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# Reducing the maximum degree of a graph by deleting vertices

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**Abstract**

We investigate the smallest number $\lambda(G)$ of vertices that need to be removed from a non-empty graph $G$ so that the resulting graph has a smaller maximum degree. We prove that if $n$ is the number of vertices, $k$ is the maximum degree, and $t$ is the number of vertices of degree $k$, then $\lambda(G) \leq \frac{n+(k-1)t}{2k}$. We also show that $\lambda(G) \leq \frac{n}{k+1}$ if $G$ is a tree. These bounds are sharp. We provide other bounds together with structural observations.

## 1 Introduction

Throughout this paper we shall use capital letters such as $X$ to denote sets or graphs, and small letters such as $x$ to denote non-negative integers or elements of a set. The set $\{1, 2, \dots\}$ of positive integers is denoted by $\mathbb{N}$. For any $n \in \mathbb{N}$, the set $\{1, \dots, n\}$ is denoted by $[n]$. For a set $X$, the set $\{\{x, y\} \colon x, y \in X, x \neq y\}$ of all 2-element subsets of $X$ is denoted by $\binom{X}{2}$. It is to be assumed that arbitrary sets are finite.

A *graph* $G$ is a pair $(X, Y)$, where $X$ is a set, called the *vertex set of* $G$, and $Y$ is a subset of $\binom{X}{2}$ and is called the *edge set of* $G$. The vertex set of $G$ and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. It is to be assumed that arbitrary graphs have non-empty vertex sets. An element of $V(G)$ is called a *vertex of* $G$, and an element of $E(G)$ is called an *edge of* $G$. We may represent an edge $\{v, w\}$ by $vw$. If $vw$ is an edge of $G$, then $v$ and $w$ are said to be *adjacent in* $G$, and we say that $w$ is a *neighbour of* $v$ *in* $G$ (and vice-versa). An edge $vw$ is said to be *incident to* $x$ if $x = v$ or $x = w$.

For any $v \in V(G)$, $N_G(v)$ denotes the set of neighbours of $v$ in $G$, $N_G[v]$ denotes $N_G(v) \cup \{v\}$ and is called the *closed neighbourhood of* $v$ *in* $G$, and $d_G(v)$ denotes $|N_G(v)|$ and is called the *degree of* $v$ *in* $G$. For $X \subseteq V(G)$, we denote $\bigcup_{v \in X} N_G(v)$ and $\bigcup_{v \in X} N_G[v]$ by $N_G(X)$ and $N_G[X]$, respectively. The *minimum degree of* $G$ is $\min\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\delta(G)$. The *maximum degree of* $G$ is

$\max\{d_G(v)\colon v \in V(G)\}$ and is denoted by $\Delta(G)$. If $G = (\emptyset, \emptyset)$, then we take both $\delta(G)$ and $\Delta(G)$ to be 0.

If $H$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then $H$ is said to be a *subgraph of* $G$, and we say that $G$ *contains* $H$. For $X \subseteq V(G)$, $(X, E(G) \cap \binom{X}{2})$ is called the *subgraph of* $G$ *induced by* $X$ and is denoted by $G[X]$. For a set $S$, $G - S$ denotes the subgraph of $G$ obtained by removing from $G$ the vertices in $S$ and all edges incident to them, that is, $G - S = G[V(G)\backslash S]$. We may abbreviate $G - \{v\}$ to $G - v$.

In this paper, we investigate the minimum number of vertices that need to be removed from a graph so that the new graph obtained has a smaller maximum degree.

Let $M(G)$ denote the set of vertices of $G$ of degree $\Delta(G)$. We call a subset $R$ of $V(G)$ a $\Delta$*-reducing set of* $G$ if $\Delta(G - R) < \Delta(G)$ or $V(G) = R$ (note that $V(G)$ is the smallest $\Delta$-reducing set of $G$ if and only if $\Delta(G) = 0$). Note that $R$ is a $\Delta$-reducing set of $G$ if and only if $M(G) \subseteq N_G[R]$. Let $\lambda(G)$ denote the size of a smallest $\Delta$-reducing set of $G$.

We provide several bounds for $\lambda(G)$. Our main results are given in the next section. Before stating our results, we need further definitions and notation.

For $A, D \subseteq V(G)$, we say that $D$ *dominates* $A$ *in* $G$ if for every $v \in A$, $v$ is in $D$ or $v$ has a neighbour in $G$ that is in $D$. Note that $D$ dominates $M(G)$ in $G$ if and only if $D$ is a $\Delta$-reducing set of $G$. Thus $\lambda(G) = \min\{|D|\colon D \text{ dominates } M(G) \text{ in } G\}$.

A *dominating set of* $G$ is a set that dominates $V(G)$ in $G$. The *domination number of* $G$, denoted by $\gamma(G)$, is the size of a smallest dominating set of $G$.

We now define some special graphs and important concepts.

If $n \geq 2$ and $v_1, v_2, \ldots, v_n$ are the distinct vertices of a graph $G$ with $E(G) = \{v_i v_{i+1}\colon i \in [n-1]\}$, then $G$ is called a $v_1v_n$*-path* or simply a *path*. The path $([n], \{\{1,2\}, \ldots, \{n-1, n\}\})$ is denoted by $P_n$. For a path $P$, the *length of* $P$, denoted by $l(P)$, is $|V(P)| - 1$ (the number of edges of $P$).

For $u, v \in V(G)$, the *distance of* $v$ *from* $u$, denoted by $d_G(u, v)$, is given by

$$d_G(u,v) = \begin{cases} 0 & \text{if } u = v; \\ \min\{l(P)\colon P \text{ is a } uv\text{-path, } G \text{ contains } P\} & \text{if } G \text{ contains a } uv\text{-path;} \\ \infty & \text{if } G \text{ contains no } uv\text{-path.} \end{cases}$$

A graph $G$ is *connected* if for every $u, v \in V(G)$ with $u \neq v$, $G$ contains a $uv$-path. A *component* of $G$ is a maximal connected subgraph of $G$ (that is, one that is not a subgraph of any other connected subgraph of $G$). It is easy to see that if $H$ and $K$ are distinct components of a graph $G$, then $H$ and $K$ have no common vertices (and therefore no common edges). If $G_1, \ldots, G_r$ are the distinct components of $G$, then we say that $G$ is the *disjoint union of* $G_1, \ldots, G_r$.

If $n \geq 3$ and $v_1, v_2, \ldots, v_n$ are the distinct vertices of a graph $G$ with $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$, then $G$ is called a *cycle*. The cycle $([n], \{\{1,2\}, \ldots, \{n-1, n\}, \{n, 1\}\})$ is denoted by $C_n$.

A graph $G$ is a *tree* if $G$ is a connected graph that contains no cycles. A graph $G$ is a *star* if $E(G) = \{uv\colon v \in V(G)\backslash\{u\}\}$ for some $u \in V(G)$. Thus a star is a tree.

The star $(\{0\} \cup [n], \{\{0, i\} : i \in [n]\})$ is denoted by $K_{1,n}$.

A graph $G$ is *complete* if every two vertices of $G$ are adjacent (that is, $E(G) = \binom{V(G)}{2}$). A graph $G$ is *empty* if no two vertices of $G$ are adjacent (that is, $E(G) = \emptyset$).

A graph $G$ is *regular* if the degrees of its vertices are the same. If $k \in \{0\} \cup \mathbb{N}$ and the degree of each vertex of $G$ is $k$, then $G$ is called *$k$-regular*.

Let $H$ be a graph. A graph $G$ is a *copy of $H$* if there exists a bijection $f : V(G) \to V(H)$ such that $E(H) = \{f(u)f(v) : uv \in E(G)\}$.

We are now ready to state our main results, given in the next section. In Section 3, we investigate $\lambda(G)$ from a structural point of view, particularly observing how this parameter changes with the removal of vertices. Some of the structural results are then used in the proofs of the main results; these proofs are given in Section 4.

## 2 Bounds

Our first result is a lower bound for $\lambda(G)$.

**Proposition 2.1** *For any graph $G$,*

$$\lambda(G) \geq \frac{|M(G)|}{\Delta(G) + 1}.$$

**Proof.** Let $k = \Delta(G)$. For any $X \subseteq V(G)$, we have $|N_G[X]| \leq \sum_{v \in X} |N_G[v]| \leq (k + 1)|X|$. Let $S$ be a $\Delta$-reducing set of $G$ of size $\lambda(G)$. Since $M(G) \subseteq N_G[S]$, $|M(G)| \leq |N_G[S]| \leq (k + 1)|S| = (k + 1)\lambda(G)$. The result follows. □

The bound above is sharp; for example, it is attained by complete graphs.

We now provide a number of upper bounds for $\lambda(G)$.

**Proposition 2.2** *For any non-empty graph $G$,*

$$\lambda(G) \leq \min \left\{ |M(G)|, \gamma(G), \frac{|E(G)|}{\Delta(G)} \right\}.$$

**Proof.** Obviously, $G - M(G)$ has no vertex of degree $\Delta(G)$. Thus $\lambda(G) \leq |M(G)|$.

Let $D$ be a dominating set of $G$. Since every vertex in $V(G) \backslash D$ is adjacent to some vertex in $D$, $d_{G-D}(v) \leq d_G(v) - 1 \leq \Delta(G) - 1$ for each $v \in V(G - D)$. Thus $\lambda(G) \leq |D|$. Consequently, $\lambda(G) \leq \gamma(G)$.

Since $G$ is non-empty, $\Delta(G) > 0$. Let $v_1$ be a vertex of $G$ of degree $\Delta(G)$. If $\Delta(G - v_1) = \Delta(G)$, then let $v_2, \dots, v_r$ be distinct vertices of $G$ such that $\Delta(G - \{v_1, \dots, v_r\}) < \Delta(G)$ and $d_{G-\{v_1,\dots,v_{i-1}\}}(v_i) = \Delta(G)$ for each $i \in [r] \backslash \{1\}$. If $\Delta(G - v_1) < \Delta(G)$, then let $r = 1$. Let $R = \{v_1, \dots, v_r\}$. By the choice of $v_1, \dots, v_r$, no two vertices in $R$ are adjacent. Thus $|E(G - R)| = |E(G)| - r\Delta(G)$, and hence $|E(G)| \geq r\Delta(G)$. Therefore, we have $\lambda(G) \leq |R| = r \leq \frac{|E(G)|}{\Delta(G)}$. □

Let $\overline{d}(G)$ denote the average degree $\frac{1}{|V(G)|}\sum_{v\in V(G)} d_G(v)$ of $G$. Proposition 2.2 and the handshaking lemma ($\overline{d}(G)|V(G)| = 2|E(G)|$) give us

$$\lambda(G) \le \frac{\overline{d}(G)|V(G)|}{2\Delta(G)}. \tag{1}$$

It immediately follows that $\lambda(G) \le \frac{1}{2}|V(G)|$. In Section 4, we characterize the cases in which the bound $\frac{1}{2}|V(G)|$ is attained.

**Theorem 2.3** *For any non-empty graph $G$,*

$$\lambda(G) \le \frac{|V(G)|}{2},$$

*and equality holds if and only if $G$ is either a disjoint union of copies of $K_2$ or a disjoint union of copies of $C_4$.*

The subsequent new theorems in this section are also proved in Section 4. The following sharp bound is our primary contribution.

**Theorem 2.4** *If $G$ is a non-empty graph, $n = |V(G)|$, $k = \Delta(G)$ and $t = |M(G)|$, then*

$$\lambda(G) \le \frac{n + (k-1)t}{2k}.$$

We point out four facts regarding Theorem 2.4. The first is that it immediately implies (1). Indeed, let $S = \{v \in V(G) \colon d_G(v) = 0\}$, $G' = G - S$ and $n' = |V(G')|$; then $\lambda(G') \le \frac{n'+(k-1)t}{2k} = \frac{kt+n'-t}{2k} \le \frac{1}{2k}\sum_{v\in V(G')} d_G(v) = \frac{1}{2k}\sum_{v\in V(G)} d_G(v) = \frac{\overline{d}(G)n}{2k}$.

Secondly, the bound in Theorem 2.4 can be attained in cases where $\lambda(G) = t$ and also in cases where $\lambda(G) < t$. If $G$ is a disjoint union of $t$ copies of $K_{1,k}$, then $\lambda(G) = t$, $n = (k+1)t$, and hence $\lambda(G) = \frac{n+(k-1)t}{2k}$. If $G$ is one of the extremal structures in Theorem 2.3, then $t = n$ and $\lambda(G) = \frac{n}{2} = \frac{n+(k-1)t}{2k}$.

Thirdly, it is immediate from the proof of Theorem 2.4 that the inequality in the result is strict if the closed neighbourhood of some vertex of $G$ contains at least 3 members of $M(G)$; see (7).

Fourthly, since $\lambda(G) \le t$, Theorem 2.4 is not useful if $t \le \frac{n+(k-1)t}{2k}$. This occurs if and only if $t \le \frac{n}{k+1}$. Thus, if $t \le \frac{n+(k-1)t}{2k}$, then $\lambda(G) \le \frac{n}{k+1}$. We have

$$\lambda(G) \le \max\left\{\frac{n}{k+1}, \frac{n+(k-1)t}{2k}\right\}, \tag{2}$$

and if $\frac{n}{k+1} < \frac{n+(k-1)t}{2k}$ and $k \ge 2$, then $n < (k+1)t$ and $\lambda(G) \le \frac{n+(k-1)t}{2k} < t$.

It turns out that if $G$ is a tree, then, although we may have $\frac{n}{k+1} < \frac{n+(k-1)t}{2k}$ (that is, $n < (k+1)t$, as in the case of trees that are paths with at least 4 vertices), $\lambda(G) \le \frac{n}{k+1}$ holds.

**Theorem 2.5** *For any tree $T$,*

$$\lambda(T) \leq \frac{|V(T)|}{\Delta(T)+1}.$$

The bound is sharp; for example, it is attained by stars.

By Proposition 2.2, any upper bound for $\gamma(G)$ is an upper bound for $\lambda(G)$. Domination is widely studied and several bounds are known for $\gamma(G)$; see [4]. The following well-known domination bound of Reed [9] gives us $\lambda(G) \leq \frac{3}{8}|V(G)|$ when $\delta(G) \geq 3$.

**Theorem 2.6 ([9])** *If $G$ is a graph with $\delta(G) \geq 3$, then*

$$\gamma(G) \leq \frac{3}{8}|V(G)|.$$

Arnautov [3], Payan [8] and Lovász [7] independently proved that

$$\gamma(G) \leq \left(\frac{1+\ln(\delta(G)+1)}{\delta(G)+1}\right)n. \tag{3}$$

Alon and Spencer [2] gave a probabilistic proof using Alon's well-known argument in [1]. By adapting the argument to our problem of dominating $M(G)$ rather than all of $V(G)$, we prove the following improved bound for $\lambda(G)$, replacing in particular $\delta(G)$ by $\Delta(G)$.

**Theorem 2.7** *If $G$ is a graph, $n = |V(G)|$, $k = \Delta(G)$ and $t = |M(G)|$, then*

$$\lambda(G) \leq \frac{n\ln(k+1)+t}{k+1}.$$

We conclude this section with a brief discussion on regular graphs. If $G$ is regular, then $M(G) = V(G)$, and hence $\lambda(G) = \gamma(G)$. For a regular graph $G$, Theorem 2.7 is given by (3) as $\delta(G) = \Delta(G)$. Kostochka and Stodolsky [6] obtained an improvement of the bound in Theorem 2.6 for 3-regular graphs.

**Theorem 2.8 ([6])** *If $G$ is a connected 3-regular graph with $|V(G)| \geq 9$, then*

$$\gamma(G) \leq \frac{4}{11}|V(G)|.$$

Also, they showed in [5] that there exists an infinite class of connected 3-regular graphs $G$ with $\gamma(G) > \left\lceil \frac{|V(G)|}{3} \right\rceil > \left\lceil \frac{|V(G)|}{\Delta(G)+1} \right\rceil$. This means that the lower bound in Proposition 2.1 is not always attained by regular graphs, and that the bound in Theorem 2.5 does not extend to the class of regular graphs. For regular graphs $G$ with $\Delta(G) \leq 2$, the problem is trivial. Indeed, if such a graph $G$ is connected, then either $G$ has only one edge or $G$ is a cycle. It is easy to check that $\{1+3t \colon 1+3t \in [n]\}$ is a $\Delta$-reducing set of $C_n$ of minimum size, and hence $\lambda(C_n) = \lceil \frac{n}{3} \rceil = \left\lceil \frac{|V(C_n)|}{\Delta(C_n)+1} \right\rceil$.

## 3 Structural results

In this section, we provide some observations on how $\lambda(G)$ is affected by the structure of $G$ and by removing vertices or edges from $G$. Some of the following facts are used in the proofs of our main results.

**Lemma 3.1** *If $G$ is a graph, $H$ is a subgraph of $G$ with $\Delta(H) = \Delta(G)$, and $R$ is a $\Delta$-reducing set of $G$, then $R \cap V(H)$ is a $\Delta$-reducing set of $H$.*

**Proof.** Let $S = R \cap V(H)$. Consider any $v \in M(H)$. Since $\Delta(H) = \Delta(G)$, $v \in M(G)$ and $N_H[v] = N_G[v]$. Since $v \in M(G)$, $u \in N_G[v]$ for some $u \in R$. Since $N_H[v] = N_G[v]$, $u \in N_H[v]$. Thus $u \in V(H)$, and hence $u \in S$. Thus $v \in N_H[S]$. The result follows. □

We point out that having $|R| = \lambda(G)$ in Lemma 3.1 does not guarantee that $|R \cap V(H)| = \lambda(H)$. Indeed, let $k \geq 2$, let $G_1$ and $G_2$ be copies of $K_{1,k}$ such that $V(G_1) \cap V(G_2) = \emptyset$, let $G$ be the disjoint union of $G_1$ and $G_2$, let $e$ be an edge of $G_2$, and let $H = (V(G), E(G)\backslash\{e\})$. For each $i \in [2]$, let $v_i$ be the vertex of $G_i$ of degree $k$. Let $R = \{v_1, v_2\}$. Then $R$ is a $\Delta$-reducing set of $G$ of size $\lambda(G)$, $\{v_1\}$ is a $\Delta$-reducing set of $H$, but $R \cap V(H) = R$.

**Proposition 3.2** *If $G$ is a graph and $G_1, \dots, G_r$ are the distinct components of $G$ whose maximum degree is $\Delta(G)$, then $\lambda(G) = \sum_{i=1}^{r} \lambda(G_i)$.*

**Proof.** Let $R$ be a $\Delta$-reducing set of $G$ of size $\lambda(G)$, and let $R_i = R \cap V(G_i)$ for each $i \in [r]$. Then $R_1, \dots, R_r$ partition $R$, so $|R| = \sum_{i=1}^{r} |R_i|$. By Lemma 3.1, $\lambda(G_i) \leq |R_i|$ for each $i \in [r]$. Suppose $\lambda(G_j) < |R_j|$ for some $j \in [r]$. Let $R'_j$ be a $\Delta$-reducing set of $G_j$ of size $\lambda(G_j)$. Then $R'_j \cup \bigcup_{i \in [r]\backslash\{j\}} R_i$ is a $\Delta$-reducing set of $G$ that is smaller than $R$, a contradiction. Therefore, $\lambda(G_i) = |R_i|$ for each $i \in [r]$. Thus we have $\lambda(G) = |R| = \sum_{i=1}^{r} |R_i| = \sum_{i=1}^{r} \lambda(G_i)$. □

**Proposition 3.3** *If $H$ is a subgraph of a graph $G$ such that $\Delta(H) = \Delta(G)$, then $\lambda(H) \leq \lambda(G)$.*

**Proof.** Let $R$ be a $\Delta$-reducing set of $G$ of size $\lambda(G)$. Let $S = R \cap V(H)$. By Lemma 3.1, $\Delta(H - S) < \Delta(G)$. Thus we have $\lambda(H) \leq |S| \leq |R| = \lambda(G)$. □

**Proposition 3.4** *If $G$ is a graph, $v \in V(G)$ and $v \notin N_G[M(G)]$, then $\lambda(G - v) = \lambda(G)$.*

**Proof.** By Proposition 3.3, $\lambda(G - v) \leq \lambda(G)$. Let $R$ be a $\Delta$-reducing set of $G - v$ of size $\lambda(G - v)$. Since $v \notin N_G[M(G)]$, $M(G - v) = M(G)$. Thus $R$ is a $\Delta$-reducing set of $G$, and hence $\lambda(G) \leq \lambda(G - v)$. Hence $\lambda(G - v) = \lambda(G)$. □

**Proposition 3.5** *If $v$ is a vertex of a graph $G$, then $\lambda(G) \leq 1 + \lambda(G - v)$.*

**Proof.** If $\Delta(G - v) < \Delta(G)$, then $\lambda(G) = 1$. Suppose $\Delta(G - v) = \Delta(G)$, so $M(G-v) \subseteq M(G)$. Let $R$ be a $\Delta$-reducing set of $G - v$ of size $\lambda(G - v)$. For any $x \in M(G)\backslash M(G-v)$, $x \in N_G[v]$. Thus $R \cup \{v\}$ is a $\Delta$-reducing set of $G$. The result follows. □

Define $M_1(G) = \{v \in M(G)\colon d_G(v, w) \leq 2 \text{ for some } w \in M(G)\backslash\{v\}\}$ and $M_2(G) = M(G)\backslash M_1(G)$. Thus $M_2(G) = \{v \in M(G)\colon d_G(v, w) \geq 3 \text{ for each } w \in M(G)\backslash\{v\}\}$.

**Proposition 3.6** *For a graph $G$, $\lambda(G) = |M(G)|$ if and only if $M_2(G) = M(G)$.*

**Proof.** Suppose $\lambda(G) = |M(G)|$ and $M_2(G) \neq M(G)$. Then $M_1(G) \neq \emptyset$. Let $v \in M_1(G)$. Then $d_G(v, w) \leq 2$ for some $w \in M(G)\backslash\{v\}$. Thus $N_G[v] \cap N_G[w] \neq \emptyset$. Let $x \in N_G[v] \cap N_G[w]$. Then $(M(G)\backslash\{v, w\}) \cup \{x\}$ is a $\Delta$-reducing set of $G$ of size $|M(G)| - 1$, a contradiction. Therefore, if $\lambda(G) = |M(G)|$, then $M_2(G) = M(G)$.

Conversely, suppose $M_2(G) = M(G)$. Let $R$ be a $\Delta$-reducing set of $G$ of size $\lambda(G)$. Then $M(G) \subseteq N_G[R]$ and $N_G[v] \cap M(G) \neq \emptyset$ for each $v \in R$. Suppose $|N_G[v] \cap M(G)| \geq 2$ for some $v \in R$. Let $x, y \in N_G[v] \cap M(G)$ with $x \neq y$. Since $x, y \in N_G[v]$, we obtain $d_G(x, y) \leq 2$, which contradicts $x, y \in M_2(G)$. Thus $|N_G[v] \cap M(G)| = 1$ for each $v \in R$. Since $M(G) \subseteq N_G[R]$, $M(G) = M(G) \cap N_G[R] = M(G) \cap \bigcup_{v \in R} N_G[v] = \bigcup_{v \in R}(N_G[v] \cap M(G))$. Thus we have $|M(G)| \leq \sum_{v \in R} |N_G[v] \cap M(G)| = \sum_{v \in R} 1 = |R|$. By Proposition 2.2, $|R| \leq |M(G)|$. Hence $|R| = |M(G)|$. □

**Proposition 3.7** *If $G$ is a graph with $M_2(G) \neq M(G)$, then $\Delta(G - M_2(G)) = \Delta(G)$ and $\lambda(G) = |M_2(G)| + \lambda(G - M_2(G))$.*

**Proof.** We use induction on $|M_2(G)|$. The result is trivial if $|M_2(G)| = 0$. Suppose $|M_2(G)| \geq 1$. Let $x \in M_2(G)$. Since $M_2(G) \neq M(G)$, $M_1(G) \neq \emptyset$. Thus we clearly have $\Delta(G - x) = \Delta(G)$, $M_1(G - x) = M_1(G)$ and $M_2(G - x) = M_2(G)\backslash\{x\} \neq M(G - x)$. By the induction hypothesis, $\lambda(G - x) = |M_2(G - x)| + \lambda((G - x) - M_2(G - x)) = |M_2(G)| - 1 + \lambda(G - (\{x\} \cup M_2(G - x))) = |M_2(G)| - 1 + \lambda(G - M_2(G))$. By Proposition 3.5, $\lambda(G) \leq 1 + \lambda(G - x)$. Suppose $\lambda(G) \leq \lambda(G - x)$. Let $R$ be a $\Delta$-reducing set of $G$ of size $\lambda(G)$. Then $x \in N_G[y]$ for some $y \in R$. Since $x \in M_2(G)$, $y \notin N_G[z]$ for each $z \in M(G)\backslash\{x\}$ (because otherwise we obtain $d_G(x, z) \leq 2$, a contradiction). We obtain that $R\backslash\{y\}$ is a $\Delta$-reducing set of $G - x$ of size $\lambda(G) - 1 \leq \lambda(G - x) - 1$, a contradiction. Thus $\lambda(G) = 1 + \lambda(G - x) = |M_2(G)| + \lambda(G - M_2(G))$. □

## 4 Proofs of the main results

We now prove Theorems 2.3, 2.4, 2.5 and 2.7.

**Proof of Theorem 2.3.** Let $n = |V(G)|$ and $k = \Delta(G)$. Since $G$ is non-empty, $k > 0$. By (1), $\lambda(G) \leq \frac{n}{2}$. It is straightforward that if $G$ is either a disjoint union of copies of $K_2$, or a disjoint union of copies of $C_4$, then $\lambda(G) = \frac{n}{2}$. We now prove the converse. Thus, suppose $\lambda(G) = \frac{n}{2}$. Then, by (1), $G$ is $k$-regular. Let $G_1, \dots, G_r$ be the distinct components of $G$. Consider any $i \in [r]$.

Applying the established bound to each of $G_1, \dots, G_r$, we have $\lambda(G_j) \le \frac{|V(G_j)|}{2}$ for each $j \in [r]$. Together with Proposition 3.2, this gives us $\sum_{j=1}^{r} \frac{|V(G_j)|}{2} \ge \sum_{j=1}^{r} \lambda(G_j) = \lambda(G) = \frac{n}{2} = \sum_{j=1}^{r} \frac{|V(G_j)|}{2}$, and hence $\lambda(G_j) = \frac{|V(G_j)|}{2}$ for each $j \in [r]$.

Suppose $k \ge 3$. Since $G$ is $k$-regular, $G_i$ is $k$-regular. Thus we have $\delta(G_i) = k \ge 3$, $\lambda(G_i) = \gamma(G_i)$, and hence, by Theorem 2.6, $\lambda(G_i) \le \frac{3|V(G_i)|}{8} < \frac{|V(G_i)|}{2}$, a contradiction.

Therefore, $k \le 2$. If $k = 1$, then $G_i$ is a copy of $K_2$. Suppose $k = 2$. Clearly, a 2-regular graph can only be a cycle. Thus, for some $p \ge 3$, $G_i$ is a copy of $C_p$. As pointed out in Section 2, $\lambda(C_p) = \lceil \frac{p}{3} \rceil$. Since $\lambda(C_p) = \lambda(G_i) = \frac{|V(G_i)|}{2} = \frac{p}{2}$, it follows that $p = 4$. The result follows. □

For any $m, n \in \{0\} \cup \mathbb{N}$, we denote $\{i \in \{0\} \cup \mathbb{N} \colon m \le i \le n\}$ by $[m, n]$. Note that $[m, n] = \emptyset$ if $m > n$.

**Proof of Theorem 2.4.** Since $G$ is non-empty, $k > 0$. Let $r = \lambda(G)$ and $G_1 = G$. Let $R$ be a $\Delta$-reducing set of $G$ of size $r$. We remove from $G_1$ a vertex $v_1$ in $R$ whose closed neighbourhood in $G_1$ contains the largest number of vertices in $M(G_1)$, and we denote the resulting graph $G_1 - v_1$ by $G_2$. If $r \ge 2$, then we remove from $G_2$ a vertex $v_2$ in $R \backslash \{v_1\}$ whose closed neighbourhood in $G_2$ contains the largest number of vertices in $M(G_2)$, and we denote the resulting graph $G_2 - v_2$ by $G_3$. If $r \ge 3$, then we remove from $G_3$ a vertex $v_3$ in $R \backslash \{v_1, v_2\}$ whose closed neighbourhood in $G_3$ contains the largest number of vertices in $M(G_3)$, and we denote the resulting graph $G_3 - v_3$ by $G_4$. Continuing this way, we obtain $v_1, \dots, v_r$ and $G_1, \dots, G_{r+1}$ such that $R = \{v_1, \dots, v_r\}$, $G_{r+1} = G - R$, $\Delta(G_i) = k$ for each $i \in [r]$ (since $|R| = r = \lambda(G)$), $\Delta(G_{r+1}) < k$ and

$$M(G) = \bigcup_{i=1}^{r} (N_{G_i}[v_i] \cap M(G_i)). \tag{4}$$

For each $i \in [r]$, let $A_i = N_{G_i}[v_i] \cap M(G_i)$. The members $v_1, \dots, v_r$ of $R$ have been labelled in such a way that

$$|A_1| \ge \cdots \ge |A_r|. \tag{5}$$

For every $i, j \in [r]$ with $i < j$, each member of $A_i \cap V(G_j)$ is of degree at most $k - 1$ in $G_j$ (as its neighbour $v_i$ in $G_i$ is not in $V(G_j)$), and hence

$$A_i \cap A_j = \emptyset. \tag{6}$$

Let $I_3 = \{i \in [r] \colon |A_i| \ge 3\}$, $I_2 = \{i \in [r] \colon |A_i| = 2\}$ and $I_1 = \{i \in [r] \colon |A_i| = 1\}$. Let $r_1 = |I_1|$, $r_2 = |I_2|$ and $r_3 = |I_3|$. Then $r = r_1 + r_2 + r_3$. By (5), we have $I_3 = [1, r_3]$, $I_2 = [r_3 + 1, r_3 + r_2]$ and $I_1 = [r_3 + r_2 + 1, r_3 + r_2 + r_1] = [r - r_1 + 1, r]$. Let $H = G_{r - r_1 + 1}$.

Suppose $r_1 = 0$. Then $I_2 \cup I_3 = [r]$. By (4), $M(G) = \bigcup_{i \in I_2 \cup I_3} A_i$. By (6), it follows that $t = \sum_{i \in I_2 \cup I_3} |A_i| \ge \sum_{i \in I_2 \cup I_3} 2 = 2r$, and hence $r \le \frac{t}{2} \le \frac{n + (k-1)t}{2k}$.

Now suppose $r_1 \neq 0$. Then $\Delta(H) = k$. By construction, $\{v_i\colon i \in I_1\}$ is a $\Delta$-reducing set of $H$, and $M(H) = \bigcup_{i \in I_1} A_i$. If we assume that $H$ has a $\Delta$-reducing set $S$ of size less than $|I_1|$, then we obtain that $(R \backslash \{v_i\colon i \in I_1\}) \cup S$ is a $\Delta$-reducing set of $G$ of size less than $|R|$, a contradiction. Thus $\lambda(H) = |I_1|$. Together with $M(H) = \bigcup_{i \in I_1} A_i$, (6) gives us $|M(H)| = \sum_{i \in I_1} |A_i| = |I_1|$. By Proposition 3.6, $M(H) = M_2(H)$. For each $i \in I_1$, let $z_i$ be the unique element of $A_i$. By (6), $z_i \neq z_j$ for every $i, j \in I_1$ with $i \neq j$. Since $M_2(H) = M(H) = \bigcup_{i \in I_1} A_i$, $M_2(H) = \{z_i\colon i \in I_1\}$. By definition of $M_2(H)$, it follows that for every $i, j \in I_1$ with $i \neq j$,

$$N_H[z_i] \cap N_H[z_j] = \emptyset.$$

Therefore,

$$|\bigcup_{i \in I_1} N_H[z_i]| = \sum_{i \in I_1} |N_H[z_i]| = (k+1)|I_1| = (k+1)r_1.$$

Let $R' = (R \backslash \{v_i\colon i \in I_1\}) \cup M(H)$. Since $|M(H)| = |I_1| = \lambda(H)$ (and $M(H)$ is a $\Delta$-reducing set of $H$), $R'$ is a $\Delta$-reducing set of $G$ of size $\lambda(G)$.

Let $B_1 = \bigcup_{i \in I_1} N_H[z_i]$, $B_2 = \{v_i\colon i \in I_2\}$ and $B_3 = \{v_i\colon i \in I_3\}$. Then $|B_1| = (k+1)r_1$, $|B_2| = r_2$ and $|B_3| = r_3$.

Suppose that there exists $j \in I_2$ such that $A_j \subseteq B_1 \cup B_2 \cup B_3$. Let $w_1$ and $w_2$ be the two members of $A_j$. Let $C = \{v_i\colon i \in I_2, i \geq j\}$. We have $w_1, w_2 \in V(G_j) = V(G) \backslash \{v_i\colon i \in [1, j-1]\}$, so $w_1, w_2 \in B_1 \cup C$. We have $w_1, w_2 \in N_{G_j}[v_j]$ and $d_{G_j}(w_1) = d_{G_j}(w_2) = k$.

Suppose $v_j = w_1$. Since $w_1, w_2 \in B_1 \cup C$, we have $w_2 \in B_1 \cup (C \backslash \{v_j\})$. Suppose $w_2 \in B_1$. Then $w_2 \in N_H[z_i]$ for some $i \in I_1$. Since $A_j \cup \{z_i\} = \{v_j, w_2, z_i\} \subseteq N_{G_j}[w_2]$, we obtain that $(R' \backslash \{v_j, z_i\}) \cup \{w_2\}$ is a $\Delta$-reducing set of $G$ of size $|R'| - 1$, which contradicts $|R'| = \lambda(G)$. Thus $w_2 \in C \backslash \{v_j\}$, meaning that $w_2 = v_i$ for some $i \in I_2$ such that $i > j$. From this we obtain that $R' \backslash \{v_j\}$ is a $\Delta$-reducing set of $G$ of size $|R'| - 1$, a contradiction.

Therefore, $v_j \neq w_1$. Similarly, $v_j \neq w_2$. If we assume that $w_1, w_2 \in C$, then we obtain that $R' \backslash \{v_j\}$ is a $\Delta$-reducing set of $G$ of size $|R'| - 1$, a contradiction. Therefore, at least one of $w_1$ and $w_2$ is in $B_1$; we may assume that $w_1 \in B_1$. Thus $w_1 \in N_H[z_i]$ for some $i \in I_1$. If we assume that $w_2 \in C$, then we obtain that $R' \backslash \{v_j\}$ is a $\Delta$-reducing set of $G$ of size $|R'| - 1$, a contradiction. Thus $w_2 \in B_1$, and hence $w_2 \in N_H[z_h]$ for some $h \in I_1$. From this we obtain that $R' \backslash \{v_j\}$ is a $\Delta$-reducing set of $G$ of size $|R'| - 1$, a contradiction.

Therefore, $A_i \nsubseteq B_1 \cup B_2 \cup B_3$ for each $i \in I_2$. For each $i \in I_2$, let $x_i \in A_i \backslash (B_1 \cup B_2 \cup B_3)$. Let $B_4 = \{x_i\colon i \in I_2\}$. Thus $B_4 \cap (B_1 \cup B_2 \cup B_3) = \emptyset$. Since $B_1, B_2$ and $B_3$ are pairwise disjoint (by construction), it follows that $|\bigcup_{i=1}^4 B_i| = \sum_{i=1}^4 |B_i|$. By (6), $x_i \neq x_j$ for every $i, j \in I_2$ with $i \neq j$. Thus $|B_4| = r_2$.

By (4) and (6), the sets $A_1, \ldots, A_r$ partition $M(G)$. Thus $t = \sum_{i=1}^r |A_i| \geq 3r_3 + 2r_2 + r_1 = 2r_3 + r_2 + r$, and hence $-r_3 - r_2 \geq r - t + r_3$.

We have

$$\begin{aligned}
n &\geq |\bigcup_{i=1}^{4} B_i| = \sum_{i=1}^{4} |B_i| = r_3 + 2r_2 + (k+1)r_1 = r_3 + 2r_2 + (k+1)(r - r_3 - r_2) \\
&= (k+1)r + (k-1)(-r_3 - r_2) - r_3 \geq (k+1)r + (k-1)(r - t + r_3) - r_3 \\
&= 2kr - (k-1)t + (k-2)r_3,
\end{aligned}$$

and hence

$$r \leq \frac{n + (k-1)t - (k-2)r_3}{2k}. \tag{7}$$

If $k = 1$, then $r_3 = 0$. Thus $(k-2)r_3 \geq 0$, and hence $r \leq \frac{n+(k-1)t}{2k}$. □

We now prove Theorem 2.5, making use of the following well-known fact.

**Lemma 4.1** *Let $x$ be a vertex of a tree $T$. Let $m = \max\{d_T(x,y) \colon y \in V(T)\}$, and let $D_i = \{y \in V(T) \colon d_T(x,y) = i\}$ for each $i \in \{0\} \cup [m]$. For each $i \in [m]$ and each $v \in D_i$, $N_G(v) \cap \bigcup_{j=0}^{i} D_j = \{u\}$ for some $u \in D_{i-1}$.*

Indeed, let $v \in D_i$. By definition of $D_i$, $v$ can only be adjacent to vertices of distance $i-1$, $i$ or $i+1$ from $x$. If $v$ is adjacent to a vertex $w$ of distance $i$, then, by considering an $xv$-path and an $xw$-path, we obtain that $T$ contains a cycle, which is a contradiction. We obtain the same contradiction if we assume that $v$ is adjacent to two vertices of distance $i-1$ from $x$.

If a vertex $v$ of a graph $G$ has only one neighbour in $G$, then $v$ is called a *leaf of $G$*.

**Corollary 4.2** *If $T$ is a tree, $x, z \in V(T)$ and $d_T(x,z) = \max\{d_T(x,y) \colon y \in V(T)\}$, then $z$ is a leaf of $T$.*

**Proof.** Let $D_0, D_1, \ldots, D_m$ be as in Lemma 4.1. Then $z \in D_m$. By Lemma 4.1, $N_G(z) = \{u\}$ for some $u \in D_{m-1}$. □

**Proof of Theorem 2.5.** Let $n = |V(T)|$ and $k = \Delta(T)$. The result is trivial for $n \leq 2$. We now proceed by induction on $n$. Thus consider $n \geq 3$. Since $T$ is a connected graph, we clearly have $k \geq 2$.

Suppose that $T$ has a leaf $z$ whose neighbour is not in $M(T)$. Then $M(T - z) = M(T)$ and, by Proposition 3.4, $\lambda(T - z) = \lambda(T)$. By the induction hypothesis, $\lambda(T - z) \leq \frac{n-1}{k+1} < \frac{n}{k+1}$. Thus $\lambda(T) < \frac{n}{k+1}$.

Now suppose that each leaf of $T$ is adjacent to a vertex in $M(T)$. Let $x$, $m$ and $D_0, D_1, \ldots, D_m$ be as in Lemma 4.1. Let $z \in D_m$. By Corollary 4.2, $z$ is a leaf of $T$. Let $w$ be the neighbour of $z$. Then $w \in M(T)$. By Lemma 4.1, $w \in D_{m-1}$.

Suppose $w = x$. Then $m = 1$ and $E(T) = \{xz_1, \ldots, xz_k\}$ for some distinct vertices $z_1, \ldots, z_k$ of $T$. Thus $\{x\}$ is a $\Delta$-reducing set of $T$, and hence $\lambda(T) = 1 = \frac{n}{k+1}$.

Now suppose $w \neq x$. Together with Lemma 4.1, this implies that $N_T(w) = \{v, z_1, \ldots, z_{k-1}\}$ for some $v \in D_{m-2}$ and some distinct vertices $z_1, \ldots, z_{k-1}$ in $D_m$.

By Corollary 4.2, $z_1, \dots, z_{k-1}$ are leaves of $T$. Let $T' = T - v$. Then each component of $T'$ is a tree. Let $\mathcal{K}$ be the set of components of $T'$ whose maximum degree is $k$, and let $\mathcal{H}$ be the set of components of $T'$ whose maximum degree is less than $k$. Let $W = \{w, z_1, \dots, z_{k-1}\}$. Note that $(W, \{wz_1, \dots, wz_{k-1}\})$ is in $\mathcal{H}$, and hence $W \cap \bigcup_{K \in \mathcal{K}} V(K) = \emptyset$. If $\mathcal{K} = \emptyset$, then $\{v\}$ is a $\Delta$-reducing set of $T$, and hence $\lambda(T) = 1 \leq \frac{n}{k+1}$. Suppose $\mathcal{K} \neq \emptyset$. For each $K \in \mathcal{K}$, let $S_K$ be a $\Delta$-reducing set of $K$ of size $\lambda(K)$. By the induction hypothesis, $|S_K| \leq \frac{|V(K)|}{k+1}$ for each $K \in \mathcal{K}$. Now $\{v\} \cup \bigcup_{K \in \mathcal{K}} S_K$ is a $\Delta$-reducing set of $T$. Therefore, we have

$$\lambda(T) \leq 1 + \sum_{K \in \mathcal{K}} |S_K| \leq \frac{|W \cup \{v\}|}{k+1} + \sum_{K \in \mathcal{K}} \frac{|V(K)|}{k+1} \leq \frac{n}{k+1},$$

as required. $\Box$

**Proof of Theorem 2.7.** We may assume that $V(G) = [n]$. Let $p = \frac{\ln(k+1)}{k+1}$. We set up $n$ independent random experiments, and in each experiment a vertex is chosen with probability $p$. More formally, for each $i \in V$, let $(\Omega_i, P_i)$ be the probability space given by $\Omega_i = \{0, 1\}$, $P_i(\{1\}) = p$ and $P_i(\{0\}) = 1 - p$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$, and let $P : 2^\Omega \to [0, 1]$ such that $P(\{\omega\}) = \prod_{i=1}^n P_i(\{\omega_i\})$ for each $\omega = (\omega_1, \dots, \omega_n) \in \Omega$, and $P(A) = \sum_{\omega \in A} P(\{\omega\})$ for each $A \subseteq \Omega$. Then $(\Omega, P)$ is a probability space.

For each $\omega = (\omega_1, \dots, \omega_n) \in \Omega$, let $S_\omega$ be the subset of $V(G)$ such that $\omega$ is the characteristic vector of $S_\omega$ (that is, $S_\omega = \{i \in [n] \colon \omega_i = 1\}$), let $T_\omega$ be the set of vertices in $M(G)$ that are neither in $S_\omega$ nor adjacent to a vertex in $S_\omega$ (that is, $T_\omega = \{v \in M(G) \colon v \notin N_G[S_\omega]\}$), and let $D_\omega = S_\omega \cup T_\omega$. Then $D_\omega$ is a $\Delta$-reducing set of $G$.

Let $X, Y : \Omega \to \mathbb{R}$ be the random variables given by $X(\omega) = |S_\omega|$ and $Y(\omega) = |T_\omega|$. For each $i \in [n]$, let $X_i : \Omega \to \mathbb{R}$ be the indicator random variable for whether vertex $i$ is in $S_\omega$; that is, for each $\omega = (\omega_1, \dots, \omega_n) \in \Omega$,

$$X_i(\omega) = \begin{cases} 1 & \text{if } i \in S_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in M(G)$, let $Y_i : \Omega \to \mathbb{R}$ be the indicator random variable for whether vertex $i$ is in $T_\omega$; that is, for each $\omega = (\omega_1, \dots, \omega_n) \in \Omega$,

$$Y_i(\omega) = \begin{cases} 1 & \text{if } i \in T_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

We have $X = \sum_{i=1}^n X_i$ and $Y = \sum_{i \in M(G)} Y_i$.

For each $i \in [n]$, $P(X_i = 1) = P_i(\{1\}) = p$. For each $i \in M(G)$,

$$\begin{aligned} P(Y_i = 1) &= P(\{\omega \in \Omega \colon \omega_j = 0 \text{ for each } j \in N_G[i]\}) \\ &= \prod_{j \in N_G[i]} P_j(\{0\}) = (1-p)^{|N_G[i]|} = (1-p)^{k+1}. \end{aligned}$$

For any random variable $Z$, let $\mathrm{E}[Z]$ denote the expected value of $Z$. By linearity of expectation,

$$\begin{aligned}\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] &= \sum_{i=1}^{n} \mathrm{E}[X_i] + \sum_{i \in M(G)} \mathrm{E}[Y_i] \\ &= \sum_{i=1}^{n} P(X_i = 1) + \sum_{i \in M(G)} P(Y_i = 1) = np + t(1-p)^{k+1}.\end{aligned}$$

By the probabilistic pigeonhole principle, there exists $\omega^* \in \Omega$ such that $X(\omega^*) + Y(\omega^*) \leq np + t(1-p)^{k+1}$. Since $X(\omega^*) + Y(\omega^*) = |S_{\omega^*}| + |T_{\omega^*}| = |D_{\omega^*}|$ and $(1-p)^{k+1} \leq e^{-p(k+1)}$, $|D_{\omega^*}| \leq np + te^{-p(k+1)} = \frac{n\ln(k+1)}{k+1} + te^{-\ln(k+1)} = \frac{n\ln(k+1)}{k+1} + \frac{t}{k+1}$. $\square$

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## References

[1] N. Alon, Transversal numbers of uniform hypergraphs, *Graphs Combin.* 6 (1990), 1–4.

[2] N. Alon and J. H. Spencer, *The Probabilistic Method*, Third Edition, Wiley, New York, 2008.

[3] V. I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices, *Prikl. Mat. i Programmirovanie* 11 (1974), 3–8.

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] A. V. Kostochka and B. Y. Stodolsky, On domination in connected cubic graphs, *Discrete Math.* 304 (2005), 45–50.

[6] A. V. Kostochka and B. Y. Stodolsky, An upper bound on the domination number of $n$-vertex connected cubic graphs, *Discrete Math.* 309 (2009), 1142–1162.

[7] L. Lovász, On the ratio of optimal and integral fractional covers, *Discrete Math.* 13 (1975), 383–390.

[8] C. Payan, Sur le nombre d'absorption d'un graphe simple, *Cahiers Centre Études Recherche Opér.* 17 (1975), 307–317.

[9] B. Reed, Paths, stars, and the number three, *Combin. Probab. Comput.* 5 (1996), 277–295.

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