Some Orthogonal Matrices Constructed by Strong Kronecker Multiplication

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Abstract

Strong Kronecker multiplication of two matrices is useful for constructing new orthogonal matrices from those known. These results are particularly important as they allow small matrices to be combined to form larger matrices, but of smaller order than the straight-forward Kronecker product would permit.

1 Introduction and Basic Definitions

Throughout this paper we use the following notation:

Notation 1 *Write* $\epsilon = \{1, -1, i, -i\}$, $X = \{x_1, \cdots, x_u, 0\}$, $Y = \{y_1, \cdots, y_v, 0\}$, $Z = \{xy \mid x \in X, y \in Y\}$, *where* $x_1, \cdots, x_u, y_1, \cdots, y_v$ *are real commuting variables, in the other words, the complex conjugate of* x_i (y_j) *is* x_i (y_j). *Let* $\Re = \{\alpha x \mid \alpha \in \epsilon, x \in X\}$, $\Im = \{\beta y \mid \beta \in \epsilon, y \in Y\}$, $\mho = \{\gamma xy \mid \gamma \in \epsilon, x \in X, y \in Y\}$. *Further we write* $\varphi = \sum_{j=1}^{u} s_j x_j^2$, $\psi = \sum_{j=1}^{v} q_j y_j^2$, *where* s_i *and* q_j *are positive integers.*

Definition 1 Let C be a $(1, -1, i, -i, 0)$ matrix of order c, satisfying $CC^* = rI$, where C^* is the Hermitian conjugate of C. We call C a *complex weighing matrix* order c and weight r, denoted by $CW(c, r)$. In particular, if C is a real matrix, we call C a *weighing matrix* denoted by $W(c, r)$. $CW(c, c)$ is called a *complex Hadamard matrix* of order c.

From Wallis [10, p.275] any complex Hadamard matrix has order 1 or order divisible by 2. Let $C = X + iY$, where X, Y consist of $1, -1, 0$ and $X \wedge Y = 0$ where

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$\wedge$ is the Hadamard product. Clearly, if C is a $CW(c,r)$ then $XX^T + YY^T = rI$, $XY^T = YX^T$.

Definition 2 *A complex orthogonal design* (see Geramita and Geramita [6]), of order n and type $(s_1, \cdots, s_u)$, denoted by $COD(m; s_1, s_2, \cdots, s_u)$ on the commuting variables $x_1, \cdots, x_u$ is a matrix of order n, say A , with elements from $\Re$, satisfying

$$AA^* = \varphi I_n.$$

In particular, if A has elements from only X, the complex orthogonal will be called an *orthogonal design* denoted by $OD(m; s_1, s_2, \cdots, s_u)$.

Note $AA^* = \varphi I_n$ implies $A^*AA^*A = \varphi A^*AI_n$ and then $A^*A = \varphi I_n$, we assert that if A is a $COD(m; s_1, s_2, \cdots, s_u)$ with elements from $\Re$ then $\pm x_j, \pm ix_j$, totally, occur s_j times in each row and column.

Let M be a matrix of order tm. Then M can be expressed as

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ & & \vdots & \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{bmatrix}$$

where M_{ij} is of order m $(i,j = 1, 2, \cdots, t)$. Analogously with Seberry and Yamada [8], we call this a t^2 *block M-structure* when M is an orthogonal matrix. Let N be a matrix of order tn. Then, write

$$N = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1t} \\ N_{21} & N_{22} & \cdots & N_{2t} \\ & & \cdots & \\ N_{t1} & N_{t2} & \cdots & N_{tt} \end{bmatrix}$$

where N_{ij} is of order n $(i,j = 1, 2, \cdots, t)$. We now define the operation $\bigcirc$ as the following:

$$M \bigcirc N = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ & & \cdots & \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix}$$

where M_{ij}, N_{ij} and L_{ij} are of order of m, n and mn, respectively and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

where $\times$ is Kronecker product, $i, j = 1, 2, \cdots, t$. We call this the *strong Kronecker* multiplication of two matrices.

The aim is to construct new orthogonal designs from those known previously. The most popular method has been the Kronecker product, so that if there exist Hadamard matrices of order $4h$ and $4n$ then there exists an Hadamard matrix of order $16hn$. Agayan [1] gave an important improvement: if there exist Hadamard matrices of order $4h$ and $4n$ then there exists an Hadamard matrix of order $8hn$. Craigen [2] introduced orthogonal pairs and disjoint weighing matrices. Seberry and Zhang [9] defined the strong Kronecker multiplication and used it to construct some orthogonal matrices. Craigen, Seberry and Zhang [3] then combined their results to show that if there exist Hadamard matrices of order $4m$, $4n$, $4p$, $4q$ then there exists an Hadamard matrix of order $8mnpq$, an extension of Agayan's result mentioned above.

In this paper, we systematically study constructions for various orthogonal matrices with special properties, including CODs, ODs, CWs, and weighing matrices, by using strong Kronecker multiplication.

2 Strong Kronecker Product

Jennifer Seberry and Xian-Mo Zhang have proved in [9]

Theorem 1 *(Strong Kronecker Product Lemma) Let $A = (A_{ij})$ satisfy $AA^T = \varphi I_{tm}$, where A_{ij} have order m and $B = (B_{ij})$ satisfy $BB^T = \psi I_{tn}$, where B_{ij} have order n then*

$$(A \bigcirc B)(A \bigcirc B)^T = \varphi\psi I_{tmn}.$$

(If A and B are orthogonal designs $A \bigcirc B$ is not an orthogonal design but an orthogonal matrix.)

We now give Theorem 1 a more general form.

Theorem 2 *Let $A = (A_{ij})$ with elements from $\Re$ satisfy $AA^* = \varphi I_{tm}$, where A_{ij} have order m and $B = (B_{ij})$ with elements from $\Im$ satisfy $BB^* = \psi I_{tn}$, where B_{ij} have order n. Then if $C = A \bigcirc B$*

$$CC^* = (A \bigcirc B)(A \bigcirc B)^* = \varphi\psi I_{tmn}.$$

($C = A \bigcirc B$ is not a complex orthogonal design but a complex orthogonal matrix.)

Proof. Proceed as in the proof of [9, Theorem 1]. We need only to replace every transpose operation for matrices in the proof for [9, Theorem 1] to the Hermitian conjugate. Note all the equalities are still valid after this replacement.

Corollary 1 *Let $A = CW(tm, p)$, and $B = CW(tn, q)$. Then, writing $C = A \bigcirc B$, $CC^* = pqI_{tmn}$.*

Proof. The orthogonality follows immediately from the theorem. □

The strong Kronecker multiplication has the potential to yield still more constructions for new orthogonal matrices as has been shown by de Launey and Seberry [4].

3 Conferred Amicability Theorem

We first proved the following lemmas and theorems for $t = 2$ but de Launey and Seberry [4] have since discovered the result is true for any t.

Lemma 1 *(Structure Lemma) Let $A = (A_{kj})$, $C = (C_{kj})$ be matrices of order tm with elements from $\Im$, where A_{kj}, C_{kj} are of order m and $B = (B_{kj})$, $D = (D_{kj})$ be matrices of order tn with elements from $\Re$, where B_{kj}, D_{kj} are of order n. Write $(A \bigcirc B)(C \bigcirc D)^* = (L_{ab})$, where $a, b = 1, \cdots, t$ then*

$$L_{ab} = \sum_{s=1}^{t}\sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}C_{bk}^* \times B_{js}D_{ks}^*.$$

In particular, if $C = A$ and $D = B$

$$L_{ab} = \sum_{s=1}^{t}\sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times B_{js}B_{ks}^*.$$

Now if B is orthogonal with $BB^* = \psi I_{tn}$, where ψ is defined in Notation 1, then

$$L_{ab} = (\sum_{j=1}^{t} A_{aj}A_{bj}^*) \times \psi I_{mn}.$$

Further if A is orthogonal with $AA^* = \varphi I_{tn}$, where φ is defined in Notation 1, then $L_{ab} = 0$, for $a \neq b$ and $L_{aa} = \varphi\psi I_{mn}$.

Proof. It is easy to calculate $L_{ab} =$

$$= \sum_{s=1}^{t}(A_{a1} \times B_{1s} + A_{a2} \times B_{2s} + \cdots + A_{at} \times B_{ts})(C_{a1}^* \times D_{1s}^* + C_{a2}^* \times D_{2s}^* + \cdots + C_{at}^* \times D_{ts}^*)$$

$$= \sum_{s=1}^{t}\sum_{j=1}^{t}\sum_{k=1}^{t}(A_{aj} \times B_{js})(C_{bk}^* \times D_{ks}^*)$$

$$= \sum_{s=1}^{t}\sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}C_{bk}^* \times B_{js}D_{ks}^*.$$

Obviously, if $C = A$ and $D = B$

$$L_{ab} = \sum_{s=1}^{t}\sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times B_{js}B_{ks}^*.$$

Further if B is orthogonal, $\sum_{j=1}^{t} B_{js}B_{ks}^* = 0$, for $j \neq k$ so

$$L_{ab} = \sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times (\sum_{s=1}^{t} B_{js}B_{ks}^*) = \sum_{s=1}^{t} A_{aj}A_{bk}^* \times \psi I_n.$$

So $L_{ab} = 0$, $a \neq b$ and $L_{aa} = \varphi\psi I_{mn}$. □

Theorem 3 *(Conferred Amicability Theorem) Suppose $A = (A_{kj})$ is a matrix of order tm with elements from $\Re$, where A_{kj} is of order m and $B = (B_{kj})$ and $C = (C_{kj})$ are matrices of order tn with elements from $\Im$, where B_{kj} and C_{kj} are of order n. Write $P = A \bigcirc B$ and $Q = A \bigcirc C$. Suppose $BC^* = CB^*$. Then P, Q are amicable i.e. $PQ^* = QP^*$.*

Proof. Let $PQ^* = (L_{ab})$ and $QP^* = (R_{ab})$, where $a, b = 1, \cdots, t$. By the Structure Lemma,

$$L_{ab} = \sum_{s=1}^{t}\sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times B_{js}C_{ks}^* = \sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times (\sum_{s=1}^{t} B_{js}C_{ks}^*).$$

Similarly,

$$R_{ab} = \sum_{s=1}^{t}\sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times C_{js}B_{ks}^* = \sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times (\sum_{s=1}^{t} C_{js}B_{ks}^*).$$

Note $BC^* = CB^*$ implies $\sum_{s=1}^{t} B_{js}C_{ks}^* = \sum_{s=1}^{t} C_{js}B_{ks}^*$, $j, k = 1, \cdots, t$. So $L_{ab} = R_{ab}$ and $PQ^* = QP^*$. □

We say matrices A and B *annihilate* one another if $AB^* = 0$.

Corollary 2 *(Conferred Annihilation) Suppose $A = (A_{kj})$ is a matrix of order tm with elements from $\Re$, where A_{kj} is of order m and $B = (B_{kj})$ and $C = (C_{kj})$ are matrices of order tn with elements from $\Im$, where B_{kj} and C_{kj} are of order n. Write $P = A \bigcirc B$ and $Q = A \bigcirc C$. Suppose $BC^* = 0$. Then $PQ^* = 0$.*

Proof. Let $PQ^* = (L_{ab})$ and $QP^* = (R_{ab})$, where $a, b = 1, \cdots, t$. By the Structure Lemma,

$$L_{ab} = \sum_{s=1}^{t}\sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times B_{js}C_{ks}^* = \sum_{j=1}^{t}\sum_{k=1}^{t} A_{aj}A_{bk}^* \times (\sum_{j=1}^{t} B_{js}C_{ks}^*).$$

Note $BC^* = 0$ implies $\sum_{j=1}^{t} B_{js}C_{ks}^* = 0$, $j, k = 1, \cdots, t$. So $L_{ab} = 0$ and then $PQ^* = 0$, also $QP^* = 0$. □

The Conferred Amicability Theorem is useful for constructing some orthogonal designs with special properties.

4 Using $COD(2n; s_1, \cdots, s_u)$

Theorem 4 *Let A be a $COD(2a; s_1, \cdots, s_u)$ with elements from $\Re$ and B be a $COD(2b; q_1, \cdots, q_v)$ with elements from $\Im$. If $A = (A_{ij})$ with blocks of order a has the additional property that $A_{ij} \wedge A_{ik} = 0$ or $A_{ji} \wedge A_{ki} = 0$, $j \neq k$, $i = 1, 2$, then there exist four matrices with elements from $\mho$, of order $2ab$, P, Q, U, V, satisfying*

(i) $PQ^* = QP^*$, $PP^* = QQ^* = \varphi\psi I_{2ab}$,

(ii) $UU^* + VV^* = \varphi\psi I_{2ab}$, $U \wedge V = 0$, $UV^* = VU^* = 0$, $U + V = P$, $U - V = Q$.

Proof. Let $B = (B_{ij})$, where B_{ij} is of order b.

Case 1, $A_{ij} \wedge A_{ik} = 0$, $j \neq k$, $i = 1, 2$. Set $P = A \bigcirc B$ and

$$Q = A \bigcirc \begin{bmatrix} B_{11} & B_{12} \\ -B_{21} & -B_{22} \end{bmatrix}.$$

Then both P and Q are of order $2ab$. By Theorem 2, we have

$$PP^* = QQ^* = \varphi\psi I_{2ab}.$$

By the orthogonality of B, $B \begin{bmatrix} B_{11} & B_{12} \\ -B_{21} & -B_{22} \end{bmatrix}^* = \begin{bmatrix} B_{11} & B_{12} \\ -B_{21} & -B_{22} \end{bmatrix} B^*$. Using the Conferred Amicability Theorem, we have $PQ^* = QP^*$. Note both P and Q have only entries in the form of γxy, where $\gamma \in \epsilon$, $x \in \Re$, $y \in \Im$ because of the additional property of A.

Set $U = A \bigcirc \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix}$ and $V = A \bigcirc \begin{bmatrix} 0 & 0 \\ B_{21} & B_{22} \end{bmatrix}$. By the orthogonality of B,

$$\begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B_{21} & B_{22} \end{bmatrix}^* = 0.$$

Clearly, both U and V have only elements from $\mho$. Using Corollary 3, we prove $UV^* = 0$ and then $VU^* = 0$. Finally, note $UU^* + VV^* = (U + V)(U + V)^* = PP^* = \varphi\psi I_{2ab}$.

Case 2, $A_{ji} \wedge A_{ki} = 0$, $j \neq k$, $i = 1, 2$. Then A^* satisfies case 1. □

Corollary 3 *Suppose there exist a $COD(2n; s_1, \cdots, s_u)$ with elements from $\Re$ and a $W(2h, r) = A = (A_{ij})$ with blocks of order h which has the additional property that $A_{ij} \wedge A_{ik} = 0$ or $A_{ji} \wedge A_{ki} = 0$, $j \neq k$, $i = 1, 2$ then there exist*

(i) two $COD(2hn; rs_1, \cdots, rs_u)$, P and Q, satisfying $PQ^* = QP^*$,

(ii) two matrices with elements from $\Re$, of order $2hn$, U and V, satisfying $UU^* + VV^* = r\varphi I_{2hn}$, $U \wedge V = 0$, $UV^* = VU^* = 0$, $U + V = P$, $U - V = Q$.

Corollary 3 is a powerful method for constructing new CODs. For example, a $W(22, 9)$ constructed from two circulant matrices with the first rows, +00000+0-0- and 0+0++00+0-0, respectively (Seberry [5, p.333] satisfies the additional property, mentioned in Corollary 3. Using the above $W(22, 9)$ and $OD(12;\ 3,\ 3,\ 3,\ 3)$, we construct $OD(12 \cdot 11;\ 3 \cdot 9,\ 3 \cdot 9,\ 3 \cdot 9,\ 3 \cdot 9)$. Using the result repeatedly we have $OD(12 \cdot 11^k;\ 3 \cdot 9^k,\ 3 \cdot 9^k,\ 3 \cdot 9^k,\ 3 \cdot 9^k)$, similarly, $OD(20 \cdot 11^k;\ 5 \cdot 9^k,\ 5 \cdot 9^k,\ 5 \cdot 9^k,\ 5 \cdot 9^k)$ and $OD(36 \cdot 11^k;\ 9^{k+1},\ 9^{k+1},\ 9^{k+1},\ 9^{k+1})$, where $k = 0, 1, \cdots$.

Corollary 4 *Suppose there exist a complex Hadamard matrix of order* $2c$ *and a* $CW(2h,s) = A = (A_{ij})$ *with blocks of order* h *which satisfy* $A_{ij} \wedge A_{ik} = 0$ *or* $A_{ji} \wedge A_{ki} = 0$, $j \neq k$, $i = 1, 2$, *then there exist*

(i) two $CW(2ch, 2cs)$, P *and* Q, *satisfying* $PQ^* = QP^*$,

(ii) two $(1,-1,0)$ *matrices,* U *and* V *of order* $2ch$, *satisfying* $UU^* + VV^* = 2csI_{2ch}$, $U \wedge V = 0$, $UV^* = VU^* = 0$, $U + V = P$, $U - V = Q$.

For example, let $2g = 2, 10, 26$. Then Golay sequences may be used to obtain $W(2h, g)$ with the additional property mentioned in Corollary 4, for all $h > g$. On the other hand, from [7, Corollary 18], there exists a complex Hadamard matrix of order $p^j(p+1)$, whenever $p \equiv 1 (mod\ 4)$. Using Corollary 4, we get a $CW(hp^j(p+1), gp^j(p+1))$, and by recursion, $CW(h^kp^j(p+1), g^kp^j(p+1))$, where $j = 1, 2, \cdots$, $k = 0, 1, \cdots$.

Corollary 5 *If there exist a* $W(2n, s)$ *and a* $W(2h, t) = A = (A_{ij})$, *with blocks of order* h, *satisfying* $A_{ij} \wedge A_{ik} = 0$ *or* $A_{ji} \wedge A_{ki} = 0$, $j \neq k$, $i = 1, 2$, *then there exist*

1. *(i) two* $W(2hn, st)$, P *and* Q, *satisfying* $PQ^T = QP^T$,
2. *(ii) two* $(1,-1,0)$ *matrices,* U *and* V *of order* $2hn$, *satisfying* $UU^T + VV^T = stI_{2hn}$, $U \wedge V = 0$, $UV^T = VU^T = 0$, $U + V = P$, $U - V = Q$.

Corollary 5 is another powerful method for constructing new weighing matrices. For example, let $2g = 2, 10, 26$. Then we have a $W(2h, g)$ mentioned in the above example, for all $h > g$. $W(2n, 2n-1)$ exist whenever $2n-1 \equiv (mod\ 4)$ is a prime power [11] so we have $W(2hn, g(2n-1))$. Further if we use Corollary 5 repeatedly, we obtain $W(2h^kn, g^k(2n-1))$, in particular, $W(2h^kn, 2n-1)$, $h > 1$, $W(2h^kn, 5^k(2n-1))$, $h > 5$ and $W(2h^kn, 13^k(2h-1))$, $h > 13$, where $k = 0, 1, \cdots$.

Corollary 6 *Let there exist a* $COD(2n; s_1, \cdots, s_u)$ *with elements from* $\Re$ *and an Hadamard matrix of order* $4h$ *then there exist*

(i) two $COD(4hn; 2hs_1, \cdots, 2hs_u)$, P *and* Q, *satisfying* $PQ^* = QP^*$,

(ii) two matrices with elements from $\Re$, *of order* $4hn$, U *and* V, *satisfying* $UU^* + VV^* = 2h\varphi I_{4hn}$, $U \wedge V = 0$, $UV^* = VU^* = 0$, $U + V = P$, $U - V = Q$.

Proof. Let $H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$ be the Hadamard matrix of order $4h$, where H_1, H_2, H_3, H_4 are of order $2h$. Let

$$N = \frac{1}{2}\begin{bmatrix} H_1 + H_2 & H_1 - H_2 \\ H_3 + H_4 & H_3 - H_4 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where we note (i) of Theorem 4 is satisfied. By the properties of H, $NN^T = 2hI_{4h}$. Using Theorem 4, we have established the corollary. □

Corollary 7 *If there exist a $CW(2n,k)$ and an Hadamard matrix of order $4h$ then there exist*

(i) two $CW(4hn;2hk)$, P and Q, satisfying $PQ^* = QP^*$,

(ii) two matrices with elements from $\{\pm1, \pm i, 0\}$, U and V of order $4hn$, satisfying $UU^* + VV^* = 2khI_{4hn}$, $U \wedge V = 0$, $UV^* = VU^* = 0$, $U + V = P$, $U - V = Q$.

Corollary 8 *If there exist a complex Hadamard matrix of order $2c$ and an Hadamard matrix of order $4h$ then there exist*

(i) two complex Hadamard matrices of order $4hc$, P and Q, satisfying $PQ^* = QP^*$,

(ii) two matrices, with elements from $\{\pm1, \pm i, 0\}$, U and V of order $4hc$, satisfying $UU^* + VV^* = 4hcI_{4hc}$, $U \wedge V = 0$, $UV^* = VU^* = 0$, $U + V = P$, $U - V = Q$.

Corollary 9 *If there exist a $W(2n,k)$ and an Hadamard matrix of order $4h$ then there exist*

(i) two $W(4hn;2hk)$, P and Q, satisfying $PQ^T = QP^T$,

(ii) two $(1,-1,0)$ matrices, U and V of order $4hn$, satisfying $UU^T + VV^T = 2hkI_{4hn}$, $U \wedge V = 0$, $UV^T = VU^T = 0$, $U = V = P$, $U - V = Q$.

Corollary 10 *If there exist a $COD(m; s_1, \cdots, s_u)$ with elements from $\Re$ and a $COD(2n; q_1, \cdots, q_v)$ with elements from $\Im$ then there exist four matrices with elements from $\mho$, of order $2mn$, P, Q, U, V, satisfying*

(i) $PQ^* = QP^*$, $PP^* = QQ^* = \varphi\psi I_{2mn}$,

(ii) $UU^* + VV^* = \varphi\psi I_{2mn}$, $U \wedge V = 0$, $UV^* = VU^* = 0$, $U + V = P$, $U - V = Q$.

Proof. Let $C = A + iB$ be the $COD(m; s_1, \cdots, s_u)$, where A, B have elements from X. Now A, B satisfy $A \wedge B = 0$, $AB^T = BA^T$, $AA^T + BB^T = \varphi I_m$. Set $N = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$. Clearly, N is an $OD(2m; s_1, \cdots, s_u)$ with elements from X. Using Theorem 4, we have established the corollary. □

Corollary 11 *If there exist a $COD(m; s_1, \cdots, s_u)$ with elements from $\Re$ and a $W(2n,k)$ then there exist*

(i) two $OD(2mn; ks_1, \cdots, ks_u)$ with elements from X, P and Q, satisfying $PQ^T = QP^T$,

(ii) two matrices, U and V with elements from X, of order $2mn$, satisfying $UU^T + VV^T = k\varphi I_{2mn}$, $U \wedge V = 0$, $UV^T = VU^T = 0$, $U + V = P$, $U - V = Q$.

For example, let q be a prime power and let m be the order of a symmetric conference matrix. Then there exists a $COD(m(q^2+q+1);\ 1,\ (m-1)q^2)$ [6, Proposition 24]. By the definition of conference matrices, we have a $W(m, m-1)$. Hence there exist two amicable $OD(m^2(q^2+q+1);\ m-1,\ (m-1)^2q^2)$, and by recursion, $OD(m^{k+1}(q^2+q+1);\ (m-1)^k,\ (m-1)^{k+1}q^2)$, where $k = 1, 2\cdots$.

Note $COD(m; s_1, \cdots, s_u) \times W(2n, k)$ gives a $COD(2mn; ks_1, \cdots, ks_u)$ but we have obtained two amicable $OD(2mn; ks_1, \cdots, ks_u)$. Thus the corollary is a non-trivial improvement of previous results.

Corollary 12 *If there exist an $OD(2n; s_1, \cdots, s_u)$ with elements from X and a $CW(c, r)$ then there exist*

(i) two $OD(2cn; rs_1, \cdots, rs_u)$ with elements from X, P and Q, satisfying $PQ^T = QP^T$,

(ii) two matrices, U and V with elements from X, of order $2cn$, satisfying $UU^T + VV^T = r\varphi I_{2cn}$, $U \wedge V = 0$, $UV^T = VU^T = 0$, $U + V = P$, $U - V = Q$.

Note $OD(2n; s_1, \cdots, s_u) \times W(c, r)$ gives $COD(2cn; rs_1, \cdots, rs_u)$ but we have obtained two amicable $OD(2cn; rs_1, \cdots, rs_u)$. Thus the corollary is a non-trivial improvement of previous results.

Corollary 13 *If there exist a $CW(c, r)$ and a $W(2n, k)$ then there exist*

(i) two $W(2cn; 2rk)$, P and Q, satisfying $PQ^T = QP^T$,

(ii) two $(1, -1, 0)$ matrices, U and V of order $2cn$, satisfying $UU^T + VV^T = rkI_{2cn}$, $U \wedge V = 0$, $UV^T = VU^T = 0$, $U + V = P$, $U - V = Q$.

5 Using $COD(4n; s_1, \cdots, s_u)$

Theorem 5 *Let A be a $COD(4a; s_1, \cdots, s_u)$ with elements from $\Re$ and B be a $COD(4b; q_1, \cdots, q_u)$ with elements from $\Im$. If $A = (A_{ij})$ with blocks of order a has the additional property that i) $A_{ij} \wedge A_{ik} = 0$ or ii) $A_{ji} \wedge A_{ki} = 0$, $(j,k) = (1,2)$, $(j,k) = (3,4)$, $i = 1,2,3,4$ then there exist four matrices U_1, U_2, U_3, U_4 with elements from $\mho$, of order $4h$, satisfying*

(i) $U_1U_1^* + U_2U_2^* + U_3U_3^* + U_4U_4^* = \varphi\psi I_{4ab}$,

(ii) $U_iU_j^* = 0$ for $i \neq j$,

(iii) $U_1 \wedge U_2 = 0$, $U_3 \wedge U_4 = 0$.

Proof. Case 1, $A_{ij} \wedge A_{ik} = 0$, $(j,k) = (1,2)$, $(j,k) = (3,4)$, $i = 1,2,3,4$. Let $B = (B_{ij})$, $i,j = 1,2,3,4$ be the $COD(4b; q_1, \cdots, q_v)$, where D_{ij} is of order b. Set

$$P_k = A \bigcirc (R_k \bigcirc B),$$

where $R_k = (r_{ij})$,

$$r_{ij} = \begin{cases} -1 & i = j = k \\ 1 & i = j \neq k \\ 0 & i \neq j \end{cases}$$

$k, i, j = 1, 2, 3, 4$. By Theorem 2, $(R_k \bigcirc B)(R_k \bigcirc B)^* = \psi I_{4b}$ and $P_i P_i^* = \varphi\psi I_{4ab}$, $i = 1, 2, 3, 4$. Define

$$U_i = \frac{1}{4}(-2P_i + \sum_{j=1}^{4} P_j), \quad i = 1, 2, 3, 4.$$

Then $\sum_{i=1}^4 U_i U_i^* = \varphi\psi I_{4ab}$. Note

$$U_k = \begin{bmatrix} A_{1k} \\ A_{2k} \\ A_{3k} \\ A_{4k} \end{bmatrix} \times [B_{k1}, B_{k2}, B_{k3}, B_{k4}],$$

$k = 1, 2, 3, 4$. Clearly, U_i has elements from $\mho$ and satisfies $U_i U_j^* = 0$ for $i \neq j$ and $U_1 \wedge U_2 = 0$, $U_3 \wedge U_4 = 0$.

Case 2, if $A_{ji} \wedge A_{ki} = 0$, $(j,k) = (1,2)$, $(j,k) = (3,4)$, $i = 1, 2, 3, 4$, then A^* satisfies Case 1. □

Corollary 14 *Let A be a $COD(4a; s_1, \cdots, s_u)$ with elements from $\Re$ and B be a $COD(4b; q_1,$
$\cdots, q_u)$ with elements from $\Im$. If $A = (A_{ij})$ with blocks of order a has the additional property that i) $A_{ij} \wedge A_{ik} = 0$ or ii) $A_{ji} \wedge A_{ki} = 0$, $(j,k) = (1,2)$, $(j,k) = (3,4)$, $i = 1, 2, 3, 4$ then there exist two matrices with elements from $\mho$, of order $4ab$, E, F, satisfying $EF^* = FE^* = 0$, $EE^* + FF^* = \varphi\psi I_{4ab}$.*

Proof. Set $E = U_1 + U_2$, $F = U_3 + U_4$, where U_i has been defined in the proof of Theorem 5. Note the properties of U_i, both E and F have elements from $\mho$ and $EF^* = FE^* = 0$, $EE^* + FF^* = \sum_{i=1}^4 U_i U_i^* = \varphi\psi I_{4ab}$. □

Corollary 15 *If there exist a $COD(4n; s_1, \cdots, s_u)$ with elements from $\Re$ and an Hadamard matrix of order $4h$ then there exist four matrices of order $4hn$ with elements from $\Re$, U_1, U_2, U_3, U_4, satisfying*

(i) $U_1 U_1^* + U_2 U_2^* + U_3 U_3^* + U_4 U_4^* = 2\varphi\psi I_{4hn}$,

(ii) $U_i U_j^* = 0$ *for* $i \neq j$,

(iii) $U_1 \wedge U_2 = 0$, $U_3 \wedge U_4 = 0$.

Proof. Let $H = (H_{ij})$, $i, j = 1, 2, 3, 4$ be the Hadamard matrix of order $4h$, where H_{ij} is of order h. We define $X_i = \frac{1}{2}(H_{i1} + H_{i2})$, $Y_i = \frac{1}{2}(H_{i1} - H_{i2})$, $Z_i = \frac{1}{2}(H_{i3} + H_{i4})$, $W_i = \frac{1}{2}(H_{i3} - H_{i4})$, where $i = 1, 2, 3, 4$. Then both $X_i \pm Y_i$ and $Z_i \pm W_i$ are $(1, -1)$ matrices and $X_i \wedge Y_i = 0$ and $Z_i \wedge W_i = 0$. Write

$$S = \begin{bmatrix} X_1 & Y_1 & Z_1 & W_1 \\ X_2 & Y_2 & Z_2 & W_2 \\ X_3 & Y_3 & Z_3 & W_3 \\ X_4 & Y_4 & Z_4 & W_4 \end{bmatrix}.$$

By the properties of S, $SS^T = 2hI_{4h}$. Using Theorem 5, we prove the corollary. □

Corollary 16 *If there exist a $COD(4n; s_1, \cdots, s_u)$ with elements from $\Re$ and an Hadamard matrix of order $4h$ then there exist two matrices of order $4hn$ with elements from $\Im$, E, F, satisfying $EF^* = FE^* = 0$, $EE^* + FE^* = 2h\varphi I_{4hn}$ also we have a $COD(8hn; 2hs_1, \cdots, 2hs_u)$.*

Proof. Let U_i, $i = 1, 2, 3, 4$, be mentioned in Corollary 14. Set $E = U_1 + U_2$, $F = U_3 + U_4$ and $H = \begin{bmatrix} E & F \\ F & E \end{bmatrix}$. Then H is a $COD(8hn; 2hs_1, \cdots, 2hs_u)$. □

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