be the unique vertex of $V(T') \cap V(W_i)$. We call u_i the *hinge* of W_i and also say that W_i *is hinged at* u_i. Let U_1 (respectively U_2) be the set of hinges of nontrivial subtrees W_i that are stars hinged at a central vertex (respectively at a leaf that is not also a central vertex). Note that $U_1 \cap U_2 = \varnothing$.

Proposition 1.5 [7] *Let T' be a subtree of the tree T. Then $\gamma(T) = \gamma(T')$ if and only if*

(*i*) *each nontrivial subtree W_i is either a star hinged at its centre or a star hinged at a leaf, and*

(*ii*) *T' has a γ-set D with $U_1 \subseteq D$ and $U_2 \subseteq D_{\mathrm{PN}}$.*

Suppose that S is a 1-cap shadow tree and we want to determine which trees T with $\gamma(T) = \gamma(S)$ have S as shadow tree, i.e. satisfy $S = S_T$. (We already know from Theorem 1.3 that $\gamma_b(T) = \gamma_b(S)$.) Let D be any γ-set of S that does not contain leaves and let $v \in D$. Using Proposition 1.5, we can join any number of new leaves to v. Furthermore, if $v \in D_{\mathrm{PN}}$ and we have not joined v to a new leaf, we can join v to a new vertex u and then join u to any number of new leaves. Then $(D - \{v\}) \cup \{u\}$ is a γ-set of the resulting tree. Note that if $v \in D_{\mathrm{PN}}$, then v is not a stem, hence this operation does not change the diameter or the branch lengths of the tree, hence $S = S_T$. We can repeat the procedure for each vertex of D and for each γ-set of S to construct all 1-cap trees that have S as shadow tree.

1.4 Overlap and branch length sequences

A shadow tree T with diametrical path $P = v_0, ..., v_d$ can be drawn in the Cartesian plane so that P lies on the x-axis with v_0 at the origin and each edge is one unit in length, where the edges not on P are drawn above the x-axis parallel to the y-axis. Thus a vertex v_i is described as being to the left of v_j , or v_j to the right of v_i, if $i < j$. A shadow tree drawn in this way is said to be *in standard representation*. We henceforth assume that all shadow trees are drawn in standard representation.

Let $H(t)$ be the tree obtained from $K_{1,3}$ by subdividing each edge $t-1$ times. If $H(t)$ is a subtree of T, then the leaves of $H(t)$ lie at the (geometric) vertices of an isosceles right triangle Δ whose hypotenuse lies on P and has length $2t$; we say that Δ has *radius* t. The triangles of the shadow tree in Fig. 2 are shown in Fig. 3. A triangle Δ' that lies inside another triangle is called a *nested triangle*; for example, the triangles Δ_4 and Δ_5 in Fig. 3 are nested triangles. As the removal of nested triangles and the branches (i.e., the vertices of the branches not on P) that form their altitudes does not affect split-sets of T, it also does not affect $\gamma_b(T)$, and we henceforth only consider shadow trees without nested triangles.

Consider a shadow tree T with d-path $P = v_0, ..., v_d$. Let $B_1, ..., B_k$ be the branches of T, in order of their occurrence on P, and let Δ_i be the triangle associated with B_i, $i = 1, ..., k$. For each i, let v_{ℓ_i} and v_{r_i} be the left and right vertices of Δ_i on P. Assuming that T has no nested triangles, $\ell_i < \ell_j$ and $r_i < r_j$ whenever

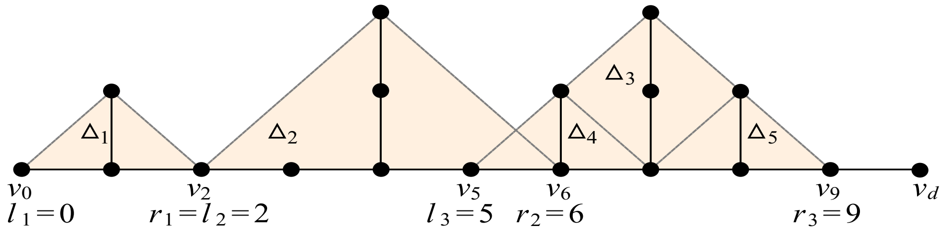

Figure 3: Triangles associated with the tree in Fig. 2

$i < j$. See Fig. 3; ignore Δ_4 and Δ_5. Notice that some triangles overlap, others just touch, and indeed, if we consider the triangles of the shadow tree of the tree in Fig. 1, some triangles are separated by edges of P (in this case the edges uv, vx, xy) that are not contained in triangles. Edges of the latter type are called *free edges*. These "overlaps" are of vital importance in deciding whether T is 1-cap or not. For $i = 1, ..., k+1$, we define the *overlaps* h_i as follows. Let

$$h_1 = -\ell_1,\ h_{k+1} = r_k - d,\ \text{and, for } i = 2, ..., k,\ h_i = r_{i-1} - \ell_i.$$

Note that $h_1, h_{k+1} \leq 0$. The *overlap sequence* $\overline{h}$ of T is defined by $\overline{h} = (h_1, ..., h_{k+1})$. For the tree in Fig. 3, $\overline{h} = (0, 0, 1, -1)$ and $v_9v_d = v_9v_{10}$ is a free edge. In addition, for $i = 1, ..., k$, say the branch B_i has length b_i, and define the *branch length sequence* $\overline{b}$ of T by $\overline{b} = (b_1, ..., b_k)$. It is worth noting that T is uniquely determined by its branch length and overlap sequences, and we also write $T = T(\overline{b}, \overline{h})$. As demonstrated in [7], whether or not a shadow tree is 1-cap depends not on the branch lengths of the triangles, but only on their least residues modulo 3 and the overlap sequence.

Theorem 1.6 [7] *Let $T = T(\overline{b}, \overline{h})$ and $T' = T'(\overline{b'}, \overline{h})$ be shadow trees without nested triangles, where $\overline{b} = (b_1, ..., b_k)$ and $\overline{b'} = (b'_1, ..., b'_k)$ such that $b'_i \equiv b_i \pmod 3$ for each $i = 1, ..., k$. Then T is 1-cap if and only if T' is 1-cap.*

Let T be a shadow tree with triangles $\Delta_1, ..., \Delta_k$. Free edges before Δ_1 are called *leading free edges*, free edges after Δ_k are *trailing free edges*, and free edges that are neither leading nor trailing free edges are called *internal free edges*. The following result was proved in [19].

Theorem 1.7 [19] *Let r, s be nonnegative integers and let T, T' be shadow trees, where T' is obtained by adding $3r$ leading and $3s$ trailing free edges to a d-path of T'. Then T is 1-cap if and only if T' is 1-cap.*

A *caterpillar* is a tree whose shadow tree only has branches of length one (if any). Seager [23] characterized radial and 1-cap caterpillars. Mynhardt and Wodlinger [21] extended Seager's results on caterpillars to the class of trees whose shadow trees have branch lengths congruent to 1 (mod 3). These results, together with Theorem 1.6, serve as our motivation for studying shadow trees with branches of length congruent to 2 (mod 3).

2 Shadow Trees with Branches of Length 2 (mod 3)

All trees in this section belong to the class $\mathcal{C}_2$, that is, they are shadow trees with branches of length congruent to 2 (mod 3). In addition, we require that our trees contain no nested triangles and no internal free edges, and denote this subclass of $\mathcal{C}_2$ by $\mathcal{C}_2'$. In Section 2.1 we define six types of trees in $\mathcal{C}_2'$. We then show that these trees are 1-cap, and in Sections 2.2 and 2.3 we show that they are in fact the only 1-cap trees of this nature. We first mention a number of further assumptions we make throughout this section.

A *stem* of a tree is a vertex adjacent to a leaf, and a *branch vertex* is a vertex of degree at least three. If f is a γ_b-broadcast of $T \neq K_2$ such that V_f^+ contains a leaf u, and v is the stem adjacent to u, then the broadcast g defined by $g(u) = 0$, $g(v) = f(u)$, and $g(w) = f(w)$ otherwise, is a γ_b-broadcast of T such that $|V_g^+| = |V_f^+|$. Therefore we consider only broadcasts without leaves as broadcast vertices.

Let $\bar{b} = (b_1, ..., b_k)$ and $\overline{h} = (h_1, ..., h_{k+1})$. Each shadow tree $T = T(\bar{b}, \overline{h})$ in this section is assumed to have a d-path $P = v_0, ..., v_d$ with branch vertices $v_{c_1}, ..., v_{c_k}$, $k \geq 1$. The branch $B_i = v_{c_i}, u_{i,1}, ..., u_{i,b_i}$ of T attached to v_{c_i} has length $b_i = 3m_i + 2$, $i = 1, ..., k$, and is covered by the triangle Δ_i, where Δ_i, Δ_{i+1} overlap by $h_{i+1} \geq 0$ edges, $i = 1, ..., k-1$. Only h_1 and h_{k+1} can be negative. Since the radius of Δ_i is b_i, the consecutive triangles Δ_i, Δ_{i+1} contain $2(b_i + b_{i+1}) - h_{i+1}$ edges of P.

Suppose $T = T(\bar{b}, \overline{h})$ is a 1-cap tree without nested triangles such that $h_i \leq 3$, $i = 2, ..., k$. Let $\overline{b'} = (b_1', ..., b_k')$, where $b_i' = 2$ for each $i = 1, ..., k$. Then $T' = T'(\overline{b'}, \overline{h})$ exists, also has no nested triangles, and, by Theorem 1.6, is 1-cap. For simplicity we sometimes assume, where appropriate, that each branch of T has length equal to two.

Given a γ-set X of T, a branch vertex v_{c_i} may or may not be in X. In either case, X contains at least $\lceil \frac{3m_i+1}{3} \rceil = m_i + 1$ vertices of $B_i - v_{c_i}$. A γ-set X such that $X \cap (V(B_i - v_{c_i})) = \{u_{i,1}, u_{i,4}, ..., u_{i,3m_i+1}\}$ for each $i = 1, ..., k$ is called a *natural γ-set.*

2.1 Six classes of 1-cap trees

Define the classes $\mathcal{T}_1 - \mathcal{T}_6$ of shadow trees as follows. For $\bar{b} = (b_1, ..., b_k)$, where $b_i \equiv 2 \pmod 3$, $i = 1, ..., k$, and $\overline{h} = (-x, h_2, ..., h_k, -y)$, let

$$\begin{aligned}
\mathcal{T}_1 &= \{T(\bar{b}, \overline{h}) : x \equiv 1 \pmod 3 \text{ and } h_i = 0,\ i = 2, ..., k\} \\
\mathcal{T}_2 &= \{T(\bar{b}, \overline{h}) : x \equiv y \equiv 1 \pmod 3,\ h_i = 1 \text{ for exactly one } i = 2, ..., k, \\
&\qquad \text{and } h_j = 0 \text{ if } j \neq i\} \\
\mathcal{T}_3 &= \{T(\bar{b}, \overline{h}) : x \equiv y \equiv 1 \pmod 3,\ h_i = 3 \text{ for exactly one } i = 2, ..., k, \\
&\qquad \text{and } h_j = 0 \text{ if } j \neq i\}
\end{aligned}$$

$\mathcal{T}_4 = \{T(\bar{b}, \bar{h}) : x \equiv y \equiv 2 \pmod 3 \text{ and } k = 1\}$
$\mathcal{T}_5 = \{T(\bar{b}, \bar{h}) : x \equiv y \equiv 2 \pmod 3,\ k = 2 \text{ and } h_2 = 1\}$
$\mathcal{T}_6 = \{T(\bar{b}, \bar{h}) : x \equiv 1 \pmod 3,\ y \equiv 2 \pmod 3,\ h_i = 0 \text{ for } i = 2, ..., k-1, \text{ and } h_k = 2\}$.

Note that the definitions of $\mathcal{T}_6$ and some instances of $\mathcal{T}_1$ (the cases $y \equiv 0$ or $2 \pmod 3$) are not symmetrical with respect to x and y; however, we also consider a tree to be in one of these classes if we can reverse its diametrical path P to fit the criteria. Let $\mathcal{T} = \bigcup_{i=1}^{6} \mathcal{T}_i \subseteq \mathcal{C}_2'$. Our aim is to prove the following result.

Theorem 2.1 *Let T be a shadow tree without internal free edges whose branches all have length congruent to* $2 \pmod 3$*. Then T is a 1-cap tree if and only if $T \in \mathcal{T}$.*

In the next four lemmas we consider shadow trees with specific overlap sequences and show that such a tree T is 1-cap if and only if it belongs to $\bigcup_{i=1}^{6} \mathcal{T}_i$. By Theorem 1.6 we may assume that each branch of T has length 2. We prove each result for $x, y \in \{0, 1, 2\}$; each lemma then follows from Theorem 1.7. A dominating set of a graph is *efficient* if each vertex is dominated exactly once, and that such a set is necessarily a γ-set.

Lemma 2.2 *Let $T = T(\bar{b}, \bar{h})$ with $\bar{h} = (-x, 0, ..., 0, -y)$ be a shadow tree with $k \geq 1$ branches. Then T is 1-cap if and only if $T \in \mathcal{T}_1 \cup \mathcal{T}_4$.*

Proof. Let T' be the subtree of T induced by all edges of T except the leading and trailing free edges. Then $\operatorname{diam} T' = 4k$ and $\operatorname{rad} T' = 2k$. Let $P' = v_0, ..., v_{4k}$ be a d-path of T'. Note that $v_2, v_6, ..., v_{4k-2}$ are the branch vertices. For each $i \in \{1, ..., k\}$, the branch B_i that starts at $v_{2+4(i-1)} = v_{4i-2}$ consists of the path $v_{4i-2}, u_{i,1}, u_{i,2}$. Define $D \subseteq V(T')$ by $D = \{u_{i,1} : i = 1, ..., k\} \cup \{v_{4i} : i = 0, ..., k\}$. Then $|D| = 2k+1$ and D is an efficient dominating set, hence $\gamma(T') = 2k + 1$. For each $x, y \in \{0, 1, 2\}$, let $T(x, y)$ be the tree obtained by adding x leading and y trailing free edges to P'. Each such $T(x, y)$ has the empty set as maximum split-set, hence, by Theorem 1.1, $\gamma_b(T(x, y)) = \operatorname{rad} T(x, y)$. We consider the admissible values of x and y below, and determine $\operatorname{rad} T(x, y)$ and $\gamma(T(x, y))$. The results are summarized in Table 1.

(*i*) If $(x, y) = (0, 0)$, then $\operatorname{rad} T(x, y) = 2k < |D|$ and $T(x, y)$ is not 1-cap.

(*ii*) If $(x, y) \in \{(1, 0), (0, 1), (1, 1)\}$, then D is a γ-set of $T(x, y)$ and $\operatorname{rad} T(x, y) = 2k + 1 = |D|$, so $T(x, y)$ is 1-cap, and $T(x, y) \in \mathcal{T}_1$ (reverse the d-path if $x = 0$ and $y = 1$).

(*iii*) If $(x, y) \in \{(0, 2), (2, 0)\}$, then $\operatorname{rad} T(x, y) = 2k + 1$. It is simple to verify that $\gamma(T(x, y)) \geq |D| + 1 = 2k+2$ for all values of $k \geq 1$, hence $T(x, y)$ is not 1-cap.

(x,y)	γ	γ_b	1-cap?
$(0,0)$	$2k+1$	$2k$	No
$(0,1)$	$2k+1$	$2k+1$	Yes
$(0,2)$	$2k+2$	$2k+1$	No
$(1,1)$	$2k+1$	$2k+1$	Yes
$(1,2)$	$2k+2$	$2k+2$	Yes
$(2,2),\ k=1$	4	4	Yes
$(2,2),\ k\geq 2$	$2k+3$	$2k+2$	No

Table 1: Possibilities for $\gamma(T(x,y))$ and $\gamma_b(T(x,y))$ in the proof of Lemma 2.2

(*iv*) If $(x,y) \in \{(1,2),(2,1)\}$, then $\operatorname{rad} T(x,y) = 2k+2$, and D together with one new vertex (incident with the first leading or the last trailing free edge) dominates $T(x,y)$. Hence $T(x,y)$ is 1-cap, and $T(x,y) \in \mathcal{T}_1$ (reverse the d-path if $x=2$ and $y=1$).

(*v*) Finally, if $(x,y)=(2,2)$, then $\operatorname{rad} T(x,y) = 2k+2$. If $k=1$, then $T(x,y)\in\mathcal{T}_4$. Let w and w' be the vertices incident with the two leading and two trailing free edges of $T(x,y)$, respectively. Then $\{w, v_2, u_{1,1}, w'\}$ dominates $T(x,y)$. Hence $\gamma(T(x,y)) = 4 = \operatorname{rad} T(x,y)$ and $T(x,y)$ is 1-cap.

If $k\geq 2$, it is simple to verify that $\gamma(T(x,y)) \geq |D|+2 = 2k+3$, hence $T(x,y)$ is not 1-cap.

Therefore, if $T(x,y)$ is 1-cap, then $T(x,y)\in\mathcal{T}_1\cup\mathcal{T}_4$. Conversely, if $T(x,y)\in\mathcal{T}_1$, then (*ii*) or (*iv*) applies, and if $T(x,y)\in\mathcal{T}_4$, then (*v*) applies with $k=1$. Thus, if $T(x,y)\in\mathcal{T}_1\cup\mathcal{T}_4$, then $T(x,y)$ is 1-cap. By Theorem 1.7, T is 1-cap if and only if $T\in\mathcal{T}_1\cup\mathcal{T}_4$. ■

Lemma 2.3 *Let $T = T(\overline{b},\overline{h})$ with $\overline{h} = (-x, h_2, ..., h_k, -y)$ be a shadow tree with $k\geq 2$ branches such that $h_i = 1$ for exactly one $i\in\{2,...,k\}$ and $h_j = 0$ if $j\neq i$. Then T is 1-cap if and only if $T\in\mathcal{T}_2\cup\mathcal{T}_5$.*

Proof. Define T', $P' = v_0, ..., v_{4k-1}$ and $T(x,y)$ as in the proof of Lemma 2.2. Say T' has $k = k_1+k_2$ branches $B_1, ..., B_k$ with triangles $\Delta_1, ..., \Delta_k$, where Δ_{k_1} and Δ_{k_1+1} overlap in one edge. Then $\operatorname{diam} T' = 4k-1$ and $\operatorname{rad} T' = 2k$. Let T_1 and T_2 be the subtrees of T' formed by $\Delta_1, ..., \Delta_{k_1}$ and $\Delta_{k_1+1}, ..., \Delta_k$ respectively. For $i=1,2$, define the efficient dominating set D_i of T_i similar to the γ-set D of T' in the proof of Lemma 2.2. Then

$$D_1 = \{u_{i,1} : i = 1, ..., k_1\} \cup \{v_{4i} : i = 0, ..., k_1\},$$
$$D_2 = \{u_{i,1} : i = k_1+1, ..., k\} \cup \{v_{4i-1} : i = k_1, ..., k\},$$

and $|D_i| = 2k_i + 1$. Note that $D_3 = (D_2 - \{v_{4k_1-1}\}) \cup \{v_{4k_1}\}$ is also a γ-set of T_2. Now $D = D_1\cup D_3$ is a γ-set of T' of cardinality $2k+1$, and $\gamma(T') = 2k+1$. Again we consider the admissible values of x and y and determine $\operatorname{rad} T(x,y)$ and $\gamma(T(x,y))$, summarizing the results in Table 2.

(x,y)	γ	γ_b	1-cap?
$(0,0)$	$2k+1$	$2k$	No
$(0,1)$	$2k+1$	$2k$	No
$(0,2)$	$2k+2$	$2k+1$	No
$(1,1)$	$2k+1$	$2k+1$	Yes
$(1,2)$	$2k+2$	$2k+1$	No
$(2,2),\ k=2$	6	6	Yes
$(2,2),\ k\geq 3$	$2k+3$	$2k+2$	No

Table 2: Possibilities for $\gamma(T(x,y))$ and $\gamma_b(T(x,y))$ in the proof of Lemma 2.3

(*i*) If $(x,y)\in\{(0,0),(1,0),(0,1)\}$, then $\operatorname{rad} T(x,y)=2k<|D|\leq\gamma(T(x,y))$ and $T(x,y)$ is not 1-cap.

(*ii*) If $(x,y)=(1,1)$, then $\operatorname{rad} T(x,y)=2k+1$ and D dominates $T(x,y)$. Hence $T(x,y)$ is 1-cap. Also, $T(x,y)\in\mathcal{T}_2$.

(*iii*) If $(x,y)\in\{(0,2),(2,0)\}$, then $\operatorname{rad} T(x,y)=2k+1$. Certainly, D does not dominate $T(x,y)$ and nor does any set of cardinality $2k+1$, as is easy to verify. Hence $T(x,y)$ is not 1-cap.

(*iv*) If $(x,y)\in\{(1,2),(2,1)\}$, then $\operatorname{rad} T(x,y)=2k+1$. By (*iii*), $\gamma(T(x,y))>2k+1$.

(*v*) If $(x,y)=(2,2)$, then $\operatorname{rad} T(x,y)=2k+2$. If $k=2$, then $\operatorname{rad} T(x,y)=6$ and $T(x,y)\in\mathcal{T}_5$. Define w and w' as in Lemma 2.2(*v*). Then $\{w,v_2,u_{1,1},v_5,u_{2,1},w'\}$ dominates $T(x,y)$, hence $T(x,y)$ is 1-cap.

If $k\geq 3$, it is simple to verify that $\gamma(T(x,y))\geq|D|+2=2k+3$, hence $T(x,y)$ is not 1-cap.

Therefore, if $T(x,y)$ is 1-cap, then $T(x,y)\in\mathcal{T}_2\cup\mathcal{T}_5$. Conversely, if $T(x,y)\in\mathcal{T}_2$, then (*ii*) applies, and if $T(x,y)\in\mathcal{T}_5$, then (*v*) applies with $k=2$. Thus, if $T(x,y)\in\mathcal{T}_2\cup\mathcal{T}_5$, then $T(x,y)$ is 1-cap. By Theorem 1.7, T is 1-cap if and only if $T\in\mathcal{T}_2\cup\mathcal{T}_5$. ■

Lemma 2.4 *Let $T=T(\overline{b},\overline{h})$ with $\overline{h}=(-x,h_2,...,h_k,-y)$ be a shadow tree with $k\geq 2$ branches such that $h_i=2$ for exactly one $i\in\{2,...,k\}$ and $h_j=0$ if $j\neq i$. Then T is 1-cap if and only if $T\in\mathcal{T}_6$.*

Proof. Proceed as in the proof of Lemma 2.3 to construct the trees T_1 and T_2, where this time Δ_{k_1} and Δ_{k_1+1} overlap in two edges, and $\operatorname{diam} T'=4k-2=2\operatorname{rad} T'$. Let $P'=v_0,...,v_{4k-2}$ be a d-path of T'. Define the efficient γ-set D_i' of T_i by

$$D_1'=\{u_{i,1}:i=1,...,k_1\}\cup\{v_{4i}:i=0,...,k_1\},$$
$$D_2'=\{u_{i,1}:i=k_1+1,...,k\}\cup\{v_{4i-2}:i=k_1,...,k\}.$$

Then $|D_i'| = 2k_i + 1$. Note that $D_1 = (D_1' - \{v_{4k_1}\}) \cup \{v_{4k_1-1}\}$ is also a γ-set of T_1, while $D_2 = (D_2' - \{v_{4k_1-2}\}) \cup \{v_{4k_1-1}\}$ is also a γ-set of T_2. Therefore $D = D_1 \cup D_2$ is a dominating set of T' of cardinality $2k+1$. To see that D is a γ-set of T', note that $D_1 - \{v_{4k_1-1}\}$ is an efficient γ-set of $H_1 = T_1 - \{v_{4k_1-1}, v_{4k_1}\}$ (hence $\gamma(H_1) = 2k_1$), and that no γ-set of H_1 contains v_{4k_1-2}. Similarly, if $H_2 = T_2 - \{v_{4k_1-2}, v_{4k_1-1}\}$, then $\gamma(H_2) = 2k_2$ and no γ-set of H_2 contains v_{4k_1}. Thus neither $D - \{v_{4k_1-1}\}$ nor the union of any γ-sets of H_1 and H_2 dominates v_{4k_1-1} in T'. Finally, since the $u_{k_1,2} - u_{k_1+1,2}$-path in T' is isomorphic to P_7, no two vertices of T' dominate $\{u_{k_1,2}, v_{4k_1-1}, u_{k_1+1,2}\}$. Hence $\gamma(T') > 2k$, from which it follows that $\gamma(T') = 2k + 1$. Since $\operatorname{rad} T' = 2k - 1$, $T(x, y)$ can only be 1-cap if $x + y \geq 3$. For all choices of x and y such that $3 \leq x + y \leq 4$, $\operatorname{rad} T(x, y) = 2k + 1$. We consider two cases.

Case 1 $x + y = 3$. Assume without loss of generality that $x = 1$ and $y = 2$. Let $v_{-1}v_0$ be the leading free edge and let $v_{4k-2}v_{4k-1}$ and $v_{4k-1}v_{4k}$ be the trailing free edges of $T(x, y)$. If $k = k_1 + 1$, that is, the 2-overlap is the last overlap, then $X = \{u_{i,1} : i = 1, ..., k\} \cup \{v_{4i} : i = 0, ..., k - 1\} \cup \{v_{4k-1}\}$ is a dominating set of $T(x, y)$ of cardinality $2k+1$, hence $\gamma(T(x, y)) = 2k+1$. Note that no γ-set of $T(x, y)$ contains v_{-1}. If $k_2 \geq 2$, that is, if the 2-overlap is not the last overlap, then it can be verified easily that $\gamma(T(x, y)) \geq 2k + 2$. Thus, $T(x, y)$ is 1-cap if and only if $T(x, y) \in \mathcal{T}_6$.

Case 2 $(x, y) = (2, 2)$. If the 2-overlap is not the last overlap, then it follows from Case 1 that $T(x, y)$ is not 1-cap. If the 2-overlap is the last overlap, the note in Case 1 that no γ-set of $T(1, 2)$ contains v_{-1} implies that $\gamma(T(2, 2)) \geq 2k+2 > \operatorname{rad} T(2, 2)$.

The lemma follows from Theorem 1.7. ■

Lemma 2.5 *Let $T = T(\overline{b}, \overline{h})$ with $\overline{h} = (-x, h_2, ..., h_k, -y)$ be a shadow tree with $k \geq 2$ branches such that $h_i = 3$ for exactly one $i \in \{2, ..., k\}$ and $h_j = 0$ if $j \neq i$. Then T is 1-cap if and only if $T \in \mathcal{T}_3$.*

Proof. Define $T'$and $T(x, y)$ as in the proof of Lemma 2.2. Say T' has $k = k_1 + k_2$ branches $B_1, ..., B_k$ with triangles $\Delta_1, ..., \Delta_k$, where Δ_{k_1} and Δ_{k_1+1} overlap in three edges. Then $\operatorname{diam} T' = 4k - 3$ and $\operatorname{rad} T' = 2k - 1$. Let $P' = v_0, ..., v_{4k-3}$ be a d-path of T' and define D by

$$D = \{u_{i,1} : i = 1, ..., k\} \cup \{v_{4i} : i = 0, ..., k_1 - 1\} \cup \{v_{4i-3} : i = k_1 + 1, ..., k\}.$$

Then D is an efficient γ-set of T' of cardinality $2k$ and $\gamma(T') = 2k$. Therefore $T(x, y)$ can only be 1-cap if $x + y \geq 2$.

If $x = y = 1$, then D is also a γ-set of $T(x, y)$ and $\operatorname{rad} T(x, y) = 2k$. The efficiency of D further implies that if $\{x, y\} = \{1, 2\}$, then $\gamma(T(x, y)) = 2k+1$ while $\operatorname{rad} T(x, y) = 2k$, and if $x = y = 2$, then $\gamma(T(x, y)) = 2k + 2$ while $\operatorname{rad} T(x, y) = 2k + 1$. Finally, if $\{x, y\} = \{0, 2\}$, then $\operatorname{rad} T(x, y) = 2k$, and it is easy to see that $\gamma(T(x, y)) = 2k + 1$. Hence $T(x, y)$ is 1-cap if and only if $x = y = 1$, that is, $T(x, y) \in \mathcal{T}_3$. ■

2.2 Clear shadow trees and pure minimum dominating sets

A shadow tree T that has at least one γ-set D that contains no branch vertices is called *clear* and D is called a *pure γ-set* of T. We now show that the only clear 1-cap shadow trees whose only free edges are leading and trailing free edges are the trees in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.

Theorem 2.6 *Let T be a clear shadow tree whose only free edges are x leading and y trailing free edges. Then T is a 1-cap tree if and only if $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.*

Proof. By Theorem 1.7 we may assume that $x, y \in \{0, 1, 2\}$ so that $\gamma_b(T) = \operatorname{rad} T$. Recall that the branch $B_i = v_{c_i}, u_{i,1}, ..., u_{i,b_i}$ of T attached to v_{c_i} has length $b_i = 3m_i + 2$, $i = 1, ..., k$, and is covered by the triangle Δ_i, where Δ_i, Δ_{i+1} overlap by h_i edges, $i = 1, ..., k-1$. Then

$$\begin{aligned}\operatorname{rad} T = \left\lceil \frac{d}{2} \right\rceil = \left\lceil \frac{1}{2}\left(x + y + \sum_{i=1}^{k} 2b_i - \sum_{i=1}^{k-1} h_i\right)\right\rceil = \sum_{i=1}^{k} b_i + \left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil \\ = 2k + 3\sum_{i=1}^{k} m_i + \left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil . \qquad (1)\end{aligned}$$

For each $i = 1, ..., k-1$, let Q_i be the path $v_{c_i+1}, ..., v_{c_{i+1}-1}$. Since $d(v_{c_i}, v_{c_{i+1}}) = c_{i+1} - c_i = b_i + b_{i+1} - h_i$, the length $\ell(Q_i)$ of Q_i is given by $\ell(Q_i) = b_i + b_{i+1} - h_i - 2$; hence Q_i contains $b_i + b_{i+1} - h_i - 1$ vertices. We determine $\gamma(T)$. Let D be a pure natural γ-set of T.

- Each branch B_i contains $\lceil \frac{b_i}{3} \rceil = m_i + 1$ vertices in D. The vertex $u_{i,1}$ is in D (since D is natural) and dominates v_{c_i}.
- The path $v_0, ..., v_{c_1-1}$ contains c_1 vertices, $\lceil \frac{c_1}{3} \rceil$ of which are in D.
- The path $v_{c_k+1}, ..., v_d$ contains $d - c_k$ vertices, $\lceil \frac{d-c_k}{3} \rceil$ of which are in D.
- Each path Q_i contains $\left\lceil \frac{b_i+b_{i+1}-h_i-1}{3} \right\rceil$ vertices in D.

Since $c_1 = x + b_1$ and $d - c_k = y + b_k$, we obtain

$$\begin{aligned}\gamma(T) &= \sum_{i=1}^{k} \left\lceil \frac{b_i}{3} \right\rceil + \left\lceil \frac{x+b_1}{3} \right\rceil + \left\lceil \frac{y+b_k}{3} \right\rceil + \sum_{i=1}^{k-1} \left\lceil \frac{b_i + b_{i+1} - h_i - 1}{3} \right\rceil \\ &= \sum_{i=1}^{k} (m_i + 1) + \left\lceil \frac{x + 3m_1 + 2}{3} \right\rceil + \left\lceil \frac{y + 3m_k + 2}{3} \right\rceil \\ &\quad + \sum_{i=1}^{k-1} \left\lceil \frac{3m_i + 3m_{i+1} - h_i + 3}{3} \right\rceil \qquad (2)\end{aligned}$$

$$= \sum_{i=1}^{k} m_i + k + m_1 + m_k + \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil + k - 1 + \sum_{i=1}^{k-1} (m_i + m_{i+1}) - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor$$

$$= 3\sum_{i=1}^{k} m_i + 2k - 1 + \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor . \tag{3}$$

Since $\operatorname{rad} T = \gamma_b(T)$ and $\gamma_b(T) \leq \gamma(T)$, (1) and (3) imply that

$$\left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil \leq \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - 1 - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor . \tag{4}$$

Assuming that T is a 1-cap tree, i.e., $\gamma_b(T) = \gamma(T)$, (4) becomes

$$\left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil = \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - 1 - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor . \tag{5}$$

Note that $\lceil \frac{x+y}{2} \rceil \leq \lceil \frac{x+2}{3} \rceil + \lceil \frac{y+2}{3} \rceil - 1$, with equality if and only if $x = 1$ or $y = 1$. Hence if $h_i = 0$ for all i, then without loss of generality $x = 1$ and $y \in \{0, 1, 2\}$. Therefore $T \in \mathcal{T}_1$.

Now assume $h_i > 0$ for at least one i. Then $\frac{1}{2}\sum_{i=1}^{k-1} h_i > \sum_{i=1}^{k-1} \lfloor \frac{h_i}{3} \rfloor$ for all $k \geq 2$. Therefore, if there are an even number of odd overlaps, then $\sum_{i=1}^{k-1} h_i$ is even, and

$$\left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil = \left\lceil \frac{x+y}{2} \right\rceil - \frac{1}{2}\sum_{i=1}^{k-1} h_i < \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - 1 - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor .$$

Hence (5) does not hold and T is not a 1-cap tree. It follows that there are an odd number of odd overlaps. Assume therefore that $h_j = t$ for some j, where t is odd. Then $\sum_{i=1, i\neq j}^{k-1} h_i$ is even. Hence, from (4),

$$\begin{aligned} \left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil &= \left\lceil \frac{x+y-t}{2} \right\rceil - \frac{1}{2}\sum_{i=1, i\neq j}^{k-1} h_i \\ &\leq \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - 1 - \left\lfloor \frac{t}{3} \right\rfloor - \sum_{i=1, i\neq j}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor . \end{aligned} \tag{6}$$

As can be seen from Table 3, $\lceil \frac{x+y-t}{2} \rceil \leq \lceil \frac{x+2}{3} \rceil + \lceil \frac{y+2}{3} \rceil - 1 - \lfloor \frac{t}{3} \rfloor$, with equality if and only if $x = y = 1$ and $t \in \{1, 3\}$. Moreover, $\frac{1}{2}\sum_{i=1, i\neq j}^{k-1} h_i \geq \sum_{i=1, i\neq j}^{k-1} \lfloor \frac{h_i}{3} \rfloor$, and this inequality is strict if $h_i > 0$ for some $i \neq j$. Hence equality holds in (6) if and only if $h_i = 0$ for all $i \neq j$, $x = y = 1$, and $t \in \{1, 3\}$. Therefore $T \in \mathcal{T}_2 \cup \mathcal{T}_3$. ■

(x,y)	$\lceil \frac{x+y-t}{2} \rceil$	$\lceil \frac{x+2}{3} \rceil + \lceil \frac{y+2}{3} \rceil - 1 - \lfloor \frac{t}{3} \rfloor$
$(0,0)$	$-\lfloor \frac{t}{2} \rfloor$	$1 - \lfloor \frac{t}{3} \rfloor$
$(0,1)$	$-\lfloor \frac{t-1}{2} \rfloor$	$1 - \lfloor \frac{t}{3} \rfloor$
$(0,2)$	$1-\lfloor \frac{t}{2} \rfloor$	$2 - \lfloor \frac{t}{3} \rfloor$
$(1,1)$	$1-\lfloor \frac{t}{2} \rfloor$	$1 - \lfloor \frac{t}{3} \rfloor$
$(1,2)$	$1-\lfloor \frac{t-1}{2} \rfloor$	$2 - \lfloor \frac{t}{3} \rfloor$
$(2,2)$	$2-\lfloor \frac{t}{2} \rfloor$	$3 - \lfloor \frac{t}{3} \rfloor$

Table 3: Comparing $\lceil \frac{x+y-t}{2} \rceil$ and $\lceil \frac{x+2}{3} \rceil + \lceil \frac{y+2}{3} \rceil - 1 - \lfloor \frac{t}{3} \rfloor$ for $x, y \in \{0,1,2\}$

2.3 Thorny trees and mixed minimum dominating sets

A shadow tree that is not clear is said to be *thorny* and its γ-sets are said to be *mixed*. Among all natural γ-sets of a thorny shadow tree T, if D is one that contains the minimum number of branch vertices, then D is a *minimally mixed γ-set* of T. In this section we show that the only thorny 1-cap shadow trees whose only free edges are leading and trailing edges are the trees in $\mathcal{T}_4 \cup \mathcal{T}_5 \cup \mathcal{T}_6$. We need two lemmas.

Lemma 2.7 *Let T be a thorny shadow tree with a minimally mixed natural γ-set D and branch vertices $v_{c_1}, ..., v_{c_k}$, $k \geq 1$. Suppose $v_{c_\alpha} \in D$. Define the vertex z to the right of v_{c_α} as follows.*

- *If $\alpha \neq k$ and $v_{c_{\alpha+1}} \in D$, let $z = v_{c_{\alpha+1}}$; if $v_{c_{\alpha+1}} \notin D$ let $z = u_{\alpha+1,1}$.*
- *If $\alpha = k$, let $z = v_d$.*

Define the vertex z' to the left of v_{c_α} similarly. Let Q be the $z' - z$ subpath of T. Then $d(v_{c_\alpha}, q) \equiv 0 \pmod 3$ for each vertex $q \in V(Q) \cap D$.

Proof. Neither $v_{c_\alpha - 1}$ nor $v_{c_\alpha + 1}$ is a branch vertex: if both were branch vertices, then $D - \{v_{c_\alpha}\}$ would be a dominating set of T, which is not the case, and if (say) $v_{c_\alpha - 1}$ were a branch vertex but not $v_{c_\alpha + 1}$, then $(D - \{v_{c_\alpha}\}) \cup \{v_{c_\alpha + 1}\}$ would be a γ-set containing fewer branch vertices than D, contrary to the choice of D.

Let Q_r (Q_ℓ, respectively) be the subpath of Q to the right (left, respectively) of v_{c_α}. Without loss of generality, consider Q_r and note that each vertex of Q_r is dominated by a vertex in $D \cap V(P)$, with the possible exception of $v_{c_{\alpha+1}}$, which may only be dominated by $z = u_{\alpha+1,1}$ if $v_{c_{\alpha+1}} \notin D$. Suppose $d(v_{c_\alpha}, q') \not\equiv 0 \pmod 3$ for some vertex $q' \in V(Q_r) \cap D$. Let q be the nearest vertex to v_{c_α} on Q_r such that

$q \in D$ and $d(v_{c_\alpha}, q) \not\equiv 0 \pmod 3$. Let $v_{r_1}, ..., v_{r_j}$ be the vertices in $V(Q) \cap D$ that lie strictly between v_{c_α} and q. If $\{v_{r_1}, ..., v_{r_j}\} \neq \varnothing$, then $d(v_{r_j}, q) \in \{1, 2\}$, otherwise $d(v_{c_\alpha}, q) \in \{1, 2\}$. Now $D' = (D - \{v_{c_\alpha}, v_{r_1}, ..., v_{r_j}\}) \cup \{v_{c_\alpha - 1}, v_{r_1 - 1}, ..., v_{r_j - 1}\}$ is a γ-set of T containing fewer branch vertices than D, a contradiction. ■

Lemma 2.8 *Let T be a thorny shadow tree with a minimally mixed γ-set D and branch vertices $v_{c_1}, ..., v_{c_k}$, $k \geq 1$. If $v_{c_1} \in D$ ($v_{c_k} \in D$, respectively), then $d(v_{c_1}, v_0) \equiv 1 \pmod 3$ ($d(v_{c_k}, v_d) \equiv 1 \pmod 3$, respectively).*

Proof. Suppose $v_{c_1} \in D$ and let w be the first vertex of P in D. By Lemma 2.7, $d(v_{c_1}, w) \equiv 0 \pmod 3$. Since w dominates v_0, $w \in \{v_0, v_1\}$. However, if $w = v_0$, then $D' = (D - \{w\}) \cup \{v_1\}$ is a γ-set of T that does not satisfy Lemma 2.7. Hence $w = v_1$ and $d(v_0, v_{c_1}) \equiv 1 \pmod 3$. Similarly, $d(v_{c_k}, v_d) \equiv 1 \pmod 3$ if $v_{c_k} \in D$. ■

If D is a natural γ-set of T, then $D \cap V(B_i) = \{u_{i,j} : j \equiv 1 \pmod 3\}$. If $v_{c_i} \in D$ and

$$D' = (D - \{u_{i,j} : j \equiv 1 \pmod 3\}) \cup \{u_{i,j} : j \equiv 0 \pmod 3\} \cup \{u_{i,b_i}\},$$

then D' is a γ-set of T, called the *i-conversion* of D. Similarly, for $i' \neq i$, if $\{v_{c_i}, v_{c_{i'}}\} \subseteq D$ and

$$D'' = (D' - \{u_{i',j} : j \equiv 1 \pmod 3\}) \cup \{u_{i',j} : j \equiv 0 \pmod 3\} \cup \{u_{i',b_{i'}}\},$$

then D'' is also a γ-set of T, called the *$\{i, i'\}$-conversion* of D.

Theorem 2.9 *Let T be a thorny 1-cap shadow tree whose only free edges are x leading and y trailing free edges. Then $T \in \mathcal{T}_4 \cup \mathcal{T}_5 \cup \mathcal{T}_6$.*

Proof. Suppose the statement of Theorem 2.9 is false. Amongst all thorny 1-cap shadow trees without internal free edges not in $\mathcal{T}_4 \cup \mathcal{T}_5 \cup \mathcal{T}_6$, let T be a smallest one. By Theorem 1.7 we may assume that $x, y \in \{0, 1, 2\}$ so that $\gamma_b(T) = \operatorname{rad} T$. Let D be a minimally mixed natural γ-set of T and let $v_{c_\alpha} \in D$. Define the vertices z and z' as in Lemma 2.7. If $z = v_d$ and $z' = v_0$, then T has exactly one branch vertex and it follows from Lemma 2.8 that $T \in \mathcal{T}_4$, so assume without loss of generality that $z \neq v_d$. We consider two cases, depending on the choice of z.

Case 1 $z = v_{c_{\alpha+1}}$. Then $z \in D$ and by Lemma 2.7, $d(v_{c_\alpha}, v_{c_{\alpha+1}}) \equiv 0 \pmod 3$. Define the vertex z'' for $v_{c_{\alpha+1}}$ similar to the vertex z for v_{c_α}.

Recall that the branches B_α and $B_{\alpha+1}$ have lengths b_α and $b_{\alpha+1}$. Now $b_\alpha \equiv b_{\alpha+1} \equiv 2 \pmod 3$ and $d(v_{c_\alpha}, v_{c_{\alpha+1}}) \equiv 0 \pmod 3$, hence $h_{\alpha+1} \equiv 1 \pmod 3$. Let X be the $\{\alpha, \alpha+1\}$-conversion of D. Then $\{u_{\alpha,b_\alpha}, u_{\alpha+1,b_{\alpha+1}}\} \subseteq X$ and for $i \in \{\alpha, \alpha+1\}$, $\operatorname{PN}(u_{i,b_i}, X) = \{u_{i,b_i}\}$.

Let $T' = T - \{u_{\alpha,b_\alpha}, u_{\alpha+1,b_{\alpha+1}}\}$ and let Δ'_α and $\Delta'_{\alpha+1}$ be the triangles of T' corresponding to the triangles Δ_α and $\Delta_{\alpha+1}$ of T. Let $h'_{\alpha+1}$ be the overlap of Δ'_α and

$\Delta'_{\alpha+1}$. Since T has no internal free edges and $h_{\alpha+1} \geq 1$, $h'_{\alpha+1} \geq -1$. If $\Delta_{\alpha-1}$ exists, let h'_α be the overlap of $\Delta_{\alpha-1}$ and Δ'_α, otherwise let h'_α be the number of leading free edges of T'. Similarly, if $\Delta_{\alpha+2}$ exists, let $h'_{\alpha+2}$ be the overlap of $\Delta'_{\alpha+1}$ and $\Delta_{\alpha+2}$, otherwise let $h'_{\alpha+2}$ be the number of trailing free edges of T'.

Since $\mathrm{PN}(u_{i,b_i}, X) = \{u_{i,b_i}\}$ for $i \in \{\alpha, \alpha+1\}$, $\gamma(T') \leq \gamma(T) - 2$ and therefore

$$\gamma_b(T') \leq \gamma(T') \leq \gamma(T) - 2 = \gamma_b(T) - 2 = \mathrm{rad}(T) - 2 = \mathrm{rad}(T') - 2. \tag{7}$$

Let m be the cardinality of a maximum split-set of T'. By Theorem 1.1, $\gamma_b(T') = \mathrm{rad}(T') - \lceil \frac{m}{2} \rceil$, hence by (7), $m \geq 3$. Since $h'_{\alpha+1} \geq -1$, the only possible free edges of T' are x leading free edges, y trailing free edges, possibly an edge to the left of Δ'_α, possibly an edge to the right of $\Delta'_{\alpha+1}$, and possibly an edge between Δ'_α and $\Delta'_{\alpha+1}$. Since none of the x leading or y trailing free edges of T is a split-edge, we deduce that $m = 3$, $h'_{\alpha+1} = -1$ and

$$h'_\alpha = \begin{cases} -1 & \text{if } \Delta_{\alpha-1} \text{ exists} \\ -x-1 & \text{otherwise} \end{cases}$$
$$h'_{\alpha+2} = \begin{cases} -1 & \text{if } \Delta_{\alpha+2} \text{ exists} \\ -y-1 & \text{otherwise} \end{cases} .$$

Suppose $\Delta_{\alpha+2}$ exists. Then $h'_{\alpha+2} = -1$ and therefore $h_{\alpha+2} = 0$. This in turn implies that $d(v_{c_{\alpha+1}}, v_{c_{\alpha+2}}) \equiv 1 \pmod 3$ and $d(v_{c_{\alpha+1}}, u_{\alpha+2,1}) \equiv 2 \pmod 3$. Now if $v_{c_{\alpha+2}} \in D$, then $z'' = v_{c_{\alpha+2}}$, otherwise $z'' = u_{\alpha+2,1} \in D$ (since D is a natural γ-set). But by Lemma 2.7, $d(v_{c_{\alpha+1}}, z'') \equiv 0 \pmod 3$, a contradiction. We deduce that $\Delta_{\alpha+2}$ does not exist. Therefore $\alpha + 1 = k$. By Lemma 2.8, $d(v_{c_{\alpha+1}}, v_d) \equiv 1 \pmod 3$, that is, $y \equiv 2 \pmod 3$ and so $y = 2$. Similarly, $\alpha = 1$ (hence $\alpha + 1 = k = 2$) and $x = 2$. Finally, $h_{\alpha+1} = h'_{\alpha+1} + 2 = 1$. Therefore $T \in \mathcal{T}_5$, contrary to the choice of T.

By symmetry, $T \in \mathcal{T}_5$ if $z' = v_{c_{\alpha-1}}$. We therefore assume henceforth that $z' \neq v_{c_{\alpha-1}}$.

Case 2 $z = u_{\alpha+1,1}$. Then $z \in D$ and by Lemma 2.7, $d(v_{c_\alpha}, z) \equiv 0 \pmod 3$ and $d(v_{c_\alpha}, v_{c_{\alpha+1}}) \equiv 2 \pmod 3$. Therefore $h_{\alpha+1} \equiv 2 \pmod 3$. If $z' = u_{\alpha-1,i}$, then similarly $d(v_{c_{\alpha-1}}, v_{c_\alpha}) \equiv 2 \pmod 3$ and $h_{\alpha-1} \equiv 2 \pmod 3$. Then $T' = T - \{u_{\alpha,b_\alpha}\}$ has no internal free edges, hence is radial, so that $\gamma_b(T') = \gamma_b(T)$. But if X is the α-conversion of D, then $X - \{u_{\alpha,b_\alpha}\}$ is a dominating set of T'. This means that

$$\gamma_b(T') \leq \gamma(T') < \gamma(T) = \gamma_b(T) = \gamma_b(T'),$$

which is impossible. Therefore $z' = v_0$ and $\alpha = 1$. By Lemma 2.8, $d(v_0, v_{c_1}) \equiv 1 \pmod 3$; hence $x \equiv 2 \pmod 3$ and so $x = 2$.

Suppose $h_2 \geq 5$. Since T has no nested triangles it follows that the branch B_1 at v_{c_1} has length at least 5. Let $T'' = T - \{u_{1,b_1}, u_{1,b_1-1}, u_{1,b_1-2}, u_{1,b_1-3}\}$. Then T'' has exactly $x + 4 = 6$ leading free edges, $y \in \{0, 1, 2\}$ trailing free edges, and no internal free edges since $h_2 \geq 5$. Moreover, $\gamma_b(T'') \leq \gamma(T'') = \gamma(T) - 2 = \mathrm{rad}(T) - 2$. Let m be the cardinality of a maximum split-set M of T''. By Theorem 1.1, $\gamma_b(T'') = \mathrm{rad}(T') - \lceil \frac{m}{2} \rceil$, hence $m \geq 3$. But it is impossible to find a split-set of cardinality three amongst six consecutive free edges (the removal of three edges would leave a component with a d-path of zero or odd length), and (for a similar reason) none of the trailing free edges is a split-edge. Hence $h_2 = 2$.

Let H be the subtree of T obtained by deleting all the vertices of B_1 except v_{c_1}. Then $D - \{u_{1,1}, u_{1,4}, ..., u_{1,b_1-1}\}$ dominates H and $\gamma(H) = \gamma(T) - m_1 - 1$. Since $\gamma_b(H) \leq \gamma(H)$, H has a maximum split-set M of cardinality $m \geq 1$. By Theorem 1.1, $\gamma_b(H) = \mathrm{rad}(H) - \lceil \frac{m}{2} \rceil = \mathrm{rad}(T) - \lceil \frac{m}{2} \rceil$. Therefore $m \geq 2m_1 + 1$. The rightmost vertex of Δ_1 is v_{2b_1+2} and, since $h_2 = 2$, the leftmost vertex of Δ_2 is v_{2b_1}. The leading free edges of H are the edges on the path $R = v_0, v_1, ..., v_{2b_1}$, and H has no internal free edges. Hence M consists of edges of R. Note that $2b_1 = 6m_1 + 4$. Since each component of $H - M$ has even positive diameter, the set $M = \{v_2v_3, v_5v_6, ..., v_{6m_1+2}v_{6m_1+3}\}$ of cardinality $m = 2m_1 + 1$ is the unique maximum split-set of H.

Let T_d be the component of $H - M$ that contains v_d; it has exactly one leading free edge and at least one branch vertex. By Theorem 1.2, T_d is a 1-cap tree. If T_d is a clear tree, then $T_d \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, and if it is a thorny tree, then, by the choice of T, $T_d \in \mathcal{T}_4 \cup \mathcal{T}_5 \cup \mathcal{T}_6$. But since T_d has exactly one leading free edge, $T_d \notin \mathcal{T}_4 \cup \mathcal{T}_5$. We examine the other possibilities.

- If $T_d \in \mathcal{T}_1$, then $y = 1$ since $\mathrm{diam}(T_d)$ is even and T_d has one leading free edge. Since $x = 2$, T is the reverse of a tree in $\mathcal{T}_6$.
- If $T_d \in \mathcal{T}_2 \cup \mathcal{T}_3$, then T_d has an odd number of leading free edges, an odd number of trailing free edges, and one odd overlap, so that $\mathrm{diam}(T_d)$ is odd, contrary to M being a nonempty split-set.
- If $T_d \in \mathcal{T}_6$, then it has one leading free edge, two trailing free edges, and only even overlaps, so that $\mathrm{diam}(T_d)$ is odd, again a contradiction.

This completes the proof of Theorem 2.9. ■

Theorem 2.1 is now an immediate consequence of Lemmas 2.2 – 2.5 and Theorems 2.6 and 2.9. Thus the 1-cap shadow trees in $\mathcal{C}_2'$ have been completely characterized.

3 Joining 1-Cap Shadow Trees

In this section we discuss joining 1-cap shadow trees in $\mathcal{C}_2$ to form new 1-cap shadow trees with internal free edges. Consider the classes $\mathcal{T}_1 - \mathcal{T}_6$ defined in Section 2.1. For $i = 1, ..., 6$, let xT_iy denote the set of all trees in $\mathcal{T}_i$ that have x leading and y trailing free edges. In order to join two trees correctly, order is important here. Thus, although $1T_12 \subseteq \mathcal{T}_1$ and $2T_11 \subseteq \mathcal{T}_1$, and a tree in $1T_12$ is isomorphic to a tree in $2T_11$, their representations are mirror images of each other. A tree in $2T_11$ is shown in Fig. 4. We further define the subset $\mathcal{T}_{1,1}$ of $\mathcal{T}_1$ by

$$\mathcal{T}_{1,1} = \{T(\overline{b}, \overline{h}) \in \mathcal{T}_1 : y \equiv 1 \ (\mathrm{mod}\ 3).$$

If $T, T' \in \mathcal{T} = \bigcup_{i=1}^{6} \mathcal{T}_i$, say $T \in kT_i\ell$ has diametrical path $v_0, ..., v_d$, and $T' \in k'T_j\ell'$ has diametrical path $v'_0, ..., v'_{d'}$, we denote the tree obtained by joining v_d to v'_0 by $T + T'$, and the set of all such trees by $kT_i\ell + k'T_j\ell'$, and say that $T + T'$ is the *sum* of T and T'. The tree T in Fig. 4 belongs to $2T_11 + 1T_11$ if we consider uv to be the joining edge, or to $2T_42 + 0T_11$ if we consider vw to be the joining edge.

It is not necessarily true that the sum of two 1-cap trees is another 1-cap tree. For example, as illustrated by the tree T with $\gamma(T) = 10$ and $\gamma_b(T) = 9$ in Fig. 5, no tree in $1T_21 + 1T_21$ is a 1-cap tree; the bold (blue) edge is a split-edge of T. Note that if $H \in 1T_21$, then $\text{diam}(H)$ is odd. In contrast, the tree T in Fig. 6 is the sum of the 1-cap trees T' and T'', both of which have even diameter and are radial, but T is not 1-cap. Again the bold (blue) edges form a split-set.

We now summarize exactly when the sum of two trees in $\mathcal{T}$ is a 1-cap tree. The proof involves examining several different cases and is given [19]. We omit it here.

Theorem 3.1 *If $F_1, F_2 \in \mathcal{T}$, then $F_1 + F_2$ is a 1-cap tree if and only if one of the following conditions holds.*

(*i*) $F_1, F_2 \in \mathcal{T}_{1,1} \cup \mathcal{T}_4$.

(*ii*) $F_i \in \mathcal{T}_{1,1}$ *and* $F_j \in \mathcal{T} - (\mathcal{T}_{1,1} \cup \mathcal{T}_4)$, $i \neq j$.

(*iii*) $F_i \in \mathcal{T}_4$ *and* $F_j \in yT_1x$, *where* $x \equiv 1 \pmod 3$ *and* $y \equiv 0 \pmod 3$, $i \neq j$.

(*iv*) $F_i \in xT_1y$ *and* $F_j \in y'T_1x$, *where* $x \equiv 1 \pmod 3$ *and* $y, y' \equiv 0$ *or* $2 \pmod 3$, $i \neq j$.

(*v*) $F_1, F_2 \in xT_1y$, *and* F_2 *has exactly one branch vertex, or* $F_1, F_2 \in yT_1x$, *and* F_1 *has exactly one branch vertex, where* $x \equiv 1 \pmod 3$ *and* $y \equiv 2 \pmod 3$.

Theorem 3.1 does not give all 1-cap trees in $\mathcal{C}_2$ with internal free edges. The radial 1-cap tree T in Fig. 7 cannot be written as the sum of two trees in $\mathcal{T}$. Note that $T = T(\overline{b}, \overline{h})$, where $\overline{b} = (2, 2)$ and $\overline{h} = (-2, -2, -2)$. The pattern does not generalize: $T(\overline{b}, \overline{h})$, where $\overline{b} = (2, 2, 2)$ and $\overline{h} = (-2, -2, -2, -2)$ is not 1-cap (and also not radial).

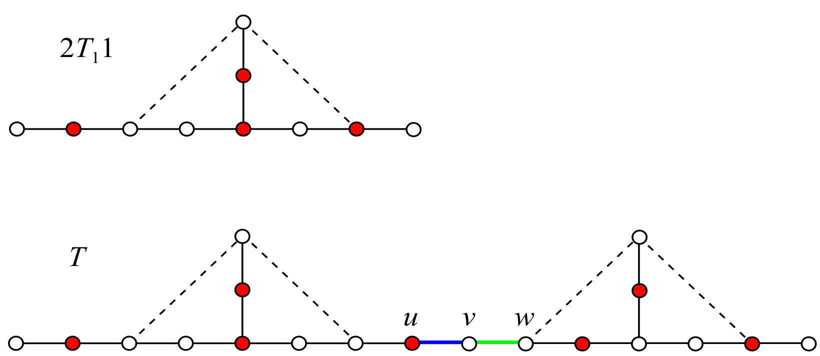

Figure 4: $T \in 2T_11 + 1T_11 = 2T_42 + 0T_11$

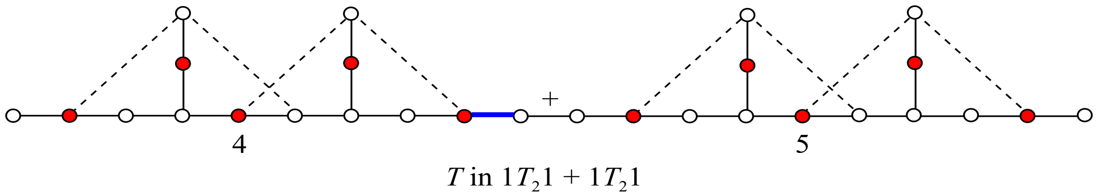

Figure 5: The tree $T \in 1T_21 + 1T_21$ is not a 1-cap tree

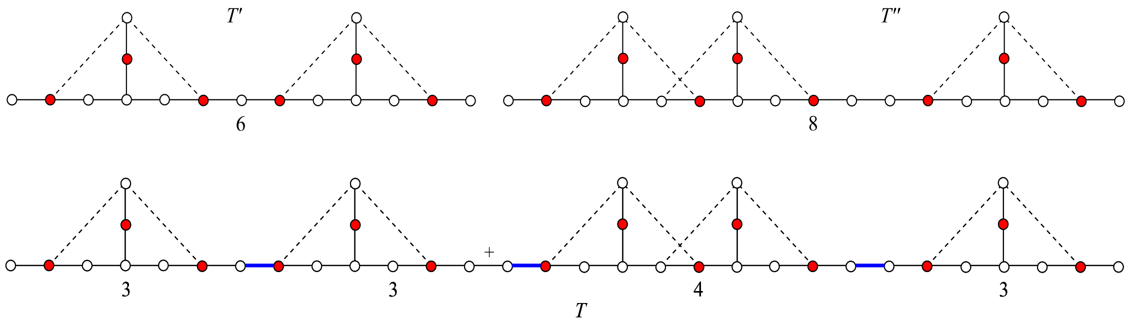

Figure 6: T' and T'' are 1-cap trees but T is not

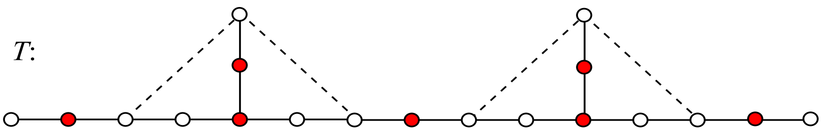

Figure 7: A 1-cap tree with internal free edges that is not the sum of trees in $\mathcal{T}$

4 Future Work

We close by briefly mention a number of open problems on 1-cap trees.

Question 1 *Is the radial* 1*-cap tree in Fig. 7 the only* 1*-cap shadow tree in* $\mathcal{C}_2$ *that cannot be written as the sum of* 1*-cap trees in the class* $\mathcal{T}$*?*

Problem 1 *Determine all* 1*-cap shadow trees in* $\mathcal{C}_2$ *that contain internal free edges.*

Since the class of 1-cap trees with branches of length congruent to 1 (mod 3) is completely characterized in [21], only the trees with branches of length congruent to 0 (mod 3) remain to be considered in the study of trees, all of whose branches have the same length (modulo 3).

Problem 2 *Determine all* 1*-cap trees with branches of length congruent to* $0 \pmod 3$*.*

Finally, the case where the shadow trees have branches of arbitrary length remains open.

Problem 3 *Characterize the class of all* 1*-cap trees.*

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