Semi-regular Sets of Matrices and Applications

Xian-Mo Zhang

Department of Computer Science
The University of Wollongong
Wollongong
NSW 2522, AUSTRALIA

Abstract

The concept of semi-regular sets of matrices was introduced by J. Seberry in "A new construction for Williamson-type matrices", *Graphs and Combinatorics,* 2(1986), 81-87.
A regular s-set of matrices of order m was first discovered by J. Seberry and A. L. Whiteman in "New Hadamard matrices and conference matrices obtained via Mathon's construction", *Graphs and Combinatorics*, 4(1988), 355-377.
In this paper we study the product of semi-regular sets of matrices and applications in various Williamson-like matrices. Using semi-regular sets of matrices we construct new classes of Willianson type matrices, new classes of complex Hadamard matrices and new Williamson type matrices with additional properties.

1 Introduction and Basic Definitions

Definition 1 Suppose $Q_1, \ldots, Q_{2s}$ are $(1,-1)$ matrices of order m satisfying

$$Q_iQ_j^T = J, \quad i-j \neq 0,\ \pm s, \quad i,j \in \{1,\ldots,2s\}, \tag{1}$$

$$Q_iQ_{i+s}^T = Q_{i+s}Q_i^T, \qquad i \in \{1,\ldots,s\}, \tag{2}$$

$$\sum_{i=1}^{2s} Q_iQ_i^T = 2smI_m. \tag{3}$$

Call $\{Q_1,\ldots,Q_{2s}\}$ a *semi-regular s-set of matrices* of order m.

Definition 2 Suppose $A_1, \ldots, A_s$ are $(1,-1)$ matrices of order m satisfying

$$A_i A_j = J, \quad i,j \in \{1,\ldots,s\}, \tag{4}$$

$$A_i^T A_j = A_j A_i^T = J, \quad i \neq j, \quad i,j \in \{1,\ldots,s\}, \tag{5}$$

$$\sum_{i=1}^{s}(A_i A_i^T + A_i^T A_i) = 2smI_m. \tag{6}$$

Call $\{A_1, \ldots, A_s\}$ a *regular s-set of matrices* of order m [9], [11].

Regular sets of matrices are special semi-regular sets of matrices. To show this, suppose $\{A_1, \ldots, A_s\}$ is a regular s-set of matrices and set $Q_j = A_j$, $Q_{j+s} = A_j^T$, $j = 1, \ldots, s$. Hence $\{Q_1, \ldots, Q_{2s}\}$ is a semi-regular s-set of matrices. J. Seberry [8] constructed a semi-regular $\frac{1}{2}(q+1)$-set of matrices of order q^2, say $S_1, \ldots, S_{q+1}$, satisfying $Q_i Q_j^T = Q_j Q_i^T = J_{q^2}$, $i \neq j$, where $q \equiv 3 (\mathrm{mod}\ 4)$ is a prime power, and a semi-regular $(p+1)$-set of matrices of order p^2, for $p \equiv 1 (\mathrm{mod}\ 4)$, a prime power. J. Seberry and A. L. Whiteman [9] proved that if $q \equiv 3 (\mathrm{mod}\ 4)$ is a prime power there exists a regular $\frac{1}{2}(q+1)$-set of matrices of order q^2, say A_i, $i = 1, \ldots, \frac{1}{2}(q+1)$, satisfying $A_i J = J A_i = qJ$.

Definition 3 Four $(1,-1)$ matrices X_1, X_2, X_3, X_4 of order n satisfying

$$X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4nI_n$$

and

$$UV^T = VU^T,$$

where $U, V \in \{X_1, X_2, X_3, X_4\}$ will be called *Williamson type matrices* of order n [11]. Circulant, symmetric Williamson type matrices will be called *Williamson matrices.*

Williamson and Williamson type matrices are discussed extensively by Baumert, Miyamoto, Seberry, Whiteman, Yamada and Yamamoto ([1], [6], [7], [8], [10] [11], [16], [18], [19], [23], [24], [25]).

Definition 4 Williamson type matrices (Williamson matrices) X_1, X_2, X_3, X_4 will be called *nice* if $X_1 X_2^T + X_3 X_4^T = 0$, *perfect* if $X_1 X_2^T + X_3 X_4^T = X_1 X_4^T + X_2 X_3^T = 0$, *special* if $X_1 X_2^T + X_3 X_4^T = X_1 X_3^T + X_2 X_4^T = X_1 X_4^T + X_2 X_3^T = 0$.

The concept of special Williamson type matrices was introduced by Turyn [15], who found symmetric, commuting and type 1 special Williamson type matrices of order 9^j for j a non-negative integer. Recently Xia [26] gave symmetric, commuting and type 1 special Williamson type matrices of order $N = 9^i \prod_{j=1}^{t} q_j^{4r_j}$, where $q_j \equiv 3 (\mathrm{mod}$ 4) is a prime power, and i, r_j are non-negative integers.

Definition 5 Type 1 $(1,-1)$ matrices A_1, A_2, A_3, A_4 of order n will be called *tight Williamson-like matrices* if $\sum_{j=1}^{4} A_j A_j^T = 4nI_n$ and $A_1A_2^T + A_2A_1^T + A_3A_4^T + A_4A_3^T = 0$.

Definition 6 Let C be a $(1,-1,i,-i)$ matrix of order c satisfying $CC^* = cI_c$, where C^* is the Hermitian adjoint of C. We call C a *complex Hadamard matrix* of order c.

From [17], any complex Hadamard matrix has order 1 or order divisible by 2. Let $C = X + iY$, where X, Y consist of $1, -1, 0$ and $X \wedge Y = 0$ where $\wedge$ is the Hadamard product. Clearly, if C is a complex Hadamard matrix then $XX^T + YY^T = cI_c$, $XY^T = YX^T$.

Definition 7 Four type 1 $(1,-1)$ matrices, say T_1, T_2, T_3, T_4 of order t will be called *T-matrices* if $T_i \wedge T_j = 0$ for $i \neq j$, where $\wedge$ is the Hadamard product, and $\sum_{j=1}^{4} T_j T_j^T = tI_t$.

Notation 1 For convenience, in this paper we write $N = 9^i \prod_{j=1}^{t} q_j^{4r_j}$, where $q_j \equiv 3(\text{mod})$ is a prime power, and i, r_j are non-negative integers.

Let $M = (M_{ij})$ and $N = (N_{gh})$ be orthogonal matrices with t^2 block M-structure [10] of order tm and tn respectively, where M_{ij} is of order m $(i,j = 1,\dots,t)$ and N_{gh} is of order n $(g, h = 1, 2, \dots, t)$. We now define the the operation $\bigcirc$ as the following:

$$M \bigcirc N = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ & & \cdots & \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix},$$

where M_{ij}, N_{ij} and L_{ij} are of order of m, n, and mn, respectively and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

where $\times$ is Kronecker product, $i, j = 1, 2, \dots, t$. We call this the *strong Kronecker* multiplication of two matrices, see [13].

2 Existence of Semi-Regular Sets of Matrices

The following results are known:

Theorem 1 *Let both $p \equiv 1(\text{mod } 4)$ and $q \equiv 3(\text{mod } 4)$ be prime powers. Then*

(i) there exists a semi-regular $(p+1)$-set of matrices of order p^2 (J. Seberry [8]),

(ii) *there exists a regular* $\frac{1}{2}(q+1)$*-set of matrices of order* q^2 *(J. Seberry and A. L. Whiteman [9]).*

Theorem 2 *If there exist a semi-regular s-set of matrices of order m and a semi-regular* $t(=sm)$*-set of matrices of order n then there exists a semi-regular s-set of matrices of order mn.*

Proof. Let $\{A_1 = (a^1_{ij}),\ A_2 = (a^2_{ij}), \ldots,\ A_{2s} = (a^{2s}_{ij})\}$ be the semi-regular s-set of matrices of order m and $\{ B_1, B_2, \ldots, B_{2t} \}$ be the semi-regular t-set of matrices of order of n.
Define $C_i = (c^i_{kj}) = (a^i_{kj}B_{(i-1)m+j+k-1)}),\ i = 1, \ldots, 2s$ so that

$$C_i = \begin{bmatrix} a^i_{11}B_{(i-1)m+1} & a^i_{12}B_{(i-1)m+2} & \cdots & a^i_{1m}B_{im} \\ a^i_{21}B_{(i-1)m+2} & a^i_{22}B_{(i-1)m+3} & \cdots & a^i_{2m}B_{(i-1)m+1} \\ & & \vdots & \\ a^i_{m1}B_{im} & a^i_{m2}B_{(i-1)m+1} & \cdots & a^i_{mm}B_{im-1} \end{bmatrix}.$$

For any $i, j,\ i-j \neq 0,\ \pm s$, there exist no B_u, B_v such that $u - v = \pm t$, B_u in C_i, B_v in C_j. Thus $C_iC_j = J_m \times J_n = J_{mn}$, for $i, j,\ i-j\ \neq 0,\ \pm s$. On the other hand, for a fixed i, write $C_iC^T_{i+s} = (D_{uv})$, where D_{uv} is of order n, $u, v = 1, \ldots, m$. Obviously, $D_{uv} = J_n$, for $u \neq v$. Note that $D_{uu} = \sum_{k=1}^{m} a^i_{uk}a^{i+s}_{vk}B_{(i-1)m+k}B^T_{(i+s-1)m+k}$. Since $B_kB^T_{k+s} = B_{k+s}B^T_k$, $D^T_{uu} = D_{uu}$. Thus $C_iC^T_{i+s}$ is symmetric, i.e. $C_iC^T_{i+s} = C_{i+s}C^T_i$.

To show

$$\sum_{i=1}^{2s} C_iC_i^T = 2smnI_{mn}, \tag{7}$$

note that $(a^i_{kj})^2 = 1$ so the diagonal element of $C_iC_i^T$ is $\sum_{j=1}^{m} B_{(i-1)m+j}B^T_{(i-1)m+j}$ and hence the diagonal element of $\sum_{i=1}^{2s} C_iC_i^T$ is

$$\sum_{j=1}^{2sm} B_jB_j^T = \sum_{j=1}^{2t} B_jB_j^T = 2tnI_n = 2smnI_n.$$

The off-diagonal elements of $C_iC_i^T$ are given by

$$\sum_{j=1}^{m}(a^i_{hj}a^i_{kj}B_{(i-1)m+j+h-1}B^T_{(i-1)m+j+k-1}) = \sum_{j=1}^{m} a^i_{hj}a^i_{kj}J\ (h \neq k).$$

Since

$$\sum_{i=1}^{s}\sum_{j=1}^{m} a^i_{hj}a^i_{kj}J = 0$$

the off-diagonal element of $\sum_{i=1}^{s} C_iC_i^T$ is zero. □

Corollary 1 *Let both $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ be prime powers.*

(i) if $(p+1)p^2 - 1$ is a prime power then there exists a semi-regular $(p+1)$-set of matrices of order $p^2((p+1)p^2-1)^2$,

(ii) if $2(p+1)p^2 - 1$ is a prime power then there exists a semi-regular $(p+1)$-set of matrices of order $p^2(2(p+1)p^2-1)^2$,

(iii) if $(q+1)q^2 - 1$ is a prime power then there exists a regular $\frac{1}{2}(q+1)$-set of matrices of order $q^2((q+1)q^2-1)^2$,

(iv) if $\frac{1}{2}(q+1)q^2 - 1$ is a prime power then there exists a semi-regular $\frac{1}{2}(q+1)$-set of matrices of order $q^2(\frac{1}{2}(q+1)q^2-1)^2$.

Proof. (i) by Theorem 1 there exists a regular $(p+1)$-set of matrices of order p^2. Since $(p+1)p^2 - 1 \equiv 1 \pmod 4$, by Theorem 1 there exists a semi-regular $(p+1)p^2$-set of matrices of order $((p+1)p^2-1)^2$. Using Theorem 2, there exists a semi-regular $(p+1)$-set of matrices of order $p^2((p+1)p^2-1)^2$.

(ii) By Theorem 1 there exists a semi-regular $(p+1)$-set of matrices of order p^2. Since $2(p+1)p^2 - 1 \equiv 3 \pmod 4$, by Theorem 1 there exists a regular $(p+1)p^2$-set of matrices of order $(2(p+1)p^2-1)^2$. Using Theorem 2, there exists a semi-regular $(p+1)$-set of matrices of order $p^2(2(p+1)p^2-1)^2$.

(iii) This is Corollary 2 of [12].

(iv) By Theorem 1 there exists a regular $\frac{1}{2}(q+1)$-set of matrices of order q^2. Case 1, $q \equiv 3 \pmod 8$. Then $\frac{1}{2}(q+1)q^2 - 1 \equiv 1 \pmod 4$. By Theorem 1 there exists a semi-regular $\frac{1}{2}(q+1)q^2$-set of matrices of order $\frac{1}{2}((q+1)q^2-1)^2$. By Theorem 2 there exists a semi-regular $\frac{1}{2}(q+1)$-set of matrices of order $q^2(\frac{1}{2}(q+1)q^2-1)^2$. Case 2, $q \equiv 7 \pmod 8$. This follows from Corollary 5 of [12]. □

3 Williamson Type Matrices and Complex Hadamard Matrices

We find new constructions for Williamson type matrices not given by Miyamoto [6] or Seberry and Yamada [10], [11]. This theorem differs from that of Seberry [8] as it does not need $A_jJ = JA_j = aJ$ where a is a constant [9].

Theorem 3 *If there exist Williamson type matrices of order n and a semi-regular $s(=2n)$-set of matrices of order m then there exist Williamson type matrices of order nm.*

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ be the Williamson type matrices of order n and let $R_1, \ldots, R_{2s}$ be the semi-regular s-set of matrices of order m. Set $E = (a_{ij}R_{j+i-1})$, $F = (b_{ij}R_{n+j+i-1})$, $G = (c_{ij}R_{2n+j+i-1})$, $H = (d_{ij}R_{3n+j+i-1})$, where $i, j = 1, \ldots, n$ and the subscripts of R are reduced modulo n. By the same reasoning as in the proof for Theorem 4 of [8], E, F, G, H are Williamson type matrices of order nm. □

Corollary 2 *If n (odd) is the order of Williamson type matrices and $2n-1$ is a prime power then there exist Williamson type matrices of order $n(2n-1)^2$.*

Proof. Since n is odd, $2n - 1 \equiv 1 (\text{mod } 4)$. By Theorem 1 there exists a semi-regular $2n$-set of matrices of order $(2n-1)^2$. By Theorem 3 we have Williamson type matrices of order $n(2n-1)^2$. □

Corollary 3 *(i) There exist Williamson type matrices of order $9^k(2 \cdot 9^k - 1)^2$ if $2 \cdot 9^k - 1$ is a prime power, where k is a non-negative integer,*

(ii) there exist Williamson type matrices of order $7 \cdot 3^k(14 \cdot 3^k - 1)^2$ if $14 \cdot 3^k - 1$ is a prime power, where k is a non-negative integer.

Proof. From the Index of [11], there exist Williamson type matrices of orders of 9^k and $7 \cdot 3^k$, where $k = 0, 1, \ldots$. Using Corollary 2, the corollary is established. □

Theorem 4 *If there exist a complex Hadamard matrix of order $2c$ and a semi-regular $s(=2c)$-set of matrices of order m then there exists a complex Hadamard matrix of order $2cm$.*

Proof. Let $\{ A_1, \ldots, A_{2s} \}$ be the semi-regular $s(=2c)$-set of matrices of order m and $C = X + iY$ be the complex Hadamard matrix of order $2c$, where both X and Y are $(0, 1, -1)$ matrices satisfying $X \wedge Y = 0$, $XX^T + YY^T = 2cI_{2c}$, $XY^T = YX^T$. Let $P = X + Y$ and $Q = X - Y$. Then both P and Q are $(1, -1)$ matrices of order $2c$ and $PP^T + QQ^T = 4cI_{2c}$, $PQ^T = QP^T$. Let $P = (p_{ij})$ and $Q = (q_{ij})$, $i, j = 1, \ldots, 2c$. Set $E = (p_{ij}A_{i+j-1})$ and $F = (q_{ij}A_{s+i+j-1})$, where $i, j = 1, \ldots, s$ and the subscripts of A are reduced modulo $s = 2c$. Clearly, both E and F are $(1, -1)$ matrices of order $2cm$, since both P and Q are $(1, -1)$ matrices of order $2c$.

We now prove

$$EE^T + FF^T = 4cmI_{2cm}.$$

Write

$$E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix} \text{ and } F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix},$$

where E_i and F_i are matrices of order $m \times sm$.

Note that

$$E_iE_i^T + F_iF_i^T = \sum_{j=1}^{s}(p_{ij}p_{ij}A_{i+j-1}A_{i+j-1}^T + q_{ij}q_{ij}A_{s+i+j-1}A_{s+i+j-1}^T)$$

$$= \sum_{j=1}^{s}(A_jA_j^T + A_{s+j}A_{s+j}^T) = \sum_{j=1}^{2s} A_jA_j^T = 2smI_m.$$

On the other hand, if $i \neq k$,

$$E_iE_k^T + F_iF_k^T = \sum_{j=1}^{s}(p_{ij}p_{kj}A_{i+j-1}A_{k+j-1}^T + q_{ij}q_{kj}A_{s+i+j-1}A_{s+k+j-1}^T)$$

$$= \sum_{j=1}^{s}(p_{ij}p_{kj} + q_{ij}q_{kj})J_m = 0.$$

Thus

$$EE^T + FF^T = 2smI_{sm} = 4cmI_{2cm}.$$

Next we prove

$$EF^T = FE^T.$$

Write $EF^T = (D_{ij})$, where D_{ij} is of order m, $i, j = 1, \ldots, 2c$. Note that $D_{ij} = \sum_{k=1}^{2c} p_{ik}q_{jk}A_{i+k-1}A_{s+j+k-1}^T$. For $i \neq j$, $D_{ij} = \sum_{k=1}^{2c} p_{ik}q_{jk}J_m$. Since $PQ^T = QP^T$, $D_{ij}^T = D_{ji}$, $i \neq j$. Note that $D_{ii} = \sum_{k=1}^{2c} p_{ik}q_{ik}A_{i+k-1}A_{s+i+k-1}^T$. From (2), Definition 1, $D_{ii}^T = D_{ii}$. Thus EF^T is symmetric, i.e. $EF^T = FE^T$. Finally, Set $U = \frac{1}{2}(E+F)$ and $V = \frac{1}{2}(E-F)$. Thus both U and V are $(1,-1,0)$ matrices of order $2cm$ satisfying $U \wedge V = 0$, $UU^T + VV^T = \frac{1}{2}(EE^T + FF^T) = 2cmI_{2cm}$. Since $EF^T = FE^T$, $UV^T = VU^T$. Thus $U + iV$ is a complex Hadamard matrix of order $2cm$. □

Corollary 4 *If both $p \equiv 1 (\mathrm{mod}\ 4)$ and $p^j(p+1) - 1$ are prime powers then there exists a complex Hadamard matrix of order $p^j(p+1)(p^j(p+1)-1)^2$, where j is a positive integer.*

Proof. Obviously, $p^j(p+1) - 1 \equiv 1 (\mathrm{mod}\ 4)$. By Theorem 1 there exists a regular $p^j(p+1)$-set of matrices of order $(p^j(p+1)-1)^2$. From Corollary 18 of [5], there exists a complex Hadamard matrix of order $p^j(p+1)$. Using Theorem 4, we have a complex Hadamard matrix of order $p^j(p+1)(p^j(p+1)-1)^2$. □

4 New Construction of Special, Perfect and Nice Williamson Type Matrices

Part (iii) of the next theorem is known in [15] where the special Williamson type matrices are symmetric and commuting. We include it here for completeness.

Theorem 5 *(i) If there exist nice Williamson type matrices of orders n and m then there exist Williamson type matrices of order nm,*

(ii) if there exist nice Williamson type matrices of order n and special Williamson type matrices of order m then there exist nice Williamson type matrices of order nm,

(iii) if there exist special Williamson type matrices of orders n and m then there exist special Williamson type matrices of order nm.

Proof. Let X_1, X_2, X_3, X_4 be nice Williamson type matrices of order n and Y_1, Y_2, Y_3, Y_4 be nice Williamson type matrices of order m. Set

$$Z_1 = \frac{1}{2}(X_1 + X_2) \times Y_1 + \frac{1}{2}(X_1 - X_2) \times Y_2, \ Z_2 = \frac{1}{2}(X_1 + X_2) \times Y_3 + \frac{1}{2}(X_1 - X_2) \times Y_4,$$

$$Z_3 = \frac{1}{2}(X_3 + X_4) \times Y_1 + \frac{1}{2}(X_3 - X_4) \times Y_2, \ Z_4 = \frac{1}{2}(X_3 + X_4) \times Y_3 + \frac{1}{2}(X_3 - X_4) \times Y_4.$$

Then Z_1, Z_2, Z_3, Z_4 are $(1, -1)$ matrices of order nm. Note that

$$\begin{aligned}
Z_1 Z_1^T &= \tfrac{1}{4}(X_1 + X_2)(X_1 + X_2)^T \times Y_1 Y_1^T + \tfrac{1}{4}(X_1 - X_2)(X_1 - X_2)^T \times Y_2 Y_2^T \\
&+ \tfrac{1}{2}(X_1 + X_2)(X_1 - X_2)^T \times Y_1 Y_2^T, \\
Z_2 Z_2^T &= \tfrac{1}{4}(X_1 + X_2)(X_1 + X_2)^T \times Y_3 Y_3^T + \tfrac{1}{4}(X_1 - X_2)(X_1 - X_2)^T \times Y_4 Y_4^T \\
&+ \tfrac{1}{2}(X_1 + X_2)(X_1 - X_2)^T \times Y_3 Y_4^T, \\
Z_3 Z_3^T &= \tfrac{1}{4}(X_3 + X_4)(X_3 + X_4)^T \times Y_1 Y_1^T + \tfrac{1}{4}(X_3 - X_4)(X_3 - X_4)^T \times Y_2 Y_2^T \\
&+ \tfrac{1}{2}(X_3 + X_4)(X_3 - X_4)^T \times Y_1 Y_2^T, \\
Z_4 Z_4^T &= \tfrac{1}{4}(X_3 + X_4)(X_3 + X_4)^T \times Y_3 Y_3^T + \tfrac{1}{4}(X_3 - X_4)(X_3 - X_4)^T \times Y_4 Y_4^T \\
&+ \tfrac{1}{2}(X_3 + X_4)(X_3 - X_4)^T \times Y_3 Y_4^T.
\end{aligned}$$

It is easy to check that

$$Z_1 Z_1^T + Z_2 Z_2^T + Z_3 Z_3^T + Z_4 Z_4^T$$

$$= \frac{1}{4}(X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T) \times (Y_1Y_1^T + Y_2Y_2^T + Y_3Y_3^T + Y_4Y_4^T) = 4nmI_{nm}.$$

Obviously, $Z_iZ_j^T = Z_jZ_i^T$, for $i, j = 1, 2, 3, 4$. Thus, Z_1, Z_2, Z_3, Z_4 are Williamson type matrices of order nm.

In particular, let X_1, X_2, X_3, X_4 be nice Williamson type matrices of order n and Y_1, Y_2, Y_3, Y_4 be special Williamson type matrices of order m. Note that

$$Z_1Z_2^T = \frac{1}{4}(X_1 + X_2)(X_1 + X_2)^T \times Y_1Y_3^T + \frac{1}{4}(X_1 - X_2)(X_1 - X_2)^T \times Y_2Y_4^T$$

$$+\frac{1}{4}(X_1 + X_2)(X_1 - X_2)^T \times Y_1Y_4^T + \frac{1}{4}(X_1 - X_2)(X_1 + X_2)^T \times Y_2Y_3^T,$$

where

$$(X_1 + X_2)(X_1 - X_2)^T \times Y_1Y_4^T + (X_1 - X_2)(X_1 + X_2)^T \times Y_2Y_3^T$$
$$= (X_1X_1^T - X_2X_2^T) \times (Y_1Y_4^T + Y_2Y_3^T) = 0.$$

Then

$$Z_1Z_2^T = \frac{1}{4}(X_1 + X_2)(X_1 + X_2)^T \times Y_1Y_3^T + \frac{1}{4}(X_1 - X_2)(X_1 - X_2)^T \times Y_2Y_4^T.$$

Similarly,

$$Z_3Z_4^T = \frac{1}{4}(X_3 + X_4)(X_3 + X_4)^T \times Y_1Y_3^T + \frac{1}{4}(X_3 - X_4)(X_3 - X_4)^T \times Y_2Y_4^T.$$

Hence

$$Z_1Z_2^T + Z_3Z_4^T = \frac{1}{4}(X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T) \times (Y_1Y_3^T + Y_2Y_4^T) = 0.$$

We have now proved Z_1, Z_2, Z_3, Z_4 are nice Williamson type matrices of order nm.

Further suppose X_1, X_2, X_3, X_4 are special Williamson type matrices of order n and Y_1, Y_2, Y_3, Y_4 are special Williamson type matrices of order m.

$$Z_1Z_3^T = \frac{1}{4}(X_1 + X_2)(X_3 + X_4)^T \times Y_1Y_1^T + \frac{1}{4}(X_1 - X_2)(X_3 - X_4)^T \times Y_2Y_2^T$$

$$+\frac{1}{4}(X_1 + X_2)(X_3 - X_4)^T \times Y_1Y_2^T + \frac{1}{4}(X_1 - X_2)(X_3 + X_4)^T \times Y_2Y_1^T.$$

Note that

$$(X_1 + X_2)(X_3 + X_4)^T = (X_1 - X_2)(X_3 - X_4)^T = 0,$$

then

$$Z_1Z_3^T = \frac{1}{4}(X_1 + X_2)(X_3 - X_4)^T \times Y_1Y_2^T + \frac{1}{4}(X_1 - X_2)(X_3 + X_4)^T \times Y_2Y_1^T.$$

Similarly,

$$Z_2Z_4^T = \frac{1}{4}(X_1+X_2)(X_3-X_4)^T \times Y_3Y_4^T + \frac{1}{4}(X_1-X_2)(X_3+X_4)^T \times Y_4Y_3^T.$$

Clearly, $Z_1Z_3^T + Z_2Z_4^T = 0$. Finally, by the same reasoning for $Z_1Z_3^T$ and $Z_2Z_4^T$, we have

$$Z_1Z_4^T = \frac{1}{4}(X_1+X_2)(X_3-X_4)^T \times Y_1Y_4^T + \frac{1}{4}(X_1-X_2)(X_3+X_4)^T \times Y_2Y_3^T$$

and

$$Z_2Z_3^T = \frac{1}{4}(X_1+X_2)(X_3-X_4)^T \times Y_3Y_2^T + \frac{1}{4}(X_1-X_2)(X_3+X_4)^T \times Y_4Y_1^T.$$

Clearly $Z_1Z_4^T + Z_2Z_3^T = 0$. Thus Z_1, Z_2, Z_3, Z_4 are special Williamson type matrices of order nm. □

Corollary 5 *If there exist nice Williamson type matrices of orders n and m then there exist Williamson type matrices of order nmN, where N was defined in Notation 1.*

Proof. From [26], there exist special Williamson type matrices of order N. By Theorem 5 there exist nice Williamson type matrices of order mN and hence Williamson type matrices of order nmN. □

Let $q \equiv 1 (\text{mod } 4)$ be a prime power and $n = \frac{1}{2}(1+q)$. By Theorem 1 there exists a semi-regular $2n = (q+1)$-set of matrices of order q^2, $Q_1, \ldots, Q_{4n}$ satisfying

$$Q_iQ_j^T = J_{q^2},\ if\ i-j \neq \pm 2n, 0,\ Q_iQ_{i+2n}^T = Q_{i+2n}Q_i^T$$

and

$$\sum_{j=1}^{4n} Q_jQ_j^T = 4q^2(1+q)I_{q^2}.$$

Suppose $A = (a_{ij})$, $B = (b_{ij})$, $C = (b_{ij})$, $D = (d_{ij})$ are Williamson type matrices of order n. Set

$$E = (a_{ij}Q_{j-i}),\ F = (b_{ij}Q_{n+j-i}),\ G = (c_{ij}Q_{2n+j-i}),\ H = (d_{ij}Q_{3n+j-i}),$$

where the subscripts of Q are reduced modulo n to the residue class $\{1, \ldots, n\}$. By the same reasoning as in the proof of Theorem 4 of [8], E, F, G, H are Williamson type matrices of order nq^2. Further suppose $AB^T + CD^T = 0$, i.e. A, B, C, D are nice Williamson type matrices of order n. Write $EF^T = (X_{ij})$, $GH^T = (Y_{ij})$, where X_{ij}, Y_{ij} are of order q^2, $i,j = 1, \ldots n$. Note that

$$X_{ij} = \sum_{k=1}^{n} a_{ik}Q_{k-i}b_{jk}Q_{n+k-j}^T = \sum_{k=1}^{n} a_{ik}b_{jk}J_{q^2},$$

since $(n+k-j)-(k-i) \neq 0,\ 2n$. Similarly,

$$Y_{ij} = \sum_{k=1}^{n} c_{ik} Q_{2n+k-i} b_{jk} Q_{3n+k-j}^T = \sum_{k=1}^{n} c_{ik} d_{jk} J_{q^2},$$

since $(3n+k-j)-(2n+k-i) \neq 0, 2n$. Note that $AB^T + CD^T = 0$ thus $X_{ij} + Y_{ij} = 0$ and then $EF^T + GH^T = 0$. Similarly, if $AD^T + BC^T = 0$ then $EH^T + FG^T = 0$. Note that if n is odd, then $2n-1 \equiv 1 (\mathrm{mod}\ 4)$. Hence we have proved

Theorem 6 *If there exist nice (perfect) Williamson type matrices of order n, where n is odd and $2n-1$ is a prime power then there exist nice (perfect) Williamson type matrices of order $n(2n-1)^2$.*

Corollary 6 *Let N, N_1 and N_2 be three products of the kind defined by Notation 1. If $2N-1$ is a prime power then there exist*

(i) perfect Williamson type matrices of order $N(2N-1)^2$,

(ii) nice Williamson type matrices of order $N(2N-1)^2 N_1$,

(iii) Williamson type matrices of order $N(2N-1)^2 N_1 (2N_1-1)^2 N_2$, if $2N_1 - 1$ is a prime power.

Proof. (i), (ii) and (iii) hold by Theorem 6, Theorem 5 and Corollary 5 respectively. □

For example, by Corollary 6 there exist perfect Williamson type matrices of order $9 \cdot 17^2$, nice Williamson type matrices of order $9 \cdot 17^2 N$ and Williamson type matrices of order $9^2 \cdot 17^4 N$.

5 Tight Williamson-like Matrices and Applications

Some tight Williamson-like matrices were found by Xia [22]. For example, from [20], we construct cyclic tight Williamson-like matrices of orders 5 and 13 with first rows

$+-++-,\ ++-++,\ --++-,\ ++++-$ and

$++---+--++-++,\ --+++-++++++-+,$
$+--+-+++----+-,\ +-+++++++--++-$ respectively.

From [20] we construct type 1 tight Williamson-like matrices of order 25. Any element in the abelian group $Z_5 \oplus Z_5$ can be expressed as (a, b), where $a, b \in Z_5$, and the addition in $Z_5 \oplus Z_5$ can be defined as $(a,b)+(c,d) = (a+b, c+d)$. Set

$$S_1 = \{(0,0), (0,1), (1,2), (3,3), (0,3), (4,4), (3,4), (2,0), (2,2), (1,0), (1,4), (0,2), (3,0)\},$$
$$S_2 = \{(0,1), (4,0), (3,1), (4,4), (0,4), (4,2), (1,0), (1,1), (3,2)\},$$
$$S_3 = \{(1,2), (3,3), (1,3), (4,1), (3,4), (2,0), (2,3), (4,3), (1,4), (0,2), (2,4), (2,1)\},$$
$$S_4 = \{(3,3), (4,1), (0,3), (2,0), (4,3), (2,2), (0,2), (2,1), (3,0)\}.$$

The type 1 $(1,-1)$ incidence matrices of S_1, S_2, S_3, S_4 form tight Williamson-like matrices of order 25.

Tight Williamson-like matrices are not Williamson type matrices but they are suitable for use in the Goethals-Seidel or Wallis-Whiteman arrays [14] with cross correlation types of properties (see Definition 4). Besides forming Hadamard matrices of Goethals-Seidel or Wallis-Whiteman type [14], tight Williamson-like matrices can be used to form Hadamard matrices in the following special array.

Let A_1, A_2, A_3, A_4 be tight Williamson-like matrices of order n. Set

$$H = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ A_3^T & A_4^T & -A_1^T & -A_2^T \\ A_4^T & A_3^T & -A_2^T & -A_1^T \end{bmatrix}.$$

Hence H is an Hadamard matrices of order $4n$ with 4×4 type 1 blocks.

Let A_1, A_2, A_3, A_4 be the tight Williamson-like matrices of order n and T_1, T_2, T_3, T_4 be T-matrices of order t.
Write

$$\begin{aligned} E_1 &= T_1 \times A_1 + T_2 \times A_2 + T_3 \times A_3^T + T_4 \times A_4^T, \\ E_2 &= T_1 \times A_2 + T_2 \times A_1 + T_3 \times A_4^T + T_4 \times A_3^T, \\ E_3 &= T_1 \times A_3 + T_2 \times A_4 - T_3 \times A_1^T - T_4 \times A_2^T, \\ E_4 &= T_1 \times A_4 + T_2 \times A_3 - T_3 \times A_2^T - T_4 \times A_1^T. \end{aligned}$$

Clearly, each E_j is a (1, -1)-matrix. It is easy to check that $\sum_{j=1}^4 E_j E_j^T = 4tnI_{tn}$. Note that the E_j are of type 1, hence we can construct an Hadamard matrix of order $4tn$ by using Theorem 3 of [14]. This proves the more general result:
If there exist tight Williamson-like matrices of order n and T-matrices of order t then there exists an Hadamard matrix of order $4tn$.

Xia proved [22] that if there exist tight Williamson-like matrices of order n and type 1 special Williamson type matrices of order m then there exist tight Williamson-like matrices of order nm. Since there exist tight Williamson-like matrices of orders 5, 13, 25 and N is the order of type 1 special Williamson type matrices , there exist tight Williamson-like matrices of orders $5N, 13N, 25N$ and thus there exist Hadamard matrices of orders $5tN$, $13tN$, $25tN$ where t is the order of the T-matrices.

Let A, B, C, D be tight Williamson-like matrices of order n. Set

$$P = \frac{1}{2}\begin{bmatrix} A+B & C+D \\ C^T+D^T & -A^T-B^T \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{2}\begin{bmatrix} A-B & C-D \\ C^T-D^T & -A^T+B^T \end{bmatrix}.$$

Thus P and Q are two disjoint $W(2n, n)$ and then we have two disjoint $W(2n, n)$ where $n = 5N,\ 13N,\ 25N$.

By Corollary 2.11 of [4] a $W(2n, n)$, where n is odd, only exists when n is a sum of two squares. Hence we have reproved that if n (odd) is the order of tight Williamson-like matrices then n is a sum of two squares (Xia [21]), in other words, the factorization of n into powers of distinct primes contains no odd powers of primes congruent to $3(\text{mod } 4)$.

Two disjoint $W(2n, n)$ are often used for constructing Hadamard matrices [2], [3].

Also we can construct two disjoint $W(2n, n)$ by using nice Williamson type matrices. Let A, B, C, D be nice Williamson type matrices of order n. Set $P = \frac{1}{2}\begin{bmatrix} A+B & C+D \\ C+D & -A-B \end{bmatrix}$ and $Q = \frac{1}{2}\begin{bmatrix} A-B & C-D \\ C-D & -A+B \end{bmatrix}$. It is easy to verify that P and Q are two disjoint $W(2n, n)$. Thus there exist two disjoint $W(2n, n)$ for $n = N(2N-1)^2N_1$ where N, N_1 were defined by Notation 1 and $2N-1$ is a prime power (see Corollary 6).

The constructions of all the above matrices P and Q were previously given in [2].

The following table shows the existence of tight Williamson-like matrices of odd orders < 60. Tight Williamson-like matrices for odd order n can only exist for $n \equiv 1(\text{mod } 4)$, where the factorization of n into powers of distinct primes contains no odd powers of primes congruent to $3(\text{mod } 4)$. Hence the following list contains only those n which exist or could possibly exist.

order	construction
5	[22], see Section 5
9^t	[15], since type 1 special Williamson type are tight Williamson-like matrices
13	[22], see Section 5
17	unknown
25	[22], see Section 5
29	unknown
37	unknown
41	unknown
45	[22], see this paper
49	unknown
53	unknown

Acknowlegement: I wish to thank Professor Jennifer Seberry for her help and encouragement.

References

[1] L. D. Baumert and M. Hall, Jr. Hadamard matrices of Williamson type. *Math. Comp.*, 19:442–447, 1965.

[2] R. Craigen. Constructing Hadamard matrices with orthogonal pairs. To appear in *Ars Combinatoria.*

[3] R. Craigen, Jennifer Seberry, and Xian-Mo Zhang. Product of four Hadamard matrices. *Journal of Combinatorial Theory*, Ser. A, 59:318–320, 1992.

[4] A. V. Geramita and J. Seberry. *Orthogonal Designs: Quadratic Forms and Hadamard Matrices.* Marcel Dekker, New York-Basel, 1979.

[5] H. Kharaghani and Jennifer Seberry. Regular complex Hadamard matrices. *Congress. Num.*, 24:149–151, 1990.

[6] Masahiko Miyamoto. A construction for Hadamard matrices. *Journal of Combinatorial Theory*, Ser. A, 57:86–108, 1991.

[7] Jennifer Seberry. Some matrices of Williamson-type. *Utilitas Math.*, 4:147–154, 1973.

[8] Jennifer Seberry. A new construction for Williamson-type matrices. *Graphs and Combinatorics*, 2:81–87, 1986.

[9] Jennifer Seberry and Albert Leon Whiteman. New Hadamard matrices and conference matrices obtained via Mathon's construction. *Graphs and Combinatorics*, 4:355–377, 1988.

[10] Jennifer Seberry and Mieko Yamada. On the products of Hadamard matrices, Williamson matrices and other orthogonal matrices using M-structures. *JCMCC*, 7:97–137, 1990.

[11] Jennifer Seberry and Mieko Yamada. Hadamard matrices, sequences and block designs. In J. Dinitz and D. Stinson, editors, *Surveys in Contemporary Design Theory*, Wiley-Interscience Series in Discrete Mathematics. John Wiley, New York, 1992.

[12] Jennifer Seberry and Xian-Mo Zhang. Regular sets of matrices and applications. To appear in *Graphs and Combinatorics.*

[13] Jennifer Seberry and Xian-Mo Zhang. Some orthogonal designs and complex Hadamard matrices by using two Hadamard matrices. *Australasian Journal of Combinatorics*, 4:93–102, 1991.

[14] J. Seberry-Wallis. On Hadamard matrices. *Journal of Combinatorial Theory*, Ser. A, 18:149–164, 1975.

[15] R. J. Turyn. A special class of Williamson matrices and difference sets. *Journal of Combinatorial Theory*, Ser. A, 36:111–115, 1984.

[16] Jennifer Seberry Wallis. Construction of Williamson type matrices. *Linear and Multilinear Algebra*, 3:197–207, 1975.

[17] W. D. Wallis, A. Penfold Street, and J. Seberry Wallis. *Combinatorics: Room Squares, sum-free sets, Hadamard Matrices*, volume 292 of Lecture Notes in Mathematics. Springer-Verlag, Berlin- Heidelberg- New York, 1972.

[18] Albert Leon Whiteman. An infinite family of Hadamard matrices of Williamson type. *Journal of Combinatorial Theory*, Ser. A. 14:334–340, 1973.

[19] Albert Leon Whiteman. Hadamard matrices of Williamson type. *J. Austral. Math. Soc.*, 21:481–486, 1976.

[20] Ming Yuan Xia. Some supplementary difference sets and Hadamard matrices. *Acta. Math. Sci.*, 4 (1):81–92, 1984.

[21] Ming Yuan Xia. On construction of some supplementary difference sets and others. *Journal of Central China Normal University (Natural Sciences), Monograph of Mathematics*, No. 1:204–208, 1989.

[22] Ming Yuan Xia. Hadamard matrices. *Combinatorial designs and applications, Lecture Notes in Pure and Appl. Math., Dekker, New York*, 126:179–181, 1990.

[23] Mieko Yamada. On the Williamson matrices of Turyn's type and and type j. *Comment. Math. Univ. San Pauli*, 31:71–73, 1982.

[24] K. Yamamoto. On a generalized Williamson equation. *Colloq. Math. Janos Bolyai*, 37:839–850, 1981.

[25] K. Yamamoto and Mieko Yamada. Williamson matrices of Turyn's type and Gauss sums. *J. Math. Soc. Japan*, 37:703–717, 1985.

[26] Ming Yuan Xia. Some infinite classes of special Williamson matrices and difference sets. To appear in *Journal of Combinatorial Theory*, Ser A.

(Received 10/9/91; revised 18/8/92)