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New bounds on the signed domination numbers of graphs

S.M. Hosseini Moghaddam

Department of Mathematics
Qom Branch, Islamic Azad University
Qom
Iran
sm.hosseini1980@yahoo.com

Abdollah Khodkar*

Department of Mathematics
University of West Georgia
Carrollton, GA 30118
U.S.A.
akhodkar@westga.edu

Babak Samadi

Department of Mathematics
Arak University
Arak
Iran
b-samadi@araku.ac.ir

Abstract

A signed dominating function of a graph G with vertex set V is a function $f : V \to \{-1, 1\}$ such that for every vertex v in V the sum of the values of f at v and at every vertex u adjacent to v is at least 1. The weight of f is the sum of the values of f at every vertex of V. The signed domination number of G is the minimum weight of a signed dominating function of G. In this paper, we study the signed domination numbers of graphs and present new sharp lower and upper bounds for this parameter. As an example, we prove that the signed domination number of a tree of order n with ℓ leaves and s support vertices is at least $(n + 4 + 2(\ell - s))/3$.

* corresponding author

1 Introduction

Throughout this paper, let G be a finite connected graph with vertex set $V = V(G)$, edge set $E = E(G)$, minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. We use [9] for terminology and notation which are not defined here. For any vertex $v \in V$, $N(v) = \{u \in G \mid uv \in E(G)\}$ denotes the *open neighborhood* of v in G, and $N[v] = N(v) \cup \{v\}$ denotes its *closed neighborhood*. A set $S \subseteq V$ is a *dominating set* in G if each vertex in $V \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A subset $B \subseteq V(G)$ is a *packing* in G if for every distinct vertices $u, v \in B$, $N[u] \cap N[v] = \emptyset$. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in G.
In [5], Harary and Haynes introduced the concept of tuple domination as a generalization of domination in graphs. Let $1 \leq k \leq \delta(G) + 1$. A set $D \subseteq V$ is a *k-tuple dominating set* in G if $|N[v] \cap D| \geq k$, for all $v \in V(G)$. The *k-tuple domination number*, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality of a k-tuple dominating set. In fact, the authors showed that every graph G with $\delta \geq k-1$ has a k-tuple dominating set and hence a k-tuple domination number. It is easy to see that $\gamma_{\times 1}(G) = \gamma(G)$. This concept has been studied by several authors including [4, 8].
Gallant et al. [4] introduced the concept of limited packing in graphs and exhibited some real-world applications of it to network security, market saturation and codes. A set of vertices $B \subseteq V$ is called a *k-limited packing set* in G if $|N[v] \cap B| \leq k$ for all $v \in V$, where $k \geq 1$. The *k-limited packing number*, $L_k(G)$, is the largest number of vertices in a k-limited packing set. When $k = 1$ we have $L_1(G) = \rho(G)$.
Let $S \subseteq V$. For a real-valued function $f : V \to R$ we define $f(S) = \sum_{v \in S} f(v)$. Also, $f(V)$ is the *weight* of f. A *signed dominating function*, abbreviated SDF, of G is defined in [2] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$, for every $v \in V$. The *signed domination number*, abbreviated SDN, of G is $\gamma_s(G) = \min\{f(V) \mid f \text{ is a SDF of } G\}$. This concept was defined in [2] and has been studied by several authors including [1, 3, 6, 7].
In this paper, we continue the study of the concept of the signed domination numbers of graphs. The authors noted that most of the existing bounds on $\gamma_s(G)$ are lower bounds except those that are related to regular graphs; for more information the reader can consult [3]. In Section 2, we prove that $\gamma_s(G) \leq n - 2\left\lfloor \dfrac{2\rho(G) + \delta(G) - 2}{2} \right\rfloor$, for a graph G of order n with $\delta(G) \geq 2$. In Section 3, we find some new sharp lower bounds on $\gamma_s(G)$ for a general graph G. The lower bound given in Part (i) of Theorem 3.2 can also be found in [6] with a much longer proof than the one presented here. We also prove that $\gamma_s(T) \geq \dfrac{n + 4 + 2(\ell - s)}{3}$, for a tree of order n with ℓ leaves and s support vertices. Furthermore we show that this bound is sharp.

2 An upper bound

We bound $\gamma_s(G)$ from above in terms of order, minimum degree and packing number of G using the concept of limited packing.

Theorem 2.1. *Let G be a graph of order n with $\delta \geq 2$. Then*

$$\gamma_s(G) \leq n - 2\left\lfloor \frac{2\rho(G) + \delta - 2}{2} \right\rfloor$$

and this bound is sharp.

Proof. Let B be a $\lfloor \frac{\delta}{2} \rfloor$-limited packing in G. Define $f : V \to \{-1, 1\}$ by

$$f(v) = \begin{cases} -1 & \text{if} \quad v \in B \\ 1 & \text{if} \quad v \in V \setminus B. \end{cases}$$

Since B is a $\lfloor \frac{\delta}{2} \rfloor$-limited packing in G, $|N[v] \cap (V \setminus B)| \geq \deg(v) - \lfloor \frac{\delta}{2} \rfloor + 1$. Therefore, for every vertex v in V,

$$f(N[v]) = |N[v] \cap (V \setminus B)| - |N[v] \cap B| \geq \deg(v) - \left\lfloor \frac{\delta}{2} \right\rfloor + 1 - \left\lfloor \frac{\delta}{2} \right\rfloor \geq 1.$$

Therefore f is a SDF of G with weight $n - 2|B|$. So, by the definition of $\lfloor \frac{\delta}{2} \rfloor$-limited packing number,

$$\gamma_s(G) \leq n - 2L_{\lfloor \frac{\delta}{2} \rfloor}(G). \tag{1}$$

We now claim that $B \neq V$. If $B = V$ and $u \in V$ such that $\deg(u) = \Delta$, then $\Delta + 1 = |N[u] \cap B| \leq \lfloor \frac{\delta}{2} \rfloor$, a contradiction.

Now let $u \in V \setminus B$. It is easy to check that $|N[v] \cap (B \cup \{u\})| \leq \lfloor \frac{\delta}{2} \rfloor + 1$, for all $v \in V$. Therefore $B \cup \{u\}$ is a $(\lfloor \frac{\delta}{2} \rfloor + 1)$-limited packing set in G. Hence, $L_{\lfloor \frac{\delta}{2} \rfloor + 1}(G) \geq |B \cup \{u\}| = L_{\lfloor \frac{\delta}{2} \rfloor}(G) + 1$. Repeating these inequalities, we obtain $L_{\lfloor \frac{\delta}{2} \rfloor}(G) \geq L_{\lfloor \frac{\delta}{2} \rfloor - 1}(G) + 1 \geq \ldots \geq L_1(G) + \lfloor \frac{\delta}{2} \rfloor - 1$, and since $L_1(G) = \rho(G)$, we conclude

$$L_{\lfloor \frac{\delta}{2} \rfloor}(G) \geq \rho(G) + \left\lfloor \frac{\delta}{2} \right\rfloor - 1 \tag{2}$$

The upper bound now follows by Inequalities (1) and (2). Moreover, The bound is sharp for the complete graph of order $n \geq 3$. $\square$

3 Lower bounds

For convenience, for the rest of the paper we make use of the following notations. Let G be a graph and $f : V(G) \longrightarrow \{-1, 1\}$ be a SDF of G. Define $V^+ = \{v \in V \mid f(v) = 1\}$ and $V^- = \{v \in V \mid f(v) = -1\}$. Let $G^+ = G[V^+]$ and $G^- = G[V^-]$ be the subgraphs of G induced by V^+ and V^-, respectively. We also let $E^+ = E(G^+)$ and $E^- = E(G^-)$. We consider $[V^+, V^-]$ as the set of edges having one end point in V^+ and the other in V^-, $V_o = \{v \in V \mid \deg(v) \text{ is odd}\}$ and $V_e = \{v \in V \mid \deg(v) \text{ is even}\}$. Also $V_o^+ = V_o \cap V^+$, $V_o^- = V_o \cap V^-$, $V_e^+ = V_e \cap V^+$ and $V_e^- = V_e \cap V^-$. Finally, $\deg_{G^+}(v) = |N(v) \cap V^+|$ and $\deg_{G^-}(v) = |N(v) \cap V^-|$. For a graph G, let $O = \{v \in V \mid \deg(v) = 0\}$, $L = \{v \in V \mid \deg(v) = 1\}$, $S = \{v \in V \mid N(v) \cap L \neq \emptyset\}$, $C(G) = V \setminus (O \cup L \cup S)$ and $\delta^* = \min\{\deg(v) \mid v \in C(G)\}$. Obviously, if $C(G) = \emptyset$, then $\gamma_s(G) = n$. Therefore, in the following discussions we assume, without loss of generality, that $C(G) \neq \emptyset$. Thus, $\delta^* \geq \max\{2, \delta\}$.

Lemma 3.1. *The following statements hold.*

(i) $\left(\left\lceil \frac{\delta^*}{2} \right\rceil + 1\right) |V^-| \leq |[V^+, V^-]| \leq \left\lfloor \frac{\Delta}{2} \right\rfloor (|V^+ \setminus L|)$,

(ii) $|V_o| + 2|V^-| \leq 2|E^+| - 2|E^-|$.

Proof. (i) Let $v \in V^-$. Since $f(N[v]) \geq 1$ and $v \in C(G)$, we have $\deg_{G^+}(v) \geq \left\lceil \frac{\deg(v)}{2} \right\rceil + 1 \geq \left\lceil \frac{\delta^*}{2} \right\rceil + 1$. Therefore, $|[V^+, V^-]| \geq (\left\lceil \frac{\delta^*}{2} \right\rceil + 1)|V^-|$. On the other hand, all leaves and support vertices belong to V^+. Now let $v \in V^+ \setminus L$. Then $\deg_{G^-}(v) \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$. Therefore, $|[V^+, V^-]| \leq \left\lfloor \frac{\Delta}{2} \right\rfloor (|V^+ \setminus L|)$.

(ii) We first derive a lower bound for $|[V^+, V^-]|$. Let $v \in V^-$. Since $f(N[v]) \geq 1$, we observe that $\deg_{G^+}(v) \geq \deg_{G^-}(v) + 2$ and $\deg_{G^+}(v) \geq \deg_{G^-}(v) + 3$ when $\deg(v)$ is odd. This leads to

$$\begin{array}{rcl} |[V^+, V^-]| & = & \sum_{v \in V_o^-} \deg_{G^+}(v) + \sum_{v \in V_e^-} \deg_{G^+}(v) \\ & \geq & \sum_{v \in V_o^-} (\deg_{G^-}(v) + 3) + \sum_{v \in V_e^-} (\deg_{G^-}(v) + 2) \\ & = & 3|V_o^-| + 2|V_e^-| + \sum_{v \in V^-} \deg_{G^-}(v) \\ & = & 2|V^-| + 2|E^-| + |V_o^-|. \end{array} \tag{3}$$

Now let $v \in V^+$. Since $f(N[v]) \geq 1$, we observe that $\deg_{G^+}(v) \geq \deg_{G^-}(v)$ and $\deg_{G^+}(v) \geq \deg_{G^-}(v) + 1$ when $\deg(v)$ is odd. It follows that

$$\begin{array}{rcl} |[V^+, V^-]| & = & \sum_{v \in V_o^+} \deg_{G^-}(v) + \sum_{v \in V_e^+} \deg_{G^-}(v) \\ & \leq & \sum_{v \in V_o^+} (\deg_{G^+}(v) - 1) + \sum_{v \in V_e^+} (\deg_{G^+}(v)) \\ & = & \sum_{v \in V^+} \deg_{G^+}(v) - |V_o^+| = 2|E^+| - |V_o^+|. \end{array} \tag{4}$$

Together inequalities (3) and (4) imply the desired inequality. □

We are now in a position to present the following lower bounds.

Theorem 3.2. *Let G be a graph of order n, size m, maximum degree Δ and ℓ leaves. Let $V_o = \{v \in V \mid \deg(v)$ is odd$\}$. Then*

(i) $$\gamma_s(G) \geq \frac{\left(\left\lceil \frac{\delta^*}{2} \right\rceil - \left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) n + 2 \left\lfloor \frac{\Delta}{2} \right\rfloor \ell}{\left\lceil \frac{\delta^*}{2} \right\rceil + \left\lfloor \frac{\Delta}{2} \right\rfloor + 1},$$

(ii) $\gamma_s(G) \geq \dfrac{\left(\left\lceil \frac{3\delta^*}{2} \right\rceil - \left\lfloor \frac{3\Delta}{2} \right\rfloor + 3\right) n + 2(\left\lfloor \frac{\Delta}{2} \right\rfloor \ell + |V_o|)}{\left\lceil \frac{3\delta^*}{2} \right\rceil + \left\lfloor \frac{3\Delta}{2} \right\rfloor + 3}$.

Furthermore these bounds are sharp.

Proof. (i) This is a straightforward result by Part (i) of Lemma 3.1, $|V^+| = \dfrac{n + \gamma_s(G)}{2}$ and $|V^-| = \dfrac{n - \gamma_s(G)}{2}$.

(ii) We have

$$\begin{aligned} 2|E^+| &= \textstyle\sum_{v \in V^+} \deg_{G^+}(v) = \sum_{v \in V^+} \deg(v) - \sum_{v \in V^+} \deg_{G^-}(v) \\ &\leq \Delta|V^+| - |[V^+, V^-]| \leq \Delta|V^+| - \left(\left\lceil \frac{\delta^*}{2} \right\rceil + 1\right)|V^-|. \end{aligned} \tag{5}$$

and

$$\begin{aligned} 2|E^-| &= \textstyle\sum_{v \in V^-} \deg_{G^-}(v) = \sum_{v \in V^-} \deg(v) - \sum_{v \in V^-} \deg_{G^+}(v) \\ &\geq \delta^*|V^-| - |[V^+, V^-]| \geq \delta^*|V^-| - \left\lfloor \frac{\Delta}{2} \right\rfloor (|V^+| - \ell). \end{aligned} \tag{6}$$

Part (ii) of Lemma 3.1 and Inequalities (5) and (6) imply the desired lower bound. The bounds are sharp for the complete graph K_n. □

The lower bound given in Part (i) of Theorem 3.2 was first found by Haas and Wexler [6] for a graph G with $\delta(G) \geq 2$ using a longer proof. The lower bound given in Part (i) of Theorem 3.2 is an improvement of the lower bound found in [6] when $\delta(G) = 1$.

As an application of the concepts of limited packing and tuple domination we give a sharp lower bound on $\gamma_s(G)$ in terms of the order of G, $\delta(G)$, $\Delta(G)$ and domination number $\gamma(G)$.

Theorem 3.3. *For any graph G of order n, minimum degree δ and maximum degree Δ,*

$$\gamma_s(G) \geq -n + 2 \max\left\{ \left\lceil \frac{\Delta + 2}{2} \right\rceil, \left\lceil \frac{\delta + 2\gamma(G)}{2} \right\rceil \right\}$$

and this bound is sharp.

Proof. We first prove the following claims.
Claim 1. $\gamma_s(G) \geq -n + 2\left\lceil \frac{\Delta+2}{2} \right\rceil$.
Let $f : V \to \{-1, 1\}$ be a SDF of G with weight $f(V(G)) = \gamma_s(G)$. Since $f(N[v]) \geq 1$, it follows that $|N[v] \cap V^-| \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$ for every vertex $v \in V(G)$. Therefore V^- is a $\left\lfloor \frac{\Delta}{2} \right\rfloor$-limited packing set in G. Thus

$$(n - \gamma_s(G))/2 = |V^-| \leq L_{\left\lfloor \frac{\Delta}{2} \right\rfloor}(G). \tag{7}$$

On the other hand, similar to the proof of Theorem 2.1, we have

$$L_{\lfloor \frac{\Delta}{2} \rfloor}(G) \leq L_{\lfloor \frac{\Delta}{2} \rfloor+1}(G) - 1 \leq \ldots \leq L_{\Delta+1}(G) - \left\lceil \frac{\Delta}{2} \right\rceil - 1 = n - \left\lceil \frac{\Delta}{2} \right\rceil - 1.$$

Now Inequality (7) implies $2(\lceil \frac{\Delta}{2} \rceil + 1) - n \leq \gamma_s(G)$, as desired.

Claim 2. $\gamma_s(G) \geq -n + 2 \left\lceil \frac{\delta + 2\gamma(G)}{2} \right\rceil$.

Since $f(N[v]) \geq 1$, it follows that $|N[v] \cap V^+| \geq \left\lceil \frac{\delta}{2} \right\rceil + 1$, for every vertex $v \in V(G)$. Therefore V^+ is a $(\lceil \frac{\delta}{2} \rceil + 1)$-tuple dominating set in G. Thus

$$(n + \gamma_s(G))/2 = |V^+| \geq \gamma_{\times(\lceil \frac{\delta}{2} \rceil + 1)}(G). \tag{8}$$

Now let D be a $(\lceil \frac{\delta}{2} \rceil + 1)$-tuple dominating set in G. Then $|N[v] \cap D| \geq \lceil \frac{\delta}{2} \rceil + 1$, for every vertex $v \in V(G)$. Let $u \in D$. It is easy to see that $|N[v] \cap (D \setminus \{u\})| \geq \lceil \frac{\delta}{2} \rceil$, for all $v \in V(G)$. Hence, $D \setminus \{u\}$ is a $\lceil \frac{\delta}{2} \rceil$-tuple dominating set in G. Hence, $\gamma_{\times(\lceil \frac{\delta}{2} \rceil+1)}(G) \geq \gamma_{\times \lceil \frac{\delta}{2} \rceil}(G) + 1$. By repeating this process, we obtain

$$\gamma_{\times(\lceil \frac{\delta}{2} \rceil+1)}(G) \geq \gamma_{\times \lceil \frac{\delta}{2} \rceil}(G) + 1 \geq \ldots \geq \gamma_{\times 1}(G) + \left\lceil \frac{\delta}{2} \right\rceil = \gamma(G) + \left\lceil \frac{\delta}{2} \right\rceil.$$

By Inequality (8),

$$(n + \gamma_s(G))/2 \geq \gamma(G) + \left\lceil \frac{\delta}{2} \right\rceil.$$

This completes the proof of Claim 2.
The result now follows by Claim 1 and Claim 2. For sharpness consider the complete graph K_n. □

We conclude this section by establishing a lower bound on the signed domination number of a tree. Dunbar et al. [2] proved that for every tree of order $n \geq 2$, $\gamma_s(T) \geq (n + 4)/3$, and showed that this bound is sharp. We now present a lower bound on $\gamma_s(T)$, which is tighter than $(n + 4)/3$.

Theorem 3.4. *Let T be a tree of order $n \geq 2$ with ℓ leaves and s support vertices. Then*

$$\gamma_s(T) \geq \frac{\left(2\lceil \frac{\delta^*}{2} \rceil - 1\right) n + 2(\ell - s + 2)}{2 \lceil \frac{\delta^*}{2} \rceil + 1}$$

and this bound is sharp.

Proof. Let $f : V \to \{-1, 1\}$ be a SDF of T with weight $f(V(T)) = \gamma_s(T)$. If $V^- = \emptyset$, then $\gamma_s(T) = n$ and the result follows. Suppose that $V^- \neq \emptyset$, and $u \in V^-$. Root the tree T at vertex u. For each vertex $v \in V^-$, let P_v denote the set of vertices w satisfying (i) w belongs to V^+, (ii) w is a descendent of v, and (iii) each vertex of the v-w path of T, except v, is in V^+. We claim that the sets P_v, $v \in V^-$, partition

the set V^+.
1. Let $v \in V^-$. Since $f[v] \geq 1$ and T is a rooted tree at u, it follows that v has a child in V^+. Hence, $P_v \neq \emptyset$ by (i).
2. Let $v_1, v_2 \in V^-$ and $v_1 \neq v_2$. Then $P_{v_1} \cap P_{v_2} = \emptyset$ by (ii).
3. Let $w \in V^+$. Assume $v \in V^-$ is the closest vertex to w on the path u-w. Then $w \in P_v$ by (iii). This proves our claim.
Let S be the set of support vertices. We define

$$W_0 = \{v \in V^- \setminus \{u\} | P_v \cap S = \emptyset\}$$

and

$$W_1 = \{v \in V^- \setminus \{u\} | P_v \cap S \neq \emptyset\}.$$

Since $f(N[u]) \geq 1$, there are at least $\lceil \frac{\delta^*}{2} \rceil + 1$ children of u that belong to V^+. Moreover, each child of u has at least one child in V^+, itself. Therefore

$$|P_u| \geq 2\left(\left\lceil \frac{\delta^*}{2} \right\rceil + 1\right) - |S \cap P_u| + |L \cap P_u|, \tag{9}$$

where L is the set of leaves in T.
Clearly, $V^- \setminus \{u\} = W_0 \cup W_1$. Every vertex v in $V^- \setminus \{u\}$ has at least $\lceil \frac{\delta^*}{2} \rceil$ children in V^+ and each child has at least one child in V^+, itself. Hence,

$$|P_v| \geq 2\left\lceil \frac{\delta^*}{2} \right\rceil. \tag{10}$$

Now let $v \in W_1$. Note that each support vertex and all leaves adjacent to it belong to only one P_v, necessarily. Also in this process we have counted just one leaf for every support vertex. This implies that

$$\sum_{v \in W_1} |P_v| \geq 2\left\lceil \frac{\delta^*}{2} \right\rceil |W_1| - |S \cap \cup_{v \in W_1} P_v| + |L \cap \cup_{v \in W_1} P_v|. \tag{11}$$

Together inequalities (9), (10) and (11) lead to

$$\begin{aligned} |V^+| &= |P_u| + \textstyle\sum_{v \in W_0} |P_v| + \sum_{v \in W_1} |P_v| \\ &\geq 2\left(\left\lceil \frac{\delta^*}{2} \right\rceil + 1\right) + 2\left\lceil \frac{\delta^*}{2} \right\rceil |W_0| + 2\left\lceil \frac{\delta^*}{2} \right\rceil |W_1| + \ell - s. \end{aligned}$$

Using $|V^- \setminus \{u\}| = |W_0| + |W_1|$ we deduce that

$$|V^+| \geq 2\left(\left\lceil \frac{\delta^*}{2} \right\rceil + 1\right) + 2\left\lceil \frac{\delta^*}{2} \right\rceil (|V^-| - 1) + (\ell - s).$$

Now by the facts that $|V^+| = \dfrac{n + \gamma_s(G)}{2}$ and $|V^-| = \dfrac{n - \gamma_s(G)}{2}$ we obtain the desired lower bound. $\square$

Since
$$\frac{\left(2\left\lceil\frac{\delta^*}{2}\right\rceil-1\right)n+2(\ell-s+2)}{2\left\lceil\frac{\delta^*}{2}\right\rceil+1}\geq\frac{n+4+2(\ell-s)}{3},$$
we conclude the following lower bound as an immediate result.

Corollary 3.5. *Let T be a tree of order $n\geq 2$, with ℓ leaves and s support vertices. Then $\gamma_s(T)\geq\frac{n+4+2(\ell-s)}{3}$.*

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