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On unique minimum dominating sets in some repeated Cartesian products

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Abstract

Unique minimum dominating sets in the Cartesian product of a graph and a Hamming graph are considered. A characterization of such sets is given, when they exist. A necessary and sufficient condition for the existence of a unique minimum dominating set is given in the special case of the Cartesian product of a tree and multiple copies of the same complete graph.

1 Introduction

Unique minimum vertex dominating sets have been studied in many classes of graphs, including trees [4], block graphs [2], and cactus graphs [3]. In [6], the author considered unique minimum dominating sets in the Cartesian product of a graph and a complete graph. In particular, a necessary and sufficient condition for the existence of a unique minimum dominating set was given for the product of a tree and a complete graph.

In the present work, we continue this study by considering unique minimum dominating sets in graphs $G \,\Box\, K_{n_1} \,\Box\, K_{n_2} \,\Box\, \cdots \,\Box\, K_{n_m}$, where $K_{n_1}, K_{n_2}, \ldots, K_{n_m}$ denote the complete graphs on $n_1, n_2, \ldots, n_m$ vertices respectively. In Section 3, we first develop a characterization of the unique minimum dominating sets in such graphs when they exist. We then consider changing the cardinalities of the complete graphs, and show that the property of having a unique minimum dominating set is preserved when the cardinalities are decreased. In Section 4, we specialize to the case of $n_i = n_j$ for $i \neq j$, and prove a necessary and sufficient condition for the existence of a unique minimum dominating set in $T \,\Box K_n^m$ where T is a tree. We conclude by noting that unique minimum dominating sets in the Cartesian product of a tree and a hypercube can be considered by setting $n_i = 2$ for $1 \leq i \leq m$.

2 Notation

In our work to follow, G denotes a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. If $v \in V(G)$, then the *open neighborhood* of v is defined by $N(v) = \{u \mid uv \in E(G)\}$ while the *closed neighborhood of* v is defined by $N[v] = N(v) \cup \{v\}$. A vertex x of G dominates every vertex in $N[x]$. Given $S \subseteq V(G)$, the open neighborhood of S, denoted $N(S)$, is the set $\cup_{v \in S} N(v)$, while the closed neighborhood, denoted $N[S]$, is the set $S \cup N(S)$. If $S \subseteq V(G)$ satisfies $N[S] = V(G)$, then S is called a *dominating set*. The cardinality of a minimum dominating set is referred to as the *domination number* of G and is denoted by $\gamma(G)$, while a dominating set of minimum cardinality is referred to as a γ-set. If D is a dominating set of G and $x \in D$, then a *private neighbor of* x *with respect to* D is any vertex u that is dominated by x and by no other vertex of D, and if $u \neq x$, then u is called an *external private neighbor of* x *with respect to* D. For notational purposes, we let $epn(x, D)$ denote the set of external private neighbors of x with respect to D. We note that $epn(x, D)$ may be empty.

Given two graphs G_1 and G_2, their Cartesian product, denoted $G_1 \,\square\, G_2$, is the graph whose vertex set is the Cartesian product of the sets $V(G_1)$ and $V(G_2)$ with two vertices (u_1, u_2) and (v_1, v_2) in $G_1 \,\square\, G_2$ adjacent if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. The projections $\pi_{G_i} : V(G_1 \,\square\, G_2) \to V(G_i)$, for $i = 1$ and $i = 2$, defined by $\pi_{G_i}((u_1, u_2)) = u_i$ will be extensively used. Finally, for $(u_1, u_2) \in V(G_1 \,\square\, G_2)$, the G_i-layer through (u_1, u_2) is defined to be the induced subgraph

$$G_i^{(u_1,u_2)} = \langle \{(v_1, v_2) \,:\, \pi_{G_{3-i}}((v_1, v_2)) = \pi_{G_{3-i}}((u_1, u_2))\} \rangle.$$

We follow [5] for any other graph product terminology.

We consider graphs $G \,\square\, K_{n_1} \,\square\, K_{n_2} \,\square \cdots \square\, K_{n_m}$ where G is a connected, finite, simple graph, and where $K_{n_1}, K_{n_2}, \ldots, K_{n_m}$ are nontrivial complete graphs on $n_1, n_2, \ldots, n_m$ vertices respectively. We note, in passing, that a Cartesian product of complete graphs is called a Hamming graph. Thus, we are considering the Cartesian product of a graph with a Hamming graph. If a Cartesian product with G and m K_n-factors is performed, we simplify our notation to $G \,\square\, K_n^m$. In particular, note that K_2^m is the m-dimensional hypercube, denoted Q_m. We assume that the vertex set of K_n is $\{1, 2, \ldots, n\}$ which we denote by $[n]$. We denote by $\mathcal{U}$ the class of graphs G which have a unique minimum dominating set. Furthermore, if $G \in \mathcal{U}$, we let $UD(G)$ denote the unique γ-set of G.

3 Repeated Products

To begin our work with repeated products, we first recall three results: one from [4] and two from [6]. We note that the proof of Proposition 1 below is as it appears in [6]. We have included the proof here for completeness.

Lemma 1 ([4]). *Let G be a graph with a unique γ-set D. Let $[u,v]$ be any edge in G other than an edge connecting a vertex in D to one of its private neighbors. Let G^- be the graph obtained from G by deleting the edge $[u,v]$. Then G^- has D as the unique γ-set.*

Lemma 2 ([6]). *If $G \,\square\, K_n \in \mathcal{U}$, then there exists $S \subseteq V(G)$ such that $UD(G \,\square\, K_n) = S \times [n]$.*

Proposition 1 ([6]). *If $G \,\square\, K_n \in \mathcal{U}$, then $G \in \mathcal{U}$. Moreover, $G \,\square\, K_m \in \mathcal{U}$ for $1 \leq m \leq n$.*

Proof. Denote $UD(G \,\square\, K_n)$ by D. By Lemma 2, there exists $S \subseteq V(G)$ such that $D = S \times [n]$. Thus, for any $(x,i) \in D$, the external private neighbors of (x,i) with respect to D all belong to $G^{(x,i)}$. Define H to be the graph

$$G \,\square\, K_n - \{(v,n)(v,j) \,:\, v \in V(G), 1 \leq j \leq n-1\}.$$

We see that H is isomorphic to $(G \,\square\, K_{n-1}) \cup G$. By Lemma 1, D is still the unique γ-set for H. The proposition follows by induction. □

Taken together, Lemma 2 and the proof of Proposition 1 imply that if $G \,\square\, K_n \in \mathcal{U}$, then $\pi_G(UD(G \,\square\, K_n)) = UD(G)$. When considering repeated products, a similar statement holds.

Lemma 3. *If $G \,\square\, K_n^m \in \mathcal{U}$, then $UD(G \,\square\, K_n^m) = UD(G) \times V(K_n^m)$.*

Proof. As noted above, if $G \,\square\, K_n \in \mathcal{U}$, then $UD(G \,\square\, K_n) = UD(G) \times [n]$. Thus, we see that

$$UD(G \,\square\, K_n^m) = UD(G \,\square\, K_n^{m-1} \,\square\, K_n) = UD(G \,\square\, K_n^{m-1}) \times [n].$$

By induction, we see that $UD(G \,\square\, K_n^m) = UD(G) \times V(K_n^m)$. □

Since the Cartesian product is both commutative and associative, Proposition 1 gives us the following result.

Proposition 2. *If $G \,\square\, K_n^m \in \mathcal{U}$, then $G \,\square\, K_{n_1} \,\square\, K_{n_2} \,\square\, \cdots \,\square\, K_{n_r} \in \mathcal{U}$ for $1 \leq n_i \leq n$ and $1 \leq i \leq r \leq m$.*

Proof. Suppose that $G \,\square\, K_n^m \in \mathcal{U}$. By associativity, $(G \,\square\, K_n^{m-1}) \,\square\, K_n \in \mathcal{U}$. By Proposition 1, we then have that $(G \,\square\, K_n^{m-1}) \,\square\, K_{n_1} \in \mathcal{U}$ so long as $1 \leq n_1 \leq n$. By commutativity, we have that $(G \,\square\, K_{n_1}) \,\square K_n^{m-1} \in \mathcal{U}$. By induction, our result follows. □

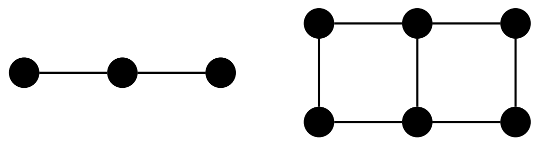
As a result of Proposition 2, in order to determine whether

$$G \,\square\, K_{n_1} \,\square\, K_{n_2} \,\square\, \cdots \,\square\, K_{n_r} \in \mathcal{U},$$

it *may* suffice to consider whether $G \,\square\, K_n^r \in \mathcal{U}$ where $n = \max\{n_1, n_2, \ldots, n_r\}$. Thus, we are motivated to define the following parameter.

Definition 1. *Let $G \in \mathcal{U}$ and let $U_n^{\square}(G)$ denote the integer m such that $G \,\square\, K_n^m \in \mathcal{U}$, but $G \,\square\, K_n^{m+1} \notin \mathcal{U}$. If $G \,\square\, K_n^m \notin \mathcal{U}$ for any $m \geq 1$, define $U_n^{\square}(G) = 0$, while if $G \,\square\, K_n^m \in \mathcal{U}$ for all $m \geq 1$, define $U_n^{\square}(G) = \infty$.*

As an illustration of this definition, consider the following examples. The graph $K_{1,2} \in \mathcal{U}$ but $K_{1,2} \,\square\, K_2 \notin \mathcal{U}$ (see Figure 1). Thus, $U_2^{\square}(K_{1,2}) = 0$. When we consider the graph $K_{1,3}$, we see that $K_{1,3} \,\square\, K_2 \in \mathcal{U}$ but $K_{1,3} \,\square\, K_2^2 \notin \mathcal{U}$. Hence, $U_2^{\square}(K_{1,3}) = 1$. Finally, when considering the graph $K_{1,4}$, we see that $K_{1,4} \,\square\, K_2^2 \in \mathcal{U}$, but $K_{1,4} \,\square\, K_2^3 \notin \mathcal{U}$. Thus, $U_2^{\square}(K_{1,4}) = 2$.



$K_{1,2}$ $\qquad$ $K_{1,2} \,\square\, K_2$

Figure 1: $K_{1,2} \in \mathcal{U}$ but $K_{1,2} \,\square\, K_2 \notin \mathcal{U}$

We now determine $U_n^{\square}(K_{1,p})$ for $n \geq 2$. For notational purposes, let $V(K_{1,p}) = \{0, 1, \ldots, p\}$ with 0 denoting the support vertex. Additionally, denote the vertices of K_n^m as strings of length m over the alphabet $[n]$. By the *jth cube* of $K_{1,p} \,\square\, K_n^m$, we mean the subgraph of $K_{1,p} \,\square\, K_n^m$ induced by $\{j\} \times V(K_n^m)$. The zeroth cube will be referred to as the *central cube*, while all other cubes will be referred to as the *outer cubes*.

Proposition 3. *If $2 \leq p \leq n$, then $U_n^{\square}(K_{1,p}) = 0$. If $p > n \geq 2$, then $U_n^{\square}(K_{1,p}) = \left\lfloor \frac{p-2}{n-1} \right\rfloor$.*

Proof. First, suppose that $2 \leq p \leq n$, and consider the graph $K_{1,p} \,\square\, K_n$. If $p < n$, then $V(K_{1,p}) \times \{1\}$ and $V(K_{1,p}) \times \{2\}$ are two distinct minimum dominating sets for $K_{1,p} \,\square\, K_n$. If $p = n$, then we see that the sets $\{0\} \times [n]$ and $\{(1,1), (2,2), \ldots, (p,p)\}$ are two distinct minimum dominating sets for $K_{1,p} \,\square\, K_n$. Thus, we see that $K_{1,p} \,\square\, K_n$ does not have a unique γ-set when $2 \leq p \leq n$, giving us the first part of our result.

Suppose then that $p > n$. Let $m = \lfloor \frac{p-2}{n-1} \rfloor$, and consider $K_{1,p} \,\square\, K_n^m$. Let D be the set $\{0\} \times V(K_n^m)$, and note that D is certainly a dominating set for $K_{1,p} \,\square\, K_n^m$. Suppose that D' is a γ-set for $K_{1,p} \,\square\, K_n^m$ and that for some $k > 0$, $|D - D'| = k$. In K_n^m, every vertex is of degree $(n-1)m$. Thus, D' contains at least $\lceil \frac{k}{(n-1)m+1} \rceil$ vertices from each of the p outer cubes of $K_{1,p} \,\square\, K_n^m$. Hence, we see that

$$|D'| \;\; \geq \;\; n^m - k + (p) \left\lceil \frac{k}{(n-1)m+1} \right\rceil .$$

Since $m < \frac{p-1}{n-1}$, we see that $(n-1)m + 1 < p$ in which case $(p) \left\lceil \frac{k}{(n-1)m+1} \right\rceil > k$. Hence $|D'| > n^m$, a contradiction. Thus, D is the unique γ-set for $K_{1,p} \,\square\, K_n^m$.

Now consider $K_{1,p} \,\Box\, K_n^{m+1}$. Once again, $D = \{0\} \times V(K_n^{m+1})$ is a dominating set for $K_{1,p} \,\Box\, K_n^{m+1}$. Construct a new set D' from D by deleting $(0, 11\cdots 1)$ and all of its neighbors in the central cube from D. Since $(m+1) \geq 2$, $|D'| > 0$. Thus, the only vertex of the central cube not dominated by D' is $(0, 11\cdots 1)$. Let $D'' = D' \cup \{(i, 11\cdots 1) \,|\, 1 \leq i \leq p\}$. D'' is a dominating set for $K_{1,p} \,\Box\, K_n^{m+1}$. Additionally, we see that

$$\begin{aligned} |D''| &= |D| - [1 + (n-1)(m+1)] + p \\ &\leq |D| - \left[1 + (n-1)\frac{p-1}{n-1}\right] + p \\ &= |D| - p + p \\ &= |D|. \end{aligned}$$

Hence, we have constructed a dominating set D'' distinct from D of cardinality at most $|D|$. Thus, $K_{1,p} \,\Box\, K_n^{m+1}$ cannot have a unique γ-set by Lemma 3. Our result now follows. $\Box$

Proposition 3 will be used in the section to follow. However, before we proceed, we note that Proposition 3 can be used to find a lower bound for $\gamma(Q_m)$. While the lower bound produced is not of practical value, it is nevertheless interesting that analysis of unique γ-sets could potentially be used to produce lower bounds for domination numbers that are otherwise difficult to obtain.

Corollary 1. *For $p \geq 2$, $\gamma(Q_{p-2}) \geq \frac{2^{p-2}}{p+1}$.*

Proof. Taking $n = 2$, Proposition 3 implies that $U_2^{\Box}(K_{1,p}) = p - 2$. That is, $K_{1,p} \,\Box\, Q_{p-2} \in \mathcal{U}$. Moreover, $|UD(K_{1,p} \,\Box\, Q_{p-2})| = 2^{p-2}$. Hence, if $\gamma(Q_{p-2}) < \frac{2^{p-2}}{p+1}$, then taking a γ-set of Q_{p-2} in each of the $p+1$ cubes of $K_{1,p} \,\Box\, Q_{p-2}$ would yield a dominating set of cardinality smaller than 2^{p-2}. Thus, our result follows. $\Box$

4 Trees

Proposition 3 provides us with the following result.

Lemma 4. *If $G \,\Box\, K_n^m \in \mathcal{U}$, then for all $v \in UD(G \,\Box\, K_n^m)$, $|epn(v, UD(G \,\Box\, K_n^m))| \geq m(n-1) + 2$.*

Proof. For notational convenience, let D denote the set $UD(G \,\Box\, K_n^m)$ and let D' denote the set $UD(G)$. Recall that by Lemma 3, $D = D' \times V(K_n^m)$. This implies that if $v \in D'$ with $epn(v, D') = \{p_1, p_2, \ldots, p_k\}$, then for all $x \in V(K_n^m)$, $(v, x) \in D$ with $epn((v, x), D) = \{(p_1, x), (p_2, x), \ldots, (p_k, x)\}$. For the sake of contradiction, suppose that $(u, w) \in D$ has $epn((u, w), D) = \{(p_1, w), (p_2, w), \ldots, (p_j, w)\}$ for some $j < m(n-1)+2$. Since $U_n^{\Box}(K_{1,j}) < m$, this implies that the subgraph of $G \,\Box\, K_n^m$ induced by $\{u, p_1, p_2, \ldots, p_j\} \times V(K_n^m)$ has a γ-set, call it B, distinct from $\{u\} \times V(K_n^m)$. In that case, $(D - (\{u\} \times V(K_n^m))) \cup B$ is a γ-set for $G \,\Box\, K_n^m$ distinct from D, a contradiction. $\Box$

Before proceeding to our main result, we recall one more theorem from [6].

Theorem 1 ([6]). *Let n be a positive integer and let T be a tree. The graph $T \square K_n \in \mathcal{U}$ if and only if T has a minimum dominating set D such that for all $v \in D$, $|epn(v, D)| \geq n + 1$.*

We are now able to classify the trees T for which $T \square K_n^m$ has a unique γ-set. For notational purposes, if $v \in V(T)$, then we let the *vth cube* of $T \square K_n^m$ denote the subgraph of $T \square K_n^m$ induced by $\{v\} \times V(K_n^m)$.

Theorem 2. *Let $n \geq 2$, $m \geq 1$, and let T be a tree. The Cartesian product $T \square K_n^m$ has a unique γ-set if and only if $T \square K_{m(n-1)+1}$ has a unique γ-set.*

Proof. First, suppose that $T \square K_n^m \in \mathcal{U}$. By Lemma 3, $UD(T \square K_n^m) = UD(T) \times V(K_n^m)$. By Lemma 4, we know that for each $v \in UD(T \square K_n^m)$, $|epn(v, UD(T \square K_n^m))| \geq m(n-1) + 2$. This implies that for each $w \in UD(T)$, $|epn(w, UD(T))| \geq m(n-1) + 2$. By Theorem 1, it follows that $T \square K_{m(n-1)+1}$ has a unique γ-set.

Now suppose that $T \square K_{m(n-1)+1} \in \mathcal{U}$. By Proposition 1 and Theorem 1, we see that T has a unique γ-set S so that every element in S has at least $m(n-1)+2$ external private neighbors with respect to S. Consider then $T \square K_n^m$. Note that the set $S \times V(K_n^m)$ is a dominating set for $T \square K_n^m$. We must show that it is a γ-set for $T \square K_n^m$, and that it is the unique γ-set for $T \square K_n^m$.

We proceed by induction on $\gamma(T)$. If $\gamma(T) = 1$, then T is a star $K_{1,p}$ with $p \geq m(n-1)+2$. By Proposition 3, we see that $T \square K_n^m$ has $UD(T) \times V(K_n^m)$ as its unique γ-set. Thus, suppose the result has been proven whenever $\gamma(T) < q$. Let T be a tree such that $\gamma(T) = q$ and such that $T \square K_{m(n-1)+1}$ has a unique γ-set. Let S be the unique γ-set for T. We know that for all $x \in S$, $|epn(x, S)| \geq m(n-1) + 2$. Consider a diametral path $x_1 x_2 \ldots x_{t-1} x_t x_{t+1}$ in T. Note that $x_t \in S$ and that $t \geq 3$.

Case One

First, suppose that $x_{t-1} \notin epn(x_t, S)$. In this case, since $|epn(x_t, S)| \geq m(n-1)+2$, we see that x_t is adjacent to at least $m(n-1)+2$ leaves. Thus, by the proof of Proposition 3, every vertex of the x_tth cube in $T \square K_n^m$ is selected for inclusion in every γ-set of $T \square K_n^m$. Let T' denote the tree obtained by removing x_t and all of its private neighbors with respect to S from T. Note that by Lemma 1, $T' \in \mathcal{U}$ with $UD(T') = S - \{x_t\}$. Additionally, observe that if $x \in S - \{x_t\}$, then $epn(x, S - \{x_t\}) \supseteq epn(x, S)$. Thus, by Theorem 1, we also see that $T' \square K_{m(n-1)+1} \in \mathcal{U}$. Since $\gamma(T') < \gamma(T)$, our induction hypothesis implies that $T' \square K_n^m \in \mathcal{U}$ and that $UD(T' \square K_n^m) = (S - \{x_t\}) \times V(K_n^m)$.

Suppose then that D is a γ-set for $T \square K_n^m$ and that $D \neq S \times V(K_n^m)$. By our observations above, we know that $\{x_t\} \times V(K_n^m) \subseteq D$. Let $B = D - (\{x_t\} \times V(K_n^m))$ and note that $B \subseteq V(T' \square K_n^m)$. If B dominates $T' \square K_n^m$, then since $UD(T' \square K_n^m) = (S - \{x_t\}) \times V(K_n^m)$ and since $B \neq (S - \{x_t\}) \times V(K_n^m)$, this implies that $|B| > |(S - \{x_t\}) \times V(K_n^m)|$. This, however, implies that $S \times V(K_n^m)$ is a smaller cardinality dominating set for $T \square K_n^m$, a contradiction.

Thus, assume that B does not dominate $T' \square K_n^m$. Since D is a dominating set of $T \square K_n^m$, this implies that B fails to dominate some subset of the x_{t-1}-cube in $T' \square K_n^m$. In particular, this implies that some subset of the x_{t-1}-cube is not contained in B. We consider two subcases.

Subcase One

Suppose that $x_{t-1} \notin S$.

- First, suppose that $N(x_{t-1}) = \{x_{t-2}, x_t\}$. Since $x_{t-1} \notin epn(x_t, S)$, this implies that $x_{t-2} \in S$. Apply Lemma 1 to T, and remove the edge $x_{t-2}x_{t-1}$. It follows that $T' - x_{t-1} \in \mathcal{U}$ and that $UD(T' - x_{t-1}) = S - \{x_t\}$. This further implies, by the same logic as above, that $(T' - x_{t-1}) \square K_n^m \in \mathcal{U}$ with unique γ-set given by $(S - \{x_t\}) \times V(K_n^m)$. Note that since B does not dominate all of the x_{t-1}-cube in $T' \square K_n^m$, this implies that B does not contain all of the x_{t-2}-cube.

  If B contains no vertices from the x_{t-1}-cube, then B is a dominating set for $(T' - x_{t-1}) \square K_n^m$ distinct from $(S - \{x_t\}) \times V(K_n^m)$. This contradicts our assumption that D was a γ-set for $T \square K_n^m$.

  Hence, we see that B contains some subset of the x_{t-1}-cube. Let $\{(x_{t-1}, p_1), (x_{t-1}, p_2), \ldots, (x_{t-1}, p_j)\} \subseteq B$. This implies that

  $$B \cap \{(x_{t-2}, p_1), (x_{t-2}, p_2), \ldots, (x_{t-2}, p_j)\} = \emptyset$$

  since otherwise D would not be a $\gamma-$set for $T \square K_n^m$. Thus, consider the set

  $$(B - \{(x_{t-1}, p_1), \ldots, (x_{t-1}, p_j)\}) \cup \{(x_{t-2}, p_1), \ldots, (x_{t-2}, p_j)\}.$$

  This is a dominating set for $(T' - x_{t-1}) \square K_n^m$ distinct from $(S - \{x_t\}) \times V(K_n^m)$, a contradiction.

- Now suppose that x_{t-1} is adjacent to a vertex, call it y, not on the diametral path. First, note that $y \in S$. If $y \notin S$, then since $x_{t-1} \notin S$, y would have a neighbor in S which, with its external private neighbors, could be used to create a longer path in T. In particular, any neighbors of x_{t-1} in T not on the diametral path are in S and have only leaf neighbors. Since our initial assumption was that each element of S has at least $m(n-1)+2$ external private neighbors, this implies that y has $m(n-1)+2$ leaf-neighbors in T. Hence, by the same logic as applied to x_t above, every vertex of the y-cube is contained in every γ-set for $T \square K_n^m$. However, this implies that $\{y\} \times V(K_n^m) \subseteq D$ which further implies that B dominates $T' \square K_n^m$, a contradiction.

Thus, in both cases, $x_{t-1} \notin S$ leads to a contradiction.

Subcase Two

Suppose now that $x_{t-1} \in S$. This implies that $|epn(x_{t-1}, S)| \geq m(n-1)+2$ by our earlier assumption. If x_{t-1} has an external private neighbor other than x_{t-2} that is not a leaf, then a longer path in T can be found. Hence, we see that x_{t-1} has at least

$m(n-1)+1$ leaf-neighbors in T, call them $l_1, l_2, \ldots, l_r$. Note that if $r \geq m(n-1)+2$, then every vertex of the x_{t-1}-cube is contained in every γ-set of $T \,\Box\, K_n^m$ implying that B is a dominating set for $T' \,\Box\, K_n^m$, a contradiction.

Thus, we see that x_{t-1} has exactly $m(n-1)+1$ leaf-neighbors and $x_{t-2} \in epn(x_{t-1}, S)$. Recall that some subset of the x_{t-1}-cube in $T \,\Box\, K_n^m$ is not contained in B. To be specific, assume k vertices of the x_{t-1}-cube are not contained in B. This implies that at least $\lceil \frac{k}{m(n-1)+1} \rceil$ vertices from each of the $l_1, l_2, \ldots, l_r$-cubes are contained in B. Additionally, the vertices in the x_{t-2}-cube that are adjacent to vertices in $(\{x_{t-1}\} \times V(K_n^m)) - B$ are dominated by vertices outside of the x_{t-1}-cube. Since

$$[m(n-1)+1] \cdot \lceil \frac{k}{m(n-1)+1} \rceil \geq k$$

we see that B contains exactly k vertices from the $l_1, l_2, \ldots, l_r$-cubes in total, since otherwise a smaller dominating set for $T \,\Box\, K_n^m$ could be constructed. Consider the set obtained from B by removing the k vertices from the $l_1, l_2, \ldots, l_r$-cubes and including the k missing vertices from the x_{t-1}-cube. This set is a dominating set for $T' \,\Box\, K_n^m$ distinct from $(S - \{x_t\}) \times V(K_n^m)$, a contradiction.

Case Two

Finally, suppose that $x_{t-1} \in epn(x_t, S)$. In this case, x_t is adjacent to at least $m(n-1)+1$ leaves, call them $l_1, l_2, \ldots, l_p$. Note that the only neighbors of x_{t-1} are x_t and x_{t-2}. If x_{t-1} had any other neighbors, either a longer path in T could be found, or x_{t-1} would not be an external private neighbor of x_t with respect to S.

Suppose that D is a γ-set of $T \,\Box\, K_n^m$ which does not contain k vertices of the x_tth cube. This implies that D contains at least $\lceil \frac{k}{(n-1)m+1} \rceil$ vertices from each of the $l_1, l_2, \ldots, l_p$-cubes. In fact, if $(m(n-1)+1)\lceil \frac{k}{(n-1)m+1} \rceil > k$, then we have reached a contradiction since a smaller dominating set for $T \,\Box\, K_n^m$ could be found simply by including every vertex of the x_tth cube. In particular, this implies that $(m(n-1)+1)\lceil \frac{k}{m(n-1)+1} \rceil = k$.

We now claim that D contains at least one vertex from the x_{t-1}-cube. To see this, first note that the tree T'' defined by $T'' = T - \{x_t, x_{t-1}, l_1, \ldots, l_p\}$ belongs to $\mathcal{U}$ with $UD(T'') = S - \{x_t\}$. Additionally, since $epn(x, S - \{x_t\}) = epn(x, S)$ for all $x \in S - \{x_t\}$, Theorem 1 implies that $T'' \,\Box\, K_{m(n-1)+1} \in \mathcal{U}$. Thus, our induction hypothesis implies that $T'' \,\Box\, K_n^m$ has a unique γ-set given by $(S - \{x_t\}) \times V(K_n^m)$. If no vertices from the x_{t-1}-cube are included in D, then

$$D \cap V(T'' \,\Box\, K_n^m) = (S - \{x_t\}) \times V(K_n^m).$$

This, however, results in at least k vertices of the x_{t-1}-cube being undominated by D since $x_{t-2} \notin S - \{x_t\}$. This is a contradiction.

Thus, D contains at least one vertex from the x_{t-1}-cube. If we "shift" these vertices to their corresponding positions in the x_{t-2}-cube, remove the vertices from D in the $l_1, l_2, \ldots, l_p$-cubes, and add in the missing vertices from the x_t-cube, we create a γ-set D' distinct from D which induces a γ-set distinct from $(S - \{x_t\}) \times V(K_n^m)$ on the subgraph $T'' \,\Box\, K_n^m$, a contradiction.

Hence, if D is a γ-set for $T \,\square\, K_n^m$, then every vertex of the x_t-cube is included in D. By the logic applied above, this implies that $S \times V(K_n^m)$ is the unique γ-set for $T \,\square\, K_n^m$.

Thus, we see that if $T \,\square\, K_{m(n-1)+1} \in \mathcal{U}$, then $T \,\square\, K_n^m \in \mathcal{U}$. □

Before we conclude, we note that Theorem 2, together with Theorem 1 above, imply the following corollary concerning hypercubes.

Corollary 2. *Let T be a tree on at least four vertices, and let $m \geq 1$. The following conditions are equivalent.*

- $T \,\square\, Q_m \in \mathcal{U}$.
- $T \,\square\, K_{m+1} \in \mathcal{U}$.
- *T has a γ-set D such that for all $v \in D$, $|epn(v, D)| \geq m + 2$.*

We note that a γ-set in a tree can be found in linear time (see [1]). Hence, the problem of determining for which m, $T \,\square\, Q_m \in \mathcal{U}$ can be solved in polynomial time.

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