AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **76(2)** (2020), Pages 232–247

Failed power domination on graphs

Abraham Glasser

Golisano College of Computing and Information Sciences
Rochester Institute of Technology
Rochester, NY 14623
U.S.A.
atg2036@rit.edu

Bonnie Jacob

Science and Mathematics Department
National Technical Institute for the Deaf
Rochester Institute of Technology
Rochester, NY 14623
U.S.A.
bcjntm@rit.edu

Emily Lederman   Stanisław Radziszowski

Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623
U.S.A.
erl3193@rit.edu   spr@cs.rit.edu

Abstract

Let G be a simple graph with vertex set V and edge set E, and let $S \subseteq V$. The sets $\mathcal{P}^i(S), i \geq 0$, of vertices *monitored* by S at the i^{th} step are given by $\mathcal{P}^0(S) = N[S]$ and $\mathcal{P}^{i+1}(S) = \mathcal{P}^i(S) \bigcup \{w : \{w\} = N[v] \backslash \mathcal{P}^i(S)$ for some $v \in \mathcal{P}^i(S)\}$. If there exists j such that $\mathcal{P}^j(S) = V$, then S is called a *power dominating set*, PDS, of G. Otherwise, S is a *failed power dominating set*, FPDS.

The *power domination number* of a simple graph G, denoted $\gamma_p(G)$ gives the minimum number of measurement devices known as phasor measurement units (PMUs) required to observe a power network represented by G, and is the minimum cardinality of any PDS of G. The *failed power domination number* of G, $\bar{\gamma}_p(G)$, is the maximum cardinality

ISSN: 2202-3518

of any FPDS of G, and represents the maximum number of PMUs that could be placed on a given power network represented by G, but fail to observe the full network. As a consequence, $\bar{\gamma}_p(G)+1$ gives the minimum number of PMUs necessary to successfully observe the full network no matter where they are placed.

We prove that $\bar{\gamma}_p(G)$ is NP-hard to compute, determine graphs in which every vertex is a PDS, and compare $\bar{\gamma}_p(G)$ to similar parameters.

1 Introduction

This paper studies power domination on graphs, which arose because of applications to electric power networks [5, 16]. We denote by $G = (V, E)$ a graph with vertex set V and edge set E. In keeping with [11], which we refer to throughout the paper for basic graph theory definitions, we assume that the graph is finite and simple. In cases where the graph in question is ambiguous, we use $V(G)$ and $E(G)$. The *open neighborhood* of a vertex $v \in V$, denoted $N_G(v)$ or $N(v)$ when the graph is understood, is the set of neighbors of v, where u and v are referred to as *neighbors* of each other if uv is an edge of G. The *closed neighborhood* of v, $N[v]$, is $N(v)\cup\{v\}$. The *open neighborhood* of a set $S \subseteq V$, denoted $N(S)$, is the union of open neighborhoods of vertices in S, and the *closed neighborhood* of S, $N[S]$, is defined as $S \cup N(S)$. A vertex v is *dominated by* S if $v \in N[S]$. A set S is a *dominating set* if $N[S] = V$. The minimum cardinality of all dominating sets of G is the *domination number* $\gamma(G)$.

Power domination differs from domination in that it contains a second step known as the *propagation step*. We use notation similar to that formalized in [1]. Let $i \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. If G is a graph and $S \subseteq V$, then the set of vertices *monitored* by S at Step i, denoted $\mathcal{P}^i(S)$, is defined as follows.

- $\mathcal{P}^0(S) = N[S]$,
- $\mathcal{P}^{i+1}(S) = \mathcal{P}^i(S) \bigcup \{w : \{w\} = N[v]\backslash\mathcal{P}^i(S) \text{ for some } v \in \mathcal{P}^i(S)\}$.

That is, Step 0 consists of finding the set of vertices dominated by S. For Step $i > 0$, if a vertex v in $\mathcal{P}^i(S)$ has exactly one neighbor w outside of $\mathcal{P}^i(S)$, then we add w to $\mathcal{P}^{i+1}(S)$. The step corresponding to $i = 0$ is known as the *domination step* and those corresponding to $i > 0$ as the *propagation steps*. Note that for any $i \geq 0$, $\mathcal{P}^i(S) \subseteq \mathcal{P}^{i+1}(S)$. Also, if $\mathcal{P}^{i_0+1}(S) = \mathcal{P}^{i_0}(S)$ for some i_0, then $\mathcal{P}^j(S) = \mathcal{P}^{i_0}(S)$ for any $j \geq i_0$, and then we write $\mathcal{P}^\infty(S) = \mathcal{P}^{i_0}(S)$.

Definition 1.1 a. A *power dominating set* (PDS) of G is a set $S \subseteq V$ such that $\mathcal{P}^\infty(S) = V$.

b. A *failed power dominating set* (FPDS) is a set $S \subseteq V$ such that S is not a PDS.

c. A *stalled power dominating set* (SPDS) is a set $S \subseteq V$ such that $\mathcal{P}^\infty(S) = \mathcal{P}^0(S)$. That is, after the domination step, no propagation steps occur.

d. The *power domination number* of G, denoted by $\gamma_p(G)$, is the minimum cardinality among all power dominating sets of G.

e. The *failed power domination number* of G, denoted by $\bar{\gamma}_p(G)$, is the maximum cardinality among all failed power dominating sets of G.

If S is an SPDS in G such that $S \cup \{u\}$ is a PDS for any vertex $u \in V \setminus S$, then we say that S is *maximally stalled*. To indicate that S is an SPDS and $\mathcal{P}^0(S) \subsetneq V$, we say that S is *properly stalled*. In [15], the authors defined a critical set in the context of directed graphs, which can be stated for undirected graphs as follows: a vertex set W is a *weakly critical set* of G if there is no vertex outside W that has exactly one neighbor in W. The results of this paper could be framed in terms of critical sets; however we use SPDS to connect these results with related work in zero forcing on undirected graphs [4, 13, 22]. Figures 1 and 2 show an example of a graph G with $\gamma_p(G) \leq 2$ but $\bar{\gamma}_p(G) \geq 20$.

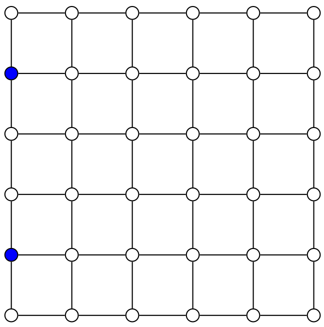


Figure 1: A PDS S in blue

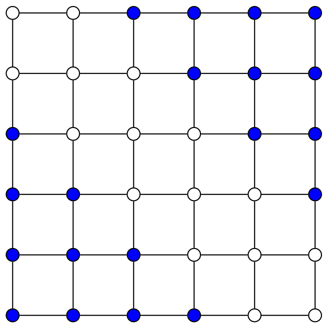


Figure 2: An FPDS / SPDS in blue

In this paper, we determine the computational complexity of the decision problem for failed power domination. We also study extremal values of $\bar{\gamma}_p(G)$ and find graphs that attain them. We present a list of graphs that have $\bar{\gamma}_p(G) = 0$, which is a particularly interesting case, since $\bar{\gamma}_p(G) = 0$ implies that any nonempty set of vertices in G is a PDS. We also discuss the relationship between $\bar{\gamma}_p(G)$ and some related parameters.

2 Motivation and related parameters

The idea of power domination on graphs is motivated by the need to monitor electric power networks. In [5], the authors describe the problem of observing a power system while minimizing the number of measurement devices known as phasor measurement units (PMUs) on the network. A PMU measures the voltage and phase angle, and allows for synchronization [20], which is one strategy described in [17] for making the power grid more robust. If a PMU measures the voltage and phase angle of vertex v (or edge e), then v (or e) is said to be *observed*. The vertex on which a PMU is placed is observed, as are its incident edges and adjacent vertices. In addition, any vertex that is incident to an observed edge is observed; any edge joining two observed vertices is observed; finally, from Kirchhoff's Law, given an observed vertex v with k incident edges, if $k-1$ of the edges are observed, then all k are observed. In [16] the authors formulate and investigate this problem as a graph theoretic problem. Later, Brueni and Heath [9] and Kneis et al. [19] independently showed that the problem can be simplified to omit any reference to edges. The formal set definition of $\mathcal{P}^i(S)$ was introduced in [1].

Under the model described in [16], the power domination number $\gamma_p(G)$ gives the minimum number of PMUs required to observe a power network represented by graph G. The power domination number has been studied for multiple families of graphs [8, 12, 24], as has the complexity of $\gamma_p(G)$ [16]. On the other hand, the failed power domination number $\bar{\gamma}_p(G)$ that we defined above gives a worst case scenario: what is the maximum number of PMUs that we could use on a given network represented by G, but fail to observe the full network? In addition, $\bar{\gamma}_p(G)+1$ gives us the minimum number of PMUs necessary to successfully observe the full network no matter where we place the PMUs.

The concept of zero forcing, while related to power domination, was introduced in 2008 in the context of minimum rank problems [2, 6] as well as quantum networks in 2007 [10]. Zero forcing acts like power domination, but without the domination step. That is, given a set S, $\mathcal{Q}^0(S) = S$, and for $i \in \mathbb{N}_0$, $\mathcal{Q}^{i+1}(S) = \mathcal{Q}^i(S) \bigcup \{N[v] : v \in \mathcal{Q}^i(S) \text{ and } |N[v]\backslash\mathcal{Q}^i(S)| = 1\}$. Note that there exists an i_0 such that for all $j > i_0$, $\mathcal{Q}^j(S) = \mathcal{Q}^{i_0}(S)$, so we write $\mathcal{Q}^\infty(S) = \mathcal{Q}^{i_0}(S)$. If $\mathcal{Q}^\infty(S) = V$, then S is a *zero forcing set*. Otherwise, S is a *failed zero forcing set*. The smallest cardinality of any zero forcing set in G is the *zero forcing number* $\mathrm{Z}(G)$, and the largest cardinality of any failed zero forcing set is the *failed zero forcing number* $\mathrm{F}(G)$ [4, 13]. Complexity results for failed zero forcing were established in [22].

Remark 2.1 *For a graph $G = (V, E)$, suppose $S \subseteq S' \subseteq V$. Then,*

1. $\mathcal{Q}^\infty(S) \subseteq \mathcal{Q}^\infty(S')$*, and*
2. $\mathcal{P}^\infty(S) \subseteq \mathcal{P}^\infty(S')$.

Proof: Suppose $S \subseteq S' \subseteq V$. Then $\mathcal{Q}^0(S) \subseteq \mathcal{Q}^0(S')$. Assume $\mathcal{Q}^k(S) \subseteq \mathcal{Q}^k(S')$. If $u \in \mathcal{Q}^{k+1}(S)$, then either $u \in \mathcal{Q}^k(S)$, implying $u \in \mathcal{Q}^{k+1}(S')$, or there exists $v \in \mathcal{Q}^k(S)$ such that $N[v]\backslash\mathcal{Q}^k(S) = \{u\}$ giving us $N[v]\backslash\{u\} \subseteq \mathcal{Q}^k(S')$. Thus $u \in \mathcal{Q}^{k+1}(S')$. Hence $\mathcal{Q}^i(S) \subseteq \mathcal{Q}^i(S')$ for any $i \in \mathbb{N}_0$, giving us $\mathcal{Q}^\infty(S) \subseteq \mathcal{Q}^\infty(S')$.

To prove 2, let $u \in \mathcal{P}^0(S)$. Then $u \in N[v]$ for some $v \in S$. Since $S \subseteq S'$, we have $u \in \mathcal{P}^0(S')$. Thus, $\mathcal{P}^0(S) \subseteq \mathcal{P}^0(S')$. The remainder of the proof is identical to the proof of 1. □

Since any set $S \subseteq V$ is a subset of the set of vertices it dominates, we have the following observation.

Observation 2.2 $\bar{\gamma}_p(G) \leq \mathrm{F}(G)$.

3 Complexity

In this section, we show that it is NP-hard to determine whether G has a failed power dominating set of cardinality at least k. We use a similar technique to the one used in [22] to show NP-completeness of failed zero forcing parameters.

We state two definitions that we use in this section and the sections that follow. A graph G is *connected* if G contains a path from u to v for every $u, v \in V(G)$, and *disconnected* otherwise. Note that the trivial graph, which is a graph with $|V(G)| = 1$ and $E(G) = \emptyset$, is connected. For $S \subseteq V(G)$ where S is nonempty, the *subgraph of G induced by S*, denoted $G[S]$ has S as its vertex set, and two vertices $u, v \in S$ are neighbors in $G[S]$ if and only if they are neighbors in G.

FAILED POWER DOMINATING SET (FPDS), (G, m)

Instance: Graph $G = (V, E)$ and a positive integer m.

Question: Does G have a proper stalled subset of cardinality at least m?

To prove that FPDS is NP-hard, we construct a polynomial reduction from the well-known NP-complete problem, INDEPENDENT SET, which remains NP-complete when restricted to connected graphs [22].

INDEPENDENT SET, (G, k)

Instance: Connected graph $G = (V, E)$ and a positive integer k.

Question: Does G contain an independent set of cardinality at least k?

The domination number of a path on k vertices, $\gamma(P_k)$, is known to be $\lceil k/3 \rceil$ [14].

Lemma 3.1 *Let G be a graph that contains an induced subgraph P_k, where $k \geq 3$, all internal vertices of P_k have degree 2 in G, and at least one end vertex of P_k has degree 1 in G. If S is an SPDS containing at least one vertex of P_k, then $|S \cap P_k| \geq \gamma(P_k) = \lceil k/3 \rceil$. If S is maximally stalled and contains at least one vertex of P_k, then $|S \cap P_k| \geq k - 1$.*

PROOF: Note that if there are at least two adjacent vertices in $\mathcal{P}^0(S) \cap P_k$, then for some $i \geq 0$, $V(P_k) \subseteq \mathcal{P}^i(S)$. If there is a vertex in $S \cap P_k$, then after the domination step, there are at least two adjacent vertices from P_k in $\mathcal{P}^0(S)$. Thus, if S is stalled, it must be that at least $\gamma(P_k)$ vertices on the path are in S; otherwise, $\mathcal{P}^1(S) \backslash \mathcal{P}^0(S)$ is nonempty.

Since at least $\gamma(P_k)$ vertices on the path P_k are in S, it follows that $\mathcal{P}^0(S)$ contains all vertices in P_k. Thus, if S is maximally stalled and contains at least one vertex of P_k, it must contain all vertices other than the end vertex that may not have degree 1. That is, $|S \cap P_k| \geq k-1$. □

To prove the following lemma, we construct a polynomial reduction from INDEPENDENT SET. An example reduction instance is shown in Figures 3 and 4.

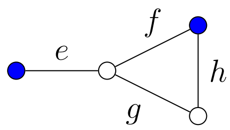


Figure 3: A graph G with independent set of cardinality 2 shown in blue

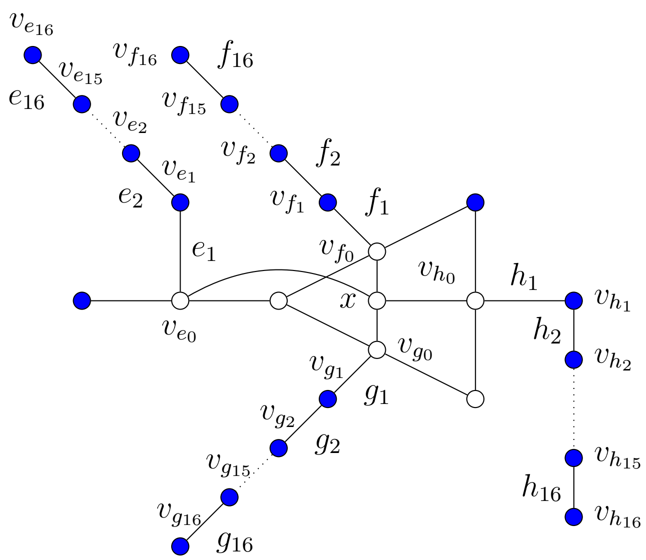


Figure 4: The graph G' with FPDS S in blue, $|S| = 66$

Lemma 3.2 *FAILED POWER DOMINATING SET is NP-hard.*

PROOF: Suppose (G, k) with $n = |V| \geq 3$, $k \geq 2$ is an instance of INDEPENDENT SET. We construct from it an instance (G', m) of FPDS for $m = n^2|E| + k$. Let U be an independent set of G. Then $U' = U \cup V'_1 \cup V'_2 \cup \cdots \cup V'_{n^2}$ is an SPDS of cardinality $n^2|E| + k$ in G', where $G' = (V', E')$ is constructed as follows.

1. $V \subseteq V'$.
2. Subdivide every edge of G. That is, for each $e = \{u, v\} \in E$, add a vertex v_{e_0} to V', and let $\{u, v_{e_0}\}, \{v_{e_0}, v\} \in E'$. Let V'_0 denote these added vertices, and E'_0 the added edges.
3. For each $e = \{u, v\} \in E$, add vertices v_{e_1} through $v_{e_{n^2}}$ to V'. For each $i = 1, 2, \ldots, n^2$, let V'_i denote all $v_i \in V'$, and add edge $e_i = \{v_{e_{i-1}}, v_{e_i}\}$ to E'. Let $P(e)$ denote the path from v_{e_0} to $v_{e_{n^2}}$. Let the set of all such paths be denoted ρ.
4. Add a vertex x to V'. For each vertex $v_{e_0} \in V'_0$, add $\{x, v_{e_0}\}$ to E'.

To see that U' is an SPDS in G', note that $\mathcal{P}^0(U') = U \cup V'_0 \cup V'_1 \cup V'_2 \cup \cdots \cup V'_{n^2}$. If U' is not an SPDS, then $\mathcal{P}^1(U') \backslash \mathcal{P}^0(U')$ is nonempty. The only vertices in $V' \backslash \mathcal{P}^0(U')$ are x and the vertices from $V \backslash U$. We know that $N_{G'}(x) = V'_0$, but each vertex in V'_0 has at least one other neighbor in $V \backslash (U \cup \mathcal{P}^0(S))$ (since U is an independent set in G). Hence, $x \notin \mathcal{P}^1(U')$. Similarly, for any vertex $v \in V \backslash U$, the neighborhood $N_{G'}(v)$ is contained in V'_0. But for each $v_{e_0} \in V'_0$, $N_{G'}(v_{e_0})$ includes x and one vertex from $V \backslash U$. Hence if $v \in V \backslash U$, then $v \notin \mathcal{P}^1(U')$, and U' is stalled.

Suppose that $S \subseteq V(G')$ is maximally stalled with $|S| \geq n^2|E|+2$. We will show that for each path $P(e) \in \rho$, $|S \cap P(e)| \geq n^2$. Since $|V'| = (n^2+1)|E| + n + 1$, there are at most $|E| + n - 1$ vertices in $V' \backslash S$. Each path $P(e)$ has $n^2 + 1$ vertices. Note that $n^2 + 1 - (|E| + n - 1) \geq n^2 - n + 2 - \frac{n(n-1)}{2} > 1$, and thus, $P(e)$ contains at least one vertex in S. By Lemma 3.1, then, $|S \cap P(e)| \geq n^2$, implying that $\cup_{i=0}^{n^2} V'_i \subseteq \mathcal{P}^0(S)$.

We show that $V'_0 \cap S = \emptyset$. Without loss of generality suppose $v_{e_0} \in S$. Then $N_{G'}(v_{e_0}) = \{u, v, x, v_{e_1}\}$ where $e = \{u, v\}$, so $\{u, v, x\} \subseteq \mathcal{P}^0(S)$. Since G is connected, G' is also connected. Thus, there is a path in G' from u to any vertex in V. Since S is properly stalled and $(V' \backslash V) \subseteq \mathcal{P}^0(S)$, it follows that there must be some vertex $y \in V$ such that $y \notin \mathcal{P}^0(S)$. Then, on the path from u to y, there exists $\hat{e} = \{w, z\} \in E$ with $w \in \mathcal{P}^0(S)$ and $z \notin \mathcal{P}^0(S)$. Consider the vertex $v_{\hat{e}_0} \in V'_0$. The set $N_{G'}[v_{\hat{e}_0}] \backslash \mathcal{P}^0(S)$ consists only of the vertex z (since we just noted that $x \in \mathcal{P}^0(S)$), so $z \in \mathcal{P}^1(S) \backslash \mathcal{P}^0(S)$, a contradiction of S being stalled. Hence, $V'_0 \cap S = \emptyset$. Since we know that for each path $P(e) \in \rho$, $|P(e) \cap S| \geq n^2$, it follows that $\cup_{i=1}^{n^2} V'_i \subseteq S$.

Now, we show that $x \notin S$. Suppose $x \in S$. Note that $V \cap S$ is nonempty, because we assumed that $|S| \geq n^2|E| + 2$. Also, $V \backslash S$ is nonempty since we showed that $\cup_{i=0}^{n^2} V'_i \subseteq \mathcal{P}^0(S)$. Since we're assuming that $x \in S$, if $V \subseteq S$, then $\mathcal{P}^0(S) = V'$, contradicting the assumption that S is properly stalled. Hence, there exists an edge $e = \{u, v\} \in E$ with $u \in S$ and $v \notin S$. Then the vertex v_{e_0} has $N_{G'}[v_{e_0}] \backslash \mathcal{P}^0(S) = \{v\}$, and $\mathcal{P}^1(S) \backslash \mathcal{P}^0(S)$ is nonempty, a contradiction of S being stalled. Hence, $x \notin S$.

Finally, we will show that $S \cap V$ is an independent set of G. Suppose there exists an edge $e = \{u, v\} \in E$ for some $u, v \in S$. Then v_{e_0} has $N_{G'}[v_{e_0}] \backslash \mathcal{P}^0(S) = \{x\}$, and $x \in \mathcal{P}^1(S) \backslash \mathcal{P}^0(S)$, a contradiction of S being stalled. Hence, $S \cap V$ is an independent set in G.

This gives us that for any maximal properly stalled subset S of V',

$$|S| = n^2|E| + t,$$

where t is the order of independent set $S \cap V$. Thus G' has an SPDS of order $m = n^2|E| + k$ if and only if G has an independent set of order k. The construction of G' is polynomial and thus this completes our proof that FPDS is NP-hard. □

For a graph G, positive integer k, and $S \subseteq V$ with $|S| \leq k$, it is verifiable in polynomial time whether or not S is a PDS [16]. Thus it is verifiable in polynomial time whether S is an FPDS, completing the proof of the following theorem.

Theorem 3.3 *FAILED POWER DOMINATING SET is NP-complete.*

4 Extreme values

In this section, we characterize n-vertex graphs G with $\bar{\gamma}_p(G) \geq n-3$. We also give some results for the case $\bar{\gamma}_p(G) = 0$.

4.1 High values of $\bar{\gamma}_p(G)$

The next observation follows from the definition of PDS.

Observation 4.1 *If S is a PDS of G, then $\mathcal{P}^0(S)\backslash S$ is a zero forcing set of $G[V\backslash S]$.*

Theorem 4.2 *Let $G = (V, E)$ be a simple graph with n vertices. Then we have the following characterization of graphs with high values of $\bar{\gamma}_p(G)$.*

1. *$\bar{\gamma}_p(G) = n-1$ if and only if G has an isolated vertex.*
2. *$\bar{\gamma}_p(G) = n-2$ if and only if G contains an isolated edge as a component, and no isolated vertices.*
3. *$\bar{\gamma}_p(G) = n-3$ if and only if G contains no components that are isolated vertices or isolated edges and G contains as an induced subgraph one of the following:*
   - *P_3, where only the middle vertex in P_3 may be adjacent to other vertices in V, or*
   - *K_3, where at most one of the vertices may be adjacent to other vertices in V.*

Proof: If G has an isolated vertex v, let $S = V\backslash\{v\}$. Then S is an FPDS, and $\bar{\gamma}_p(G) = n-1$. Conversely, let $\bar{\gamma}_p(G) = n-1$, and let S be an FPDS. If the single vertex $v \in V\backslash S$ has an edge to some vertex $u \in S$, then $v \in \mathcal{P}^0(S)$. Hence, v is isolated, completing the proof of part 1.

If G contains no isolated vertices, and one component is K_2 with vertices u, v, then let $S = V\backslash\{u, v\}$. Then S is an FPDS, and $\bar{\gamma}_p(G) = n-2$.

Conversely, suppose $\bar{\gamma}_p(G) = n-2$. We know G contains no isolated vertices. Let S be an FPDS with $|S| = n-2$. Let u, v be the two vertices in $V\backslash S$. If u is adjacent to some vertex $w \in S$, then $u \in \mathcal{P}^0(S)$, giving us that all vertices except possibly v are in $\mathcal{P}^0(S)$. But then, $v \in \mathcal{P}^1(S)$, implying that S is a PDS. Therefore, neither u nor v is adjacent to any vertex in S, but since there are no isolated vertices, uv forms a copy of K_2, completing the proof of part 2.

If G does not contain any isolated vertex or component that is K_2, then $\bar{\gamma}_p(G) \leq n-3$. If G contains an induced copy of $P_3 = \{u, v, w\}$ with edges uv, vw, note that only v may be adjacent to other vertices in V. Let $S = V\backslash\{u, v, w\}$. Then it is possible that $v \in \mathcal{P}^0(S)$, but $u, w \notin \mathcal{P}^i(S)$ for any $i \geq 0$ since $N(u) = N(w) = \{v\}$. The same holds if $G[\{u, v, w\}]$ forms a copy of K_3.

Conversely, suppose $\bar{\gamma}_p(G) = n - 3$, and let S be an FPDS with $|S| = n - 3$. Let $\{u, v, w\} = V \backslash S$. Suppose $\{u, v\} \subseteq N(S)$. Then $w \notin N(S)$, because $w \in N(S)$ implies $\{u, v, w\} \subseteq \mathcal{P}^0(S)$. However, since w cannot be an isolated vertex, $w \in N(u)$ (without loss of generality) but then $w \in \mathcal{P}^1(S)$, implying that S is a PDS. Hence, only one of $\{u, v, w\}$ may be in $N(S)$. Without loss of generality, say it is v. Since G has no isolated vertices or K_2 component, and vertices u and w have no neighbors outside of $\{u, v, w\}$, then $G[\{u, v, w\}]$ is either K_3 or P_3. If it is K_3, we are done. If it is P_3, and v has any other neighbors in G, note that v must be the middle vertex. If not, $\{u, w\} \subseteq \mathcal{P}^2(S)$, implying S is not an FPDS. This completes the proof of part 3. □

4.2 Graphs in which every vertex is a PDS

In this subsection, we present some results on graphs that have $\bar{\gamma}_p(G) = 0$. Note that if $\bar{\gamma}_p(G) = 0$, then any single vertex is a PDS of G. We use the notation $\mathcal{P}^i_G(S)$ to indicate $\mathcal{P}^i(S)$ in G only when the graph in question is ambiguous. We also use the following graph operation. Given graphs G_1 and G_2, the *join* $G = G_1 \vee G_2$ of G_1 and G_2, has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{\{u, v\} : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Lemma 4.3 *For $n \geq 2$, $\bar{\gamma}_p(G_1 \vee G_2 \vee \cdots \vee G_n) = 0$ if and only if for each $i = 1, 2, \ldots, n$, either $\bar{\gamma}_p(G_i) = 0$ or $G_i = \overline{K_2}$.*

PROOF: Let G_1 and G_2 be graphs, and let $v \in V(G_1)$. Then $\mathcal{P}^0_{G_1 \vee G_2}(\{v\}) = \mathcal{P}^0_{G_1}(\{v\}) \cup V(G_2)$, and as a result, $\mathcal{P}^i_{G_1 \vee G_2}(\{v\}) = \mathcal{P}^i_{G_1}(\{v\}) \cup V(G_2)$ for any $i \geq 0$, unless $G_1 = \overline{K_2}$, in which case $\mathcal{P}^1_{G_1 \vee G_2}(\{v\}) = V(G_1) \cup V(G_2)$. Hence, $\{v\}$ is a PDS in $G_1 \vee G_2$ if and only if $\{v\}$ is a PDS in G_1 or $G_1 = \overline{K_2}$, and similarly for G_2. That is, $\bar{\gamma}_p(G_1 \vee G_2) = 0$ if and only if $\bar{\gamma}_p(G_1) = 0$ or $G_1 = \overline{K_2}$, and $\bar{\gamma}_p(G_2) = 0$ or $G_2 = \overline{K_2}$. We can use the same argument if G_1 or G_2 is itself the join of two graphs. Hence, by induction, $\bar{\gamma}_p(G_1 \vee G_2 \vee \cdots \vee G_n) = 0$ if and only if $\bar{\gamma}_p(G_i) = 0$ or $G_i = \overline{K_2}$ for each $i = 1, 2, \ldots, n$. □

In a poster [23], Tostado-Marquez listed several families of graphs that have $\bar{\gamma}_p = 0$. For $n \geq 4$, a *wheel on n vertices*, W_n, is defined by $W_n = C_{n-1} \vee \{v\}$.

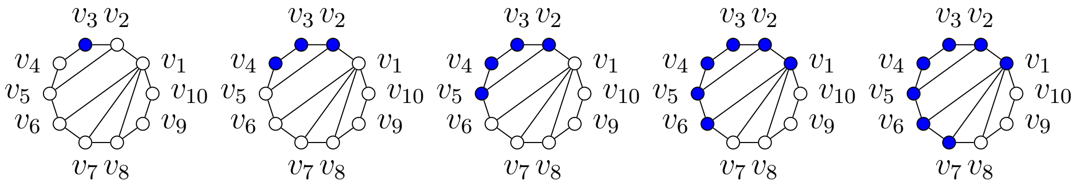
Example 4.4 *The following graphs have $\bar{\gamma}_p = 0$ [23].*

1. *a path on n vertices, P_n for $n \geq 1$,*
2. *a cycle on n vertices, C_n for $n \geq 3$,*
3. *a complete graph on n vertices, K_n, for $n \geq 1$,*
4. *a wheel on n vertices, W_n for $n \geq 4$.*

Note that the proofs that $\bar{\gamma}_p(W_n) = 0$ and $\bar{\gamma}_p(K_n) = 0$ follow immediately by noting that $\bar{\gamma}_p(G) = 0$ if G consists of a single isolated vertex or $G = C_n$ for $n \geq 3$, and applying Lemma 4.3. We add several families of graphs to this list. An example for item 4 from Theorem 4.6 below is shown in Figure 5. In the proof of Theorem 4.6, we use a property that follows from the definitions of PDS and zero forcing sets:

Lemma 4.5 *Suppose G is a graph, and $S \subseteq V$. Suppose that for some $i \geq 0$, a subset S' of $\mathcal{P}^i(S)$ is a zero forcing set of the graph induced by $(V \backslash \mathcal{P}^i(S)) \cup S'$. Then S is a PDS of G.*

Proof: For each $j \geq 0$, $\mathcal{Q}^j(S') \subseteq \mathcal{P}^{i+j}(S)$ in G. Thus, if $\mathcal{Q}^\infty(S') = V(G)$, then $\mathcal{P}^\infty(S') = V(G)$. □



(a) S (b) $\mathcal{P}^0(S)$ (c) $\mathcal{P}^1(S)$ (d) $\mathcal{P}^2(S)$ (e) $\mathcal{P}^3(S)$

Figure 5: A graph G with $\bar{\gamma}_p(G) = 0$ as in Theorem 4.6, item 4. $S = \{v_3\}$ is shown in blue on the left, followed by $\mathcal{P}^0(S)$ through $\mathcal{P}^3(S)$. Continuing, $\mathcal{P}^4(S) = \{v_1, v_2, \ldots, v_8\}$, and $\mathcal{P}^5(S) = V$.

Theorem 4.6 *If G is any of the following graphs, then $\bar{\gamma}_p(G) = 0$.*

1. *$\overline{C_n}$ for $n \geq 5$,*
2. *$\overline{P_n}$ for $n \geq 4$,*
3. *$C_n = v_1 v_2 \ldots v_n v_1$ with k chords: $\{v_1, v_i\}, \{v_1, v_{i+1}\}, \ldots, \{v_1, v_{i+k-1}\}$, where $i \geq 3$, $n \geq 4$, and $i + k \leq n - 1$.*
4. *$C_n = v_1 v_2 \ldots v_n v_1$ with $k+1$ chords: $\{v_1, v_i\}, \{v_1, v_{i+1}\}, \ldots, \{v_1, v_{i+k-1}\}$, and $\{v_2, v_{i-1}\}$ where $i \geq 5$, $n \geq 6$, and $i + k \leq n - 1$.*

Proof: To prove 1, let $G = \overline{C_n}$ with $n \geq 5$, and $S = \{v\}$ for any vertex $v \in V(G)$. Let the two neighbors of v in $C_n = \overline{G}$ be u and w. In $G = \overline{C_n}$, then $\mathcal{P}^0(S) = V(G) \backslash \{u, w\}$, and it follows easily that $\mathcal{P}^1(S) = V(G)$.

To prove 2, note that if $G = \overline{P_n}$ with $n \geq 4$, and $S = \{v\}$ for any vertex v with $\deg(v) = 2$ in P_n, then we can use the same argument as for $\overline{C_n}$. Otherwise, if

$\deg(v) = 1$ in P_n, then $\mathcal{P}^0(S) = V(G)\backslash\{u\}$ where u is the unique neighbor of v in P_n. Since $n \geq 4$, we see that $\mathcal{P}^1(S) = V$, and $S = \{v\}$ is a PDS.

To prove 3, let P_1 and P_2 be the unique paths from v_1 to v_i and from v_1 to v_{i+k-1}, respectively, whose internal vertices all have degree 2. Note that $\{v_1, v_j, v_{j+1}\}$ is a zero forcing set of G for $2 \leq j \leq n-1$. If $S = \{v_1\}$, then $\mathcal{P}^0(S)$ consists of v_1, v_2, v_n, and v_i through v_{i+k-1}, which is a zero forcing set of G. Hence, $S = \{v_1\}$ is a PDS. Similarly, if $S = \{v_j\}$ where $i \leq j \leq i+k-1$, then $\mathcal{P}^0(S)$ includes v_1, v_{j-1}, v_j, and v_{j+1}, a zero forcing set of G. Hence $S = \{v_j\}$ is a PDS. Finally, suppose $S = \{u\}$ for an internal vertex of P_1 or P_2. There exists j such that $v \in \mathcal{P}^j(S)$, and both neighbors of u on the cycle are also in $\mathcal{P}^j(S)$, so $\mathcal{P}^j(S)$ is a zero forcing set, and it follows by Lemma 4.5 that $S = \{u\}$ is a PDS.

To prove 4, note that $\{v_1, v_j, v_{j+1}\}$ is a zero forcing set of G for $2 \leq j \leq n-1$. Let P_1 denote the path with all internal vertices of degree 2 from v_2 to v_{i-1}, and P_2 the similar path from v_{i+k-1} to v_1. If $S = \{v_1\}$, then $\mathcal{P}^0(S)$ contains v_1 and $\{v_j, i \leq j \leq i+k-1\}$, which is a zero forcing set of G; hence $S = \{v_1\}$ is a PDS. Similarly, if $S = \{v_\ell\}$ for $\ell \in \{2, i, i+1, \ldots, i+k-1, n\}$, then $\mathcal{P}^0(S)$ contains $\{v_1, v_j, v_{j+1}\}$ for some j, a zero forcing set. Thus, $S = \{v_\ell\}$ for $\ell \in \{2, i, i+1, \ldots, i+k-1, n\}$ is a PDS. Suppose $S = \{v_{i-1}\}$. Then $\mathcal{P}^0(S) = \{v_{i-2}, v_{i-1}, v_i, v_2\}$. Since v_{i-2} has a unique neighbor v_{i-3} outside of $P^0(S)$, the next vertex along P_1, $\mathcal{P}^1(S) = \{v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_2\}$. This continues for all internal vertices of P_1, giving us that for some ℓ, $\mathcal{P}^\ell(S) = V(P_1) \cup \{v_i\}$. Since v_1 is the only neighbor of v_2 outside of $\mathcal{P}^\ell(S)$, $v_1 \in \mathcal{P}^{\ell+1}(S)$, so $\mathcal{P}^{\ell+1}(S)$ includes at least two adjacent vertices in G as well as v_1, which is a zero forcing set. Hence, $S = \{v_{i-1}\}$ is a PDS. If $S = \{u\}$ where $u \in V(P_1)$ or $u \in V(P_2)$, there exists ℓ such that $\mathcal{P}^\ell(S)$ contains all vertices in P_1 and the vertex v_1, or all vertices in P_2 (which includes v_1). This is a zero forcing set; hence $S = \{u\}$ is a PDS for any $u \in V(P_1)$ or $V(P_2)$. For an example, see Figure 5. □

Note that if G is disconnected, then the vertices of any single component form an FPDS of G, giving us the following observation.

Observation 4.7 *If $\bar{\gamma}_p(G) = 0$, then G is connected.*

In a connected graph G, a vertex $v \in V$ is a *cut-vertex* if $G[V\backslash\{v\}]$ is disconnected. The *path cover number* of a graph G, denoted $\mathrm{P}(G)$, is the minimum number of vertex disjoint paths, each of which is an induced subgraph of G, that contain all vertices of G. Hogben [18, Theorem 2.13] showed that $\mathrm{P}(G) \leq \mathrm{Z}(G)$, which leads to the next theorem.

Theorem 4.8 *Suppose $\bar{\gamma}_p(G) = 0$ and G has either a vertex of degree one or a cut-vertex. Then $G = P_n$ for some $n \geq 1$.*

Proof: Suppose G has a vertex v of degree one, and $\bar{\gamma}_p(G) = 0$. Then $\mathcal{P}^0(\{v\}) = \{u, v\}$ where u is the unique neighbor of v. By Observation 4.1, $\{u\}$ is a zero forcing set of $G[V\backslash\{v\}]$. Thus $\mathrm{Z}(G[V\backslash\{v\}]) = 1$, and by [18, Theorem 2.13], since

$\mathrm{P}(G) \leq \mathrm{Z}(G)$, $\mathrm{P}(G[V\backslash\{v\}]) = 1$. Hence, $G[V\backslash\{v\}]$ is a path, and consequently, G is as well.

Suppose G has a cut-vertex v. Let $u \in V$ with $u \neq v$, and let $\mathcal{K}_u$ be the component of $G[V\backslash\{v\}]$ containing u. Let vertex w be in a different component, $\mathcal{K}_w$ of $G[V\backslash\{v\}]$. By assumption, both $\{u\}$ and $\{w\}$ are PDS. Then there exists some j such that $v \in \mathcal{P}^j(\{u\})$, and all other vertices in $\mathcal{P}^j(\{u\})$ are in the component $\mathcal{K}_u$. Then $|N(v)\backslash V(\mathcal{K}_w)| = 1$, because we assumed that $|N(v)\backslash V(\mathcal{K}_w)| \geq 1$, and if $|N(v)\backslash V(\mathcal{K}_w)| \geq 2$, then $\{u\}$ is an FPDS. Since we can make the same argument using w instead of u, we know that v has exactly two neighbors: $u' \in V(\mathcal{K}_u)$ and $w' \in V(\mathcal{K}_w)$. The set $S' = \{v\}$ is a PDS by assumption. Then $\mathcal{P}^0(S') = \{u', v, w'\}$. $G[V\backslash\{v\}]$ consists of two components, $\mathcal{K}_u$ and $\mathcal{K}_w$, so $\{u'\}$ is a zero forcing set of $\mathcal{K}_u$ and $\{w'\}$ is a zero forcing set of $\mathcal{K}_w$. Thus, $\mathcal{K}_u$ is a path with end vertex u', and $\mathcal{K}_w$ is a path with end vertex w'. It follows that $G = P_n$. □

5 Values of $\bar{\gamma}_p(G)$ for special graphs

In this section, we determine the value of $\bar{\gamma}_p(G)$ for some specific graph families.

Theorem 5.1 *The failed power domination number of the complete bipartite graph $K_{m,n}$ with $m \geq n \geq 1$ is given by*

$$\bar{\gamma}_p(K_{m,n}) = \begin{cases} m-2 & \text{if } m \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof: If $m = 1$, $G = K_2$, clearly resulting in $\bar{\gamma}_p(G) = 0$. If $m = n = 2$, $K_{m,n} = C_4$, so $\bar{\gamma}_p(K_{m,n}) = 0$ by Example 4.4, item 2. If $n = 1$ but $m \geq 2$, then $\bar{\gamma}_p(K_{m,n}) = m-2$ by Theorem 4.2, item 3.

Assume $m \geq n \geq 2$ and let $S \subseteq V_1$ with $|S| = m-2$. Let u, v be the vertices in $V_1\backslash S$. Then $\mathcal{P}^0(S) = S \cup V_2$, and $V\backslash\mathcal{P}^0(S) = \{u, v\}$. Since $N(u) = N(v) = V_2$, $\mathcal{P}^\infty(S) = \mathcal{P}^0(S)$, S is stalled, and $\bar{\gamma}_p(G) \geq m-2$. Note that if S consists of vertices in both V_1 and V_2, then $\mathcal{P}^0(S) = V$, so if S is an FPDS, $S \subseteq V_1$ or $S \subseteq V_2$. Then $|S| \leq m-2$ because if $|S| > m-2$, then $\mathcal{P}^1(S) = V$. Hence $\bar{\gamma}_p(G) = m-2$. □

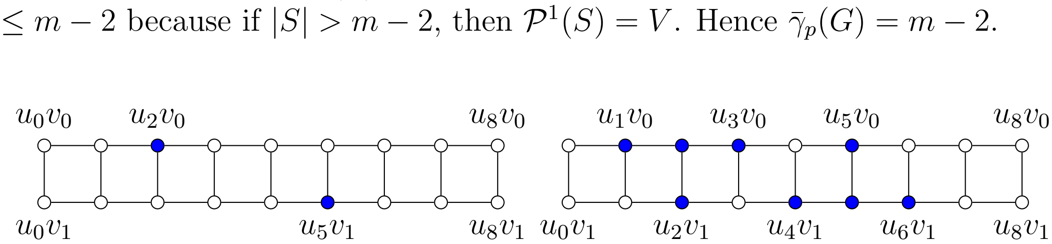


Figure 6: A ladder graph, $P_9 \square P_2$ with FPDS S in blue on the left and $\mathcal{P}^0(S)$ in blue on the right.

For graphs G and H, we denote by $G \square H$ the *Cartesian product* of G and H, where $V(G \square H) = V(G) \times V(H)$. If $u_1, u_2 \in V(G)$ and $v_1, v_2 \in V(H)$, then (u_1, v_1)

is adjacent to (u_2, v_2) in $G \Box H$ if and only if $u_1 = u_2$ in G and $\{v_1, v_2\} \in E(H)$, or $\{u_1, u_2\} \in E(G)$ and $v_1 = v_2$ in H. Here, we write u_iv_j for vertex (u_i, v_j) for brevity.

A *ladder graph* is the graph $P_n \Box P_2$ for $n \geq 2$. Each copy of P_2 is called a *rung*.

Theorem 5.2 *For the ladder graph $G = P_k \Box P_2$ with $k \geq 4$, $\bar{\gamma}_p(G) = \left\lceil \frac{k-4}{3} \right\rceil$.*

Proof: Let the vertices of P_k be denoted by u_i, $0 \leq i \leq k-1$, and the vertices of P_2 by v_j, $j = 0$ or 1. Define $S \subseteq V(P_k \Box P_2)$ by $u_iv_j \in S$ if and only if $i \equiv 2 \mod 3$ with $2 \leq i \leq k-3$ and $j \equiv i \mod 2$, as in Figure 6. Then $|S| = \left\lceil \frac{k-4}{3} \right\rceil$. We show that S is an FPDS. Suppose that $u_iv_j \in \mathcal{P}^0(S)$. If $u_iv_j \in S$, then $N(u_iv_j) \subseteq \mathcal{P}^0(S)$. Otherwise, if $u_iv_j \in \mathcal{P}^0(S) \backslash S$, then u_iv_j has exactly two neighbors in $V \backslash \mathcal{P}^0(S)$, namely $u_{i-1}v_j$ or $u_{i+1}v_j$, and $u_iv_{j'}$ where $j' \equiv (j+1) \mod 2$. Thus $\mathcal{P}^\infty(S) = \mathcal{P}^0(S)$; that is, S is an SPDS, giving us $\bar{\gamma}_p(G) \geq \lceil \frac{k-4}{3} \rceil$.

To show that $\bar{\gamma}_p(G) \leq \lceil \frac{k-4}{3} \rceil$, note that if $\{u_0v_0, u_0v_1\} \subseteq Z$, then Z is a zero forcing set, and similarly for $\{u_{k-1}v_0, u_{k-1}v_1\}$. If $u_0v_j \in S$, note that $\{u_0v_0, u_0v_1\} \subseteq \mathcal{P}^0(S)$, which implies that $\mathcal{P}^0(S)$ is a zero forcing set, and S is a PDS, and similarly for the case that $u_{k-1}v_j \in S$. Further, if $u_1v_j \in S$, then $\{u_0v_j, u_1v_j\} \subseteq \mathcal{P}^0(S)$, and $\{u_0v_0, u_0v_1, u_1v_j\} \subseteq \mathcal{P}^1(S)$, meaning that $\mathcal{P}^1(S)$ is a zero forcing set, and S is a PDS, and similarly for $u_{k-2}v_j \in S$. Thus, if S is an FPDS, $u_iv_j \notin S$ for $i \in \{0, 1, k-2, k-1\}$ and $j \in \{0, 1\}$. That is, no vertices from the first two or last two rungs of the ladder are in any FPDS.

Also, if $\{u_iv_0, u_iv_1, u_{i+1}v_0, u_{i+1}v_1\} \subseteq Z$ for any $i \leq k-2$ (that is, if all vertices from two consecutive rungs are in Z), then Z is a zero forcing set. Thus, if S is an FPDS with $u_iv_j \in S$, then $u_{i-1}v_j, u_{i-1}v_{j'}, u_{i+1}v_j, u_{i+1}v_{j'} \notin S$, and further, $u_{i-2}v_{j'}, u_{i+2}v_{j'} \notin S$ for $j' \equiv j \mod 2$. Suppose that $u_iv_j, u_{i+2}v_j \in S$ for $2 \leq i \leq k-5$ and $j = 1$ or 2. Then $\{u_iv_j, u_iv_{j'}, u_{i+1}v_j, u_{i+2}v_j, u_{i+2}v_{j'}\ \} \subseteq \mathcal{P}^0(S)$ where $j' = (j+1) \mod 2$. Since $u_{i+1}v_{j'}$ is the only neighbor of $u_{i+1}v_j$ outside of $\mathcal{P}^0(S)$, $u_{i+1}v_j \in \mathcal{P}^1(S)$, giving us that $\{u_iv_j, u_iv_{j'}, u_{i+1}v_j, u_{i+1}v_{j'}, u_{i+2}v_j, u_{i+2}v_{j'}\ \} \subseteq \mathcal{P}^1(S)$. This forms a zero forcing set of G; hence, S is a PDS. Thus, if S is an FPDS with $u_iv_j, u_{i'}v_{j'} \in S$, then $|i - i'| \geq 3$. That is, $\bar{\gamma}_p(G) \leq \lceil \frac{k-4}{3} \rceil$. □

In the proof of the next theorem, we refer to copies of K_k in $K_k \Box P_\ell$ as *layers*.

Theorem 5.3 *For the graph $G = K_k \Box P_\ell$ with $k, \ell \geq 3$, $\bar{\gamma}_p(G) = (k-2) \left\lfloor \frac{\ell-1}{2} \right\rfloor$.*

Proof: Let the vertices of K_k be denoted w_i, $0 \leq i \leq k-1$, and the vertices of P_ℓ denoted x_j, $0 \leq j \leq \ell-1$. Define $S \subseteq V(K_k \Box P_\ell)$ by $w_ix_j \in S$ if and only if $i \leq k-3$ and j is odd with $j < \ell - 1$. Then $\mathcal{P}^0(S) = V(G \Box H) \backslash (\{w_ix_j | i \geq k-2 \text{ and } j \text{ is even}\} \cup \{w_ix_{\ell-1} | \text{ for any } i \text{ if } \ell \text{ is even}\})$. If $w_ix_j \in \mathcal{P}^0(S)$ with j even, then w_ix_j is adjacent to $w_{k-2}x_j$ and $w_{k-1}x_j$; if $w_ix_j \in S$ with j odd and $i < k-2$, then $N(w_ix_j) \subseteq \mathcal{P}^0(S)$; if $w_ix_j \in \mathcal{P}^0(S)$ with j odd and $i \geq k-2$, then w_ix_j is adjacent to vertices w_ix_{j-1} and w_ix_{j+1}, both of which are not in $\mathcal{P}^0(S)$. Hence $\mathcal{P}^1(S) = \mathcal{P}^0(S)$, and S is an FPDS with $|S| = (k-2) \left\lfloor \frac{\ell-1}{2} \right\rfloor$, giving us $\bar{\gamma}_p(G) \geq (k-2) \left\lfloor \frac{\ell-1}{2} \right\rfloor$.

Next we show that $\bar{\gamma}_p(G) \leq (k-2) \left\lfloor \frac{\ell-1}{2} \right\rfloor$. Consider any $S' \subseteq V(K_k \Box P_\ell)$. Note that if any of the following conditions holds, then S' is a PDS.

(c1) $w_i x_0 \in S'$ or $w_i x_{\ell-1} \in S'$ for any i.

(c2) $w_i x_1 \in S'$ for all but one i, or $w_i x_{\ell-2} \in S'$ for all but one i.

(c3) $w_h x_j \in S'$ and $w_i x_{j+1} \in S'$ for any h, i, j.

(c4) $w_i x_j \in S'$ for all i and for some j.

(c5) $w_i x_j \in S'$ for some j and all but one i, and $w_h x_{j-2} \in S'$ or $w_h x_{j+2} \in S'$ for any h.

(c6) $w_i x_j \in S'$ for some j and all i except $i = \hat{\imath}$, and $w_h x_{j-3} \in S'$ or $w_h x_{j+3} \in S'$ for any h, where $h \neq \hat{\imath}$.

If $\ell = 3$, the result follows immediately by (c1) and (c2), so we assume $\ell \geq 4$. We first assume $k \geq 4$. Let H be a subgraph of G induced by the vertices of four consecutive layers of G and let S'_H denote S' restricted to H. We will show that if S' is an FPDS, then $|S'_H| \leq 2(k-2)$. Note that if S'_H is a PDS of H, then S' is a PDS of G. By (c3), at most two layers of H can have vertices in S'. If each has less than $k-1$ vertices in S', then $|S'_H| \leq 2(k-2)$. By (c4), the only remaining possibility is that some layer j of H has $k-1$ vertices in S', but by (c5) and (c6), then there is at most one vertex outside of layer j in S'_H, giving us $|S'_H| \leq k$. Since we assume for this part of the proof that $k \geq 4$, it follows that $|S'_H| \leq 2(k-2)$, and since H consists of any consecutive four layers of G, it follows that any FPDS S of G has $S \leq (k-2)\left\lfloor \frac{\ell-1}{2} \right\rfloor$.

Finally, if $k = 3$ we show that $\bar{\gamma}_p(G) \leq \lfloor \frac{\ell-1}{2} \rfloor$. Call a layer with s vertices from S' an *s-layer*. If S' is an FPDS and all layers are 0- or 1-layers, then (c1) and (c 3) imply that $|S'| \leq \lfloor \frac{\ell-1}{2} \rfloor$. If there are any 3-layers, then S' is a PDS by (c4). The only remaining case is that some layer is a 2-layer. By (c3) and (c5), two layers immediately before the first occurring 2-layer and immediately after the last occurring 2-layer are 0-layers. Also, any closest pair of 2-layers (that is, a pair of 2-layers with no other 2-layers between them) must have at least three 0-layers between them. Together, these constraints imply that if S' is an FPDS, then $|S'| \leq \lfloor \frac{\ell-1}{2} \rfloor$. □

6 Future work

While we were able to produce a list of graphs that have $\bar{\gamma}_p(G) = 0$ (where every single vertex is itself a PDS), a complete description of all such graphs is still open. The zero forcing number of trees has been related to other parameters such as the path cover number [2], and a technique for determining the zero forcing number of a graph with a cut-vertex was also described [21]. Achieving similar results for the failed power domination number of a graph should be a feasible task. Many parameters in zero forcing, especially related to minimum rank, are investigated for their adherence to a property known as the *Graph Complement Conjecture* which states that the sum of the parameter on G and on the complement graph $\overline{G}$ is bounded by $|V(G)|$ plus a small constant. For minimum rank, $\mathrm{mr}(G)$, the conjecture is: $\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) = |V(G)| + 2$. Originally mentioned at an American Institute for Mathematics workshop [3], it formally appeared in [7]. It is natural, and likely challenging, to investigate whether there is any similar relationship among power dominating numbers or failed power dominating numbers of graphs and their complements.

References

[1] A. Aazami, Domination in graphs with bounded propagation: algorithms, formulations and hardness results, *J. Comb. Optim.* **19** (4) (2010), 429–456.

[2] AIM Minimum Rank-Special Graphs Work Group, Zero forcing sets and the minimum rank of graphs, *Linear Algebra Appl.* **428** (7) (2008), 1628–1648.

[3] American Institute of Mathematics Workshop, *Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns*, Palo Alto, CA, (2006), Webpage: `http://aimath.org/pastworkshops/matrixspectrum.html`.

[4] T. Ansill, B. Jacob, J. Penzellna and D. Saavedra, Failed skew zero forcing on a graph, *Linear Algebra Appl.* **509** (2016), 40–63.

[5] T. L. Baldwin, L. Mili, M. B. Boisen and R. Adapa, Power system observability with minimal phasor measurement placement, *IEEE Trans. Power Syst.* **8** (2) (1993), 707–715.

[6] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche and H. van der Holst, Zero forcing parameters and minimum rank problems, *Linear Algebra Appl.* **433** (2) (2010), 401–411.

[7] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben and H. van der Holst, On the graph complement conjecture for minimum rank, *Linear Algebra Appl.* **436** (12) (2012), 4373–4391.

[8] K. F. Benson, D. Ferrero, M. Flagg, V. Furst, L. Hogben, V. Vasilevska and B. Wissman, Zero forcing and power domination for graph products, *Australas. J. Combin.* **70** (2018), 221–235.

[9] D. J. Brueni and L. S. Heath, The PMU placement problem, *SIAM J. Discrete Math.* **19** (3) (2005), 744–761.

[10] D. Burgarth and V. Giovannetti, Full control by locally induced relaxation, *Phys. Rev. Lett.* **99** (10) (2007), 100501.

[11] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs*, Chapman and Hall/CRC (2010).

[12] D. Ferrero, L. Hogben, F. H. J. Kenter and M. Young, Note on power propagation time and lower bounds for the power domination number, *J. Comb. Optim.* **34** (3) (2017), 736–741.

[13] K. Fetcie, B. Jacob and D. Saavedra, The failed zero forcing number of a graph, *Involve* **8** (1) (2015), 99–117.

[14] A. Frendrup, M. A. Henning, B. Randerath and P. D. Vestergaard, An upper bound on the domination number of a graph with minimum degree 2, *Discrete Math.* **309** (4) (2009), 639–646.

[15] C. Grigorious, T. Kalinowski and S. Stephen, On the power domination number of de Bruijn and Kautz digraphs, In *Combinatorial Algorithms*, *Lec. Notes in Comp. Sci.* **10765**, Springer, Cham (2018), 264–272.

[16] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi and M. A. Henning, Domination in graphs applied to electric power networks, *SIAM J. Discrete Math.* **15** (4) (2002), 519–529.

[17] G. T. Heydt, C.-C. Liu, A. G. Phadke and V. Vittal, Solution for the crisis in electric power supply, *IEEE Comput. Appl. Power* **14** (3) (2001), 22–30.

[18] L. Hogben, Minimum rank problems, *Linear Algebra Appl.* **432** (8) (2010), 1961–1974.

[19] J. Kneis, D. Mölle, S. Richter and P. Rossmanith, Parameterized power domination complexity, *Inform. Process. Lett.* **98** (4) (2006), 145–149.

[20] R. F. Nuqui and A. G. Phadke, Phasor measurement unit placement techniques for complete and incomplete observability, *IEEE Trans. Power Del.* **20** (4) (2005), 2381–2388.

[21] D. D. Row, A technique for computing the zero forcing number of a graph with a cut-vertex, *Linear Algebra Appl.* **436** (12) (2012), 4423–4432.

[22] Y. Shitov, On the complexity of failed zero forcing, *Theoret. Comput. Sci.* **660** (2017), 102–104.

[23] J. Tostado-Marquez, Failed power domination, In *Society for Advancement of Chicanos/Hispanics and Native Americans in Science (SACNAS) National Diversity in STEM Conference*, (2017).

[24] S. Varghese, A. Vijayakumar and A. M. Hinz, Power domination in Knödel graphs and Hanoi graphs, *Discuss. Math. Graph Theory* **38** (1) (2018), 63–74.

(Received 7 Aug 2018; revised 24 Oct 2019, 15 Dec 2019)