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On maximal isolation sets in the uniform intersection matrix

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Abstract

Let $A_{k,t}$ be the 0, 1-matrix of size $\binom{k}{t} \times \binom{k}{t}$, whose rows and columns are labeled by all t-uniform subsets of $\{1, 2, \ldots, k\}$, such that there is a one in the entry on row x and column y if and only if the two subsets x, y intersect. We give constructions of large isolation sets in $A_{k,t}$, where our constructions are the best possible for large enough k. We first prove that the largest identity submatrix in $A_{k,t}$ is of size $k - 2t + 2$, for $k \geq 2t$. Then we provide constructions of isolation sets in $A_{k,t}$, for any $t \geq 2$:

- If $k = 2t + r$ and $0 \leq r \leq 2t - 3$, there exists an isolation set of size $2r + 3 = 2k - 4t + 3$.
- If $k \geq 4t - 3$, there exists an isolation set of size k.

The construction is maximal for $k \geq 4t - 3$, since the Boolean rank of $A_{k,t}$ is k in this case. We prove that the construction is maximal also for $k = 2t, 2t+1$. Finally, we consider the problem of the maximal triangular isolation submatrix of $A_{k,t}$ that has ones in every entry on or below the main diagonal, and zeros elsewhere. We give an optimal construction of such a submatrix of size $(\binom{2t}{t} - 1) \times (\binom{2t}{t} - 1)$, for any $t \geq 1$ and large enough k. This construction is tight, as there is a matching upper bound which can be derived from a theorem of Frankl about skew matrices.

1 Introduction

Intersecting families of subsets have been studied extensively over the years (see, for example, [2, 3, 4, 9]). Some of these results can be inferred as statements about families of maximal submatrices of the 0, 1-matrix $A_{k,t}$ of size $\binom{k}{t} \times \binom{k}{t}$, whose rows and columns are labeled by all t-uniform subsets of $[k] \stackrel{\text{def}}{=} \{1, 2, \ldots, k\}$, such that there is a one in the entry on row x and column y if and only if the two subsets x, y intersect.

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For example, Pyber's work in [9] about maximal cross-intersecting families of t-uniform subsets implies that if $k \geq 2t$ then the largest all-ones square submatrix of $A_{k,t}$ is of size $\binom{k-1}{t-1}^2$. Using a theorem of Bollobás [2], it is possible to show that the largest submatrix representing a crown graph in $A_{k,t}$ is of size $\binom{2t}{t} \times \binom{2t}{t}$, where a crown graph is a complete bipartite graph from which the edges of a perfect matching have been removed.

Here we suggest to continue and explore various families of maximal submatrices of $A_{k,t}$. In particular, we would like to find small submatrices of $A_{k,t}$ whose Boolean rank is large. The *Boolean rank* of a $0,1$-matrix B of size $n \times m$ is equal to the smallest integer r, such that B can be factorized as a product of two $0,1$-matrices, $X \cdot Y = B$, where X is a matrix of size $n \times r$ and Y is a matrix of size $r \times m$, and all additions and multiplications are Boolean (that is, $1+1=1$, $1+0=0+1=1$, $1 \cdot 1 = 1, 1 \cdot 0 = 0 \cdot 1 = 0$). The Boolean rank is also equal to the minimal number of monochromatic combinatorial rectangles required to cover all of the ones of B, and it is equal to the minimal number of bicliques needed to cover the edges of the bipartite graph whose adjacency matrix is B (see [5]). Lastly, the Boolean rank is also tightly related to the notion of nondeterministic communication complexity [6].

The Boolean rank of $A_{k,t}$ was shown to be k for any $1 \leq t \leq k/2$ (see [8]). Furthermore, it was proved in [8] that there exists a family of submatrices of $A_{k,t}$, each of size $(m \cdot s) \times (m \cdot s)$, where $m = \binom{2t-2}{t-1}$ and $s = k - 2t + 2$, whose Boolean rank is also k, for a large range of values of k, t. These submatrices are rather large, and a question that arises is if there are smaller submatrices of $A_{k,t}$ whose Boolean rank is k, or as close as possible to k. We answer this question and prove that for large enough k, there are, in fact, submatrices of size $k \times k$ of $A_{k,t}$, whose Boolean rank is k.

Natural candidates for small matrices with a large Boolean rank are *isolation sets* (or *fooling sets* as they are called in communication complexity). An isolation set for a Boolean matrix B is a subset of entries F in B that are all ones of B, such that no two ones in F are in the same row or column of B, and no two ones in F are contained in an all-one submatrix of size 2×2 of B. Throughout the paper we represent an isolation set of a given matrix B as a submatrix F of B, where the ones of the isolation set are on the main diagonal of F, and F is called an *isolation matrix*. The Boolean rank of an isolation matrix of size $f \times f$ is equal to f, and therefore, the size of the maximal isolation set in a given matrix is a lower bound on the Boolean rank of that matrix (see for example [1, 6]). Hence, finding large isolation sets in $A_{k,t}$ answers partially the question of finding small submatrices of $A_{k,t}$ with a large Boolean rank.

If $k < 2t$, then $A_{k,t}$ is just the all-ones matrix, since every two subsets of size t intersect, and thus, the largest isolation set is of size 1. Therefore, the question of finding large isolation sets in $A_{k,t}$ is interesting only for $k \geq 2t$. The simplest form of an isolation matrix is the identity matrix. Hence, we first consider the problem of determining the size of the largest identity submatrix in $A_{k,t}$, and prove in Section 2 the following theorem:

Theorem 1.1 *Let $k \geq 2t$. The largest identity submatrix in $A_{k,t}$ is of size $s \times s$, where $s = k - 2t + 2$.*

Recall that the complement of $A_{k,t}$ is the adjacency matrix of the Kneser graph $KG_{k,t}$, in which the vertices are all t-uniform subsets of $[k]$, and there is an edge between two subsets x, y if and only if $x \cap y = \emptyset$. Furthermore, the complement of the identity matrix is the adjacency matrix of the crown graph of the same size. Thus, from Theorem 1.1, we immediately get that the largest submatrix representing a crown graph in $KG_{k,t}$ is of size $s = k - 2t + 2$. Note that $k - 2t + 2$ is also the chromatic number of $KG_{k,t}$ (see [7]).

Another simple isolation matrix is the *triangular matrix* with ones in every entry on and below the main diagonal, and zeros elsewhere. In Section 3 we give an optimal construction of such a triangular matrix in $A_{k,t}$, and prove the next theorem:

Theorem 1.2 *For any $t \geq 1$ and large enough k, the maximal triangular submatrix of $A_{k,t}$ is of size $m \times m$, where $m = \binom{2t}{t} - 1$.*

The construction presented in Theorem 1.2 uses ideas similar to those of Tuza [10], and is shown to be optimal using a result of Frankl [4] that proved a skew version of a theorem of Bollobás [2].

As can be seen, the size of the maximal triangular submatrix of $A_{k,t}$ given in Theorem 1.2, does not depend on k (as long as k is large enough). Thus, for large enough k, the maximal identity submatrix I_s promised by Theorem 1.1, is a larger isolation submatrix in $A_{k,t}$. But is I_s the largest isolation matrix in $A_{k,t}$? If $t = 1$ then $A_{k,t} = I_k = I_s$, and in this case, this is, of course, the maximal isolation set.

As we prove in Section 4, for $2 \leq t \leq k/2$, there are larger isolation sets, and the submatrix I_s is not the largest isolation matrix for these values of t and k. In fact, when k is large enough, there exists an isolation set of size k in $A_{k,t}$:

Theorem 1.3 *For any $t \geq 2$, the matrix $A_{k,t}$ has an isolation set of the following size:*

- *If $k = 2t + r$ and $0 \leq r \leq 2t - 3$, there exists an isolation set of size $2r + 3 = 2k - 4t + 3$.*
- *If $k \geq 4t - 3$ there exists an isolation set of size k.*

Notice that for any fixed given t, the size of the isolation set stated in Theorem 1.3, starts at 3 when $k = 2t$, and then grows by an additive term of two when k is increased by one, until the point that $k = 4t - 3$. Then, we get an isolation set of maximal size k.

It is also not hard to verify that our construction is maximal for $k = 2t$ (see also [1] which discuss the case of $k = 2t$). We conclude by proving in Section 5 that our construction is maximal also for $k = 2t + 1$:

Theorem 1.4 *If $k = 2t + 1$ and $t \geq 2$, then the size of any isolation set in $A_{k,t}$ is at most 5.*

It remains an open problem to prove whether the construction proved in Theorem 1.3 is maximal also for $2t + 2 \leq k \leq 4t - 4$.

2 The maximal identity submatrix in $A_{k,t}$

In all that follows we denote the identity matrix of size $n \times n$ by I_n, and refer to the subsets representing a row or column of $A_{k,t}$ as row or column indices. Therefore, each row or column index is a subset of $\binom{[k]}{t}$. We now prove Theorem 1.1, and show that the maximal identity submatrix of $A_{k,t}$ is of size $s \times s$, where $s = k - 2t + 2$.

First notice that there exists such a large identity submatrix in $A_{k,t}$. Just take s row indices of the form $\{1, 2, \ldots, t-1\} \cup \{i\}$ and column indices of the form $\{t, t+1, \ldots, 2t-2\} \cup \{i\}$, for $i = 2t-1, 2t, \ldots, k$. This defines an identity submatrix of $A_{k,t}$ of size $s \times s$.

We next show that this is the largest identity submatrix possible in $A_{k,t}$. Clearly this is true for a submatrix on the main diagonal of $A_{k,t}$. Assume, by contradiction, that there exists an identity submatrix I_{s+1} on the main diagonal of $A_{k,t}$, and let $x_1, \ldots, x_{s+1}$ be the row and column indices of I_{s+1}, where we have that $x_i \cap x_j = \emptyset$ if and only if $i \neq j$. But then we get an independent set of size $s + 1$ in $A_{k,t}$ that includes $x_1, \ldots, x_{s+1}$. Thus, the complement of $A_{k,t}$, that is, the Kneser graph $KG_{k,t}$, has a clique of size $s + 1$. This is in contradiction to the fact that the chromatic number of $KG_{k,t}$ is s (see [7]). In general though, the identity submatrix does not have to be on the main diagonal of $A_{k,t}$, and thus, a different proof is needed.

Proof of Theorem 1.1: Let I_n be a submatrix of $A_{k,t}$, where $n \geq 2$, and $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the row and column indices of I_n, respectively.

Let $m = \min_{1 \leq i \leq n} |x_i \cap y_i| \geq 1$, and assume without loss of generality that the minimum is attained for $i = 1$. Note that by the structure of I_n, the subsets $x_1 \setminus y_1$, $y_1 \setminus x_1$, $x_1 \cap y_1$, $x_2 \cap y_2$, $\ldots$, $x_n \cap y_n$ are all pairwise disjoint. Thus the sum of their sizes is at most k, and using the definition of m we obtain the following:

$$k \geq |x_1 \setminus y_1| + |y_1 \setminus x_1| + \sum_{i=1}^{n} |x_i \cap y_i| \geq 2(t - m) + nm \geq 2t + (n-2)m \geq 2t + n - 2.$$

Hence $n \leq k - 2t + 2$ as claimed. □

The following bound on the largest crown graph that is a submatrix of $KG_{k,t}$ is an immediate consequence of Theorem 1.1.

Corollary 2.1 *The largest submatrix representing a crown graph in $KG_{k,t}$, is of size $s \times s$, where $s = k - 2t + 2$.*

3 Maximal triangular matrices in $A_{k,t}$

As stated in the introduction, a theorem of Bollobás [2] allows one to show that the largest submatrix representing a crown graph in $A_{k,t}$, is of size $\binom{2t}{t} \times \binom{2t}{t}$, and this result is tight. Specifically, Bollobás proved that if (A_i, B_i) are m pairs of sets, such that $|A_i| = a, |B_i| = b$ for $1 \leq i \leq m$, and $A_i \cap B_j = \emptyset$ if and only if $i = j$, then $m \leq \binom{a+b}{a}$.

This theorem of Bollobás has several generalizations; among them is a result of Frankl [4] that considers the skew version of the problem, and shows that the same bound holds even under the following relaxed assumptions: Let (A_i, B_i) be pairs of sets such that $|A_i| = a$, $|B_i| = b$ for $1 \leq i \leq m$, $A_i \cap B_i = \emptyset$ for every $1 \leq i \leq m$, and $A_i \cap B_j \neq \emptyset$ if $i > j$. Then $m \leq \binom{a+b}{a}$. Note that for this formulation of the problem, all entries below the main diagonal are ones, but above the main diagonal there can be either zeros or ones.

In this section we prove Theorem 1.2, which addresses the following special case. What is the maximal number m of pairs of subsets (A_i, B_i) such that $|A_i| = |B_i| = t$ for every $1 \leq i \leq m$, with $A_i \cap B_j \neq \emptyset$ if and only if $i \geq j$?

Such a set of m pairs of subsets defines a *triangular* submatrix of $A_{k,t}$ of size $m \times m$, with ones on and below the main diagonal, and zeros elsewhere. Denote such a matrix by D_m, and notice that D_m is an isolation matrix. We first show how the result of Frankl [4], stated above, can be used to give a simple upper bound on the size of any triangular submatrix D_m of $A_{k,t}$.

Claim 3.1 *Let D_m be a triangular submatrix of $A_{k,t}$. Then $m \leq \binom{2t}{t} - 1$.*

Proof: To verify this, simply add to any maximal triangular submatrix an additional first row and last column that are all zero (for large enough k, it is always possible to define one more row index and column index that do not intersect with any of the given row and column indices of the submatrix). Thus, we get a matrix in which the main diagonal is all-zero, and below the main diagonal all elements are one. By the result of Frankl, the size of such a matrix is at most $\binom{2t}{t} \times \binom{2t}{t}$. The claim follows. □

We now proceed to prove Theorem 1.2, and show a construction of a triangular submatrix D_m of $A_{k,t}$, which matches the above upper bound. The construction we describe is recursive, using an idea similar to that of Tuza in [10].

Proof of Theorem 1.2: Let $f(a, b)$ be the maximal m such there exists a triangular matrix of size $m \times m$, defined by row indices that are subsets of size a and column indices that are subsets of size b. As we show below in Lemma 3.2, we have $f(a, b) \geq g(a, b)$, where $g(a, b)$ is the following recursive function:

$$g(a,b) = \begin{cases} a, & \text{if } b = 1 \\ b, & \text{if } a = 1 \\ g(a, b-1) + g(a-1, b) + 1, & \text{otherwise.} \end{cases}$$

It is easy to verify that the solution of this recursion is $g(a,b) = \binom{a+b}{a} - 1$.

Since $f(t,t)$ is the size of the maximal triangular submatrix D_m of $A_{k,t}$ (for large enough k), we have $f(t,t) \geq g(t,t) = \binom{2t}{t} - 1$. By Claim 3.1, it also holds that $f(t,t) \leq \binom{2t}{t} - 1$. The theorem follows. □

Lemma 3.2 *Let $a, b \geq 1$ and let $f(a,b)$ be as defined in the proof of Theorem 1.2. Then:*

$$f(a,b) = \begin{cases} a, & \text{if } b = 1 \\ b, & \text{if } a = 1 \end{cases}$$

and otherwise,

$$f(a,b) \geq f(a,b-1) + f(a-1,b) + 1.$$

Proof: The proof is by induction on a and b. The base of the induction is $a = 1$ or $b = 1$. Asumme first that $b = 1$. To see that $f(a,1) \geq a$, take as row indices the subsets $\{1, a+1, \ldots, 2a-1\}, \{1, 2, a+1, \ldots, 2a-2\}, \ldots \{1, 2, \ldots, a\}$, and as column indices the subsets $\{1\}, \{2\}, \ldots, \{a\}$.

For the lower bound on $f(a,1)$, assume by contradiction that $f(a,1) \geq a+1$, and let the first $a+1$ column indices be $\{1\}, \{2\}, \ldots, \{a+1\}$. Since the last row of the matrix is all-ones, then the index of the last row intersects with all column indices. Thus, it contains the subset $\{1, 2, \ldots, a+1\}$, in contradiction to the fact that the size of the row indices is a. Hence, $f(a,1) = a$ as claimed. Similar arguments hold for $f(1,b)$, while exchanging the row and column indices.

Assume now that $a, b > 1$, and using the induction hypothesis, let $D'_{a,b-1}$ be a triangular submatrix of size $f(a, b-1)$, with row indices of size a and column indices of size $b-1$, and let $D''_{a-1,b}$ be a triangular submatrix of size $f(a-1,b)$, with row indices of size $a-1$ and column indices of size b.

Assume that each row index of $D'_{a,b-1}$ is disjoint from all column indices of $D''_{a-1,b}$ (this is always possible for a large enough range of elements for the indices), and let x be a new element that does not appear in any of the row or column indices of $D'_{a,b-1}$ or of $D''_{a-1,b}$. Add x to each column index of $D'_{a,b-1}$ and to each row index of $D''_{a-1,b}$. Therefore, the row and column indices of these two matrices are now subsets of size a and b, respectively, and each column index of $D'_{a,b-1}$ intersects all row indices of $D''_{a-1,b}$ (as they all contain x).

Now add to $D'_{a,b-1}$ one more row and column, as a last row and column, defined by the row index $\{x\} \cup S$, and the column index $\{x\} \cup T$, where S is a subset of size $a-1$ and T is a subset of size $b-1$, and S and T are disjoint from all row and column indices of $D'_{a,b-1}$. Denote the resulting matrix by $\tilde{D}'_{a,b-1}$.

Consider the following triangular matrix $D_{a,b}$ defined by all row and column indices of $\tilde{D}'_{a,b-1}$ and $D''_{a-1,b}$ (after adding x and the additional row and column as described above):

$$D_{a,b} = \begin{pmatrix} \tilde{D}'_{a,b-1} & O \\ J & D''_{a-1,b} \end{pmatrix},$$

where J is the all-ones matrix and O is the all-zeros matrix. The size of $D_{a,b}$ is $f(a, b-1) + f(a-1, b) + 1$, and as stated, the row and column indices are subsets of size a and b, respectively. Hence, $f(a, b) \geq f(a, b-1) + f(a-1, b) + 1$ as claimed. □

4 Constructions of large isolation sets for $k \geq 2t$

In this section we prove Theorem 1.3, and give constructions of families of large isolation sets in $A_{k,t}$, where for a large enough k, the constructions are the best possible, as we get an isolation set of size k.

The proof of the theorem contains several parts, according to the range of values of k compared to t. We first provide a basic construction of isolation sets of size $k - t + 1$ for $k \geq 3t - 2$, and then use this construction to build large isolation sets for $2t \leq k \leq 3t-3$ (Lemma 4.3), for $3t-2 \leq k \leq 4t-3$ (Lemma 4.4), and finally for $k \geq 4t - 3$ (Lemma 4.5). Theorem 1.3 follows immediately from these three results.

Throughout this section, denote by I_n the identity matrix of size $n \times n$, by O the all-zero matrix and by J the all-one matrix. If we want to specify the number of rows in O, we will simply write $O_{n,m}$ for the all-zero matrix of size $n \times m$, and similarly for J.

4.1 A construction of isolation sets of size $k - t + 1$ for $k \geq 3t - 2$

We now prove that if $k \geq 3t - 2$ then there exists an isolation set of size $k - t + 1$ in $A_{k,t}$. We first need to show that there exists an isolation matrix, not necessarily in $A_{k,t}$, of a certain structure, such that each row and column of this matrix has the same number of ones.

Claim 4.1 *For any $q \geq p - 1$, there exists an isolation matrix $F_{p,q}$ of size $(p + q) \times (p + q)$, such that there are p ones and q zeros in each column of $F_{p,q}$.*

Proof: Take the circulant matrix $F_{p,q}$, whose first column is $(\overbrace{1, 1, \cdots, 1}^{p}, \overbrace{0, 0, \cdots, 0}^{q})$. It is not hard to verify that $F_{p,q}$ is an isolation matrix when $q \geq p - 1$. Also, each column of $F_{p,q}$ is a cyclic permutation of the first column, and thus, each column contains p ones and q zeros. See Figure 1 for an example. □

Lemma 4.2 *If $k \geq 3t - 2$ and $t \geq 2$, there exists an isolation set of size $k - t + 1$ in $A_{k,t}$.*

Proof: Let $F_{p,q}$ be the isolation matrix described in Claim 4.1, with $p = t$ and $q = k-2t+1 \geq 3t-2-2t+1 = t-1 = p-1$. Let X and Y be the following matrices achieved by concatenating I_{q+p} and $J_{q+p,p-1}$, and $F_{p,q}$ and $O_{p-1,q+p}$, as follows:

$$X = [I_{q+p} J_{q+p,p-1}], \qquad Y = \begin{bmatrix} F_{p,q} \\ O_{p-1,q+p} \end{bmatrix}.$$

$$F_{5,4} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \mathbf{1} \end{pmatrix}$$

Figure 1: An isolation matrix $F_{5,4}$ of size $(p+q) \times (p+q) = 9 \times 9$, with $p = 5$ ones and $q = 4$ zeros in each column. The isolation set contains the ones in bold on the diagonal of $F_{5,4}$.

Observe that $X \cdot Y = F_{p,q}$. Furthermore, since each row of X and each column of Y are vectors of length $q + 2p - 1 = k$ with exactly $p = t$ ones, then we can view them as the characteristic vectors of subsets in $\binom{[k]}{t}$. Thus, $X \cdot Y = F_{p,q}$ is an isolation submatrix of $A_{k,t}$ of size $(q + p) \times (q + p) = (k - t + 1) \times (k - t + 1)$ as required. □

4.2 A construction of large isolation sets for $2t \leq k \leq 3t - 3$

Lemma 4.3 *Let $t \geq 2$ and $k = 2t+r$, where $0 \leq r \leq t-3$. There exists an isolation matrix in $A_{k,t}$ of size $(2r + 3) \times (2r + 3)$.*

Proof: Let $t' = r + 2$ and $k' = 2t' + r$. Thus, $k' = 2t' + r = 3t' - 2$, and therefore, by Lemma 4.2, there exists an isolation matrix F' of size $(k' - t' + 1) \times (k' - t' + 1)$ in $A_{k',t'}$, where the row and column indices of F' are subsets of size t' of $[k']$.

Since $k - k' = 2t + r - 2t' - r = 2(t - r - 2)$, there are still $2(t - r - 2)$ elements from $[k]$ that were not used to construct the row and column indices of F'. Add to each row index of F' half of these elements, and to each column index the other half.

Now the row and column indices are subsets of $[k]$ of size $t' + t - r - 2 = t$, and the resulting matrix is an isolation matrix of size $(2r + 3) \times (2r + 3)$ in $A_{k,t}$, as $2r + 3 = k' - t' + 1$. □

4.3 A construction of large isolation sets for $3t - 2 \leq k \leq 4t - 3$

Lemma 4.4 *Let $t \geq 2$ and $k = 2t + r$, where $t - 2 \leq r \leq 2t - 3$. There exists an isolation matrix in $A_{k,t}$ of size $(2r + 3) \times (2r + 3)$.*

Proof: If $k = 3t - 2$ then by Lemma 4.2, there exists an isolation matrix of size $(2r+3) \times (2r+3)$ as required, since $k-t+1 = 2t-1 = 2r+3$. Otherwise, $k > 3t-2$ and $r > t - 2$ and define $k' = 3t - 2$. Also let F' be the isolation submatrix of $A_{k',t}$ of size $(k' - t + 1) \times (k' - t + 1) = (2t - 1) \times (2t - 1)$, as promised by Lemma 4.2.

Finally, let F'' be another isolation matrix of size $(2r-2t+4)\times(2r-2t+4)$ that has the following structure:

$$F'' = \left(\begin{array}{c|c} U & L_1 \\ \hline L_2 & U \end{array}\right)$$

where U is an upper triangular matrix with ones on and above the main diagonal and zeros elsewhere, L_1 is a lower triangular matrix with ones on and below the main diagonal and zeros elsewhere, and L_2 is a strictly lower triangular matrix with ones below the main diagonal and zeros elsewhere, and such that U, L_1, L_2 are all of size $(r-t+2)\times(r-t+2)$.

We next show how to construct an isolation matrix F of size $(2r+3)\times(2r+3)$, which has the matrices F' and F'' on its main diagonal, and such that F is a submatrix of $A_{k,t}$. See Figure 2 for the exact structure of F. Since the sum of dimensions of F' and F'' is $(2t-1)+(2r-2t+4) = 2r+3$, it follows that F is a matrix of size $(2r+3)\times(2r+3)$ as claimed.

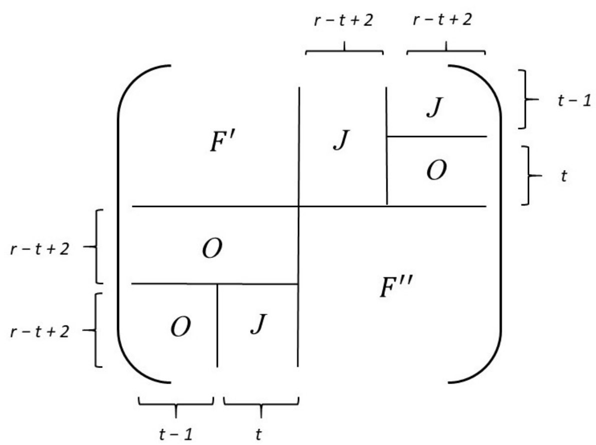

Figure 2: The structure of the matrix F presented in the proof of Lemma 4.4. The dimensions of the submatrices of F are specified alongside the figure.

In what follows we show that there is a way to assign row and column indices that are all subsets of $\binom{[k]}{t}$, such that we get the above structure of F', F'' and F. Then we can conclude that F is an isolation submatrix of $A_{k,t}$, since this structure of F', F'' and F, guaranties that any two ones on the diagonal of F are not in an all-ones submatrix of size 2×2.

The row and column indices of F': Denote the row and column indices of F' by $R_1,\ldots,R_{2t-1}$ and $C_1,\ldots,C_{2t-1}$, respectively. According to the construction described in Lemma 4.2, both the row and column indices of F' are subsets of size t of $[k']$ defined as follows:

- For $i = 1,\ldots,2t-1$: $R_i = \{i\}\cup S'$, where $S' = \{2t, 2t+1,\ldots,3t-2\}$.

- For $i = 0, \ldots, 2t-2$:
  $C_{i+1} = \{i \bmod (2t-1)+1,\ (i+1) \bmod (2t-1)+1,\ \ldots, (i+t-1) \bmod (2t-1)+1\}$.

Note that the largest element in a column index of F' is $2t-1$. Furthermore, it appears in exactly the last t column indices of F'.

The row and column indices of F'': Let $r' = r - t + 2$ and denote the row and column indices of F'' by $R_{2t}, \ldots, R_{2t+2r'-1}$ and $C_{2t}, \ldots, C_{2t+2r'-1}$, where:

- For $i = 0, \ldots, 2r'-1$: $C_{2t+i} = \{k-2r'+1+i\} \cup S''$, where $S'' = \{1, 2, \ldots, t-1\}$.
- For $i = 0, \ldots, r'-1$:

  $$R_{2t+i} = \{k - 2r' + 1 + i, k - 2r' + 2 + i, \ldots, k - 2r' + r' + 1 + i\} \cup T,$$

  where $T = \emptyset$ if $r = 2t-3$, and otherwise, $T = \{2t, 2t+1 \ldots, 4t-r-4\} \subset S'$. Note that the indices are well defined because $k - 2r' + 1 = 4t - r - 3$, and the maximal element in T is $4t - r - 4$. Furthermore, each index is a subset of size $r' + 1 + |T| = r - t + 3 + (2t - r - 3) = t$ as required.
- $R_{2t+r'} = (R_{2t+r'-1} \setminus \{k - r'\}) \cup \{2t-1\}$.
- For $i = 1, \ldots, r'-1$: $R_{2t+r'+i} = (R_{2t+r'+i-1} \setminus \{k - r' + i\}) \cup \{k - 2r' + i\}$.

It is not hard to verify that F'' has the structure described above, and that all row and column indices are subsets of $\binom{[k]}{t}$. Therefore, F'' is an isolation submatrix of $A_{k,t}$ of size $(2r') \times (2r')$ as required.

Now if we consider the matrix defined by all the row and column indices $R_1, \ldots,$ $R_{2t+2r'-1}$ and $C_1, \ldots, C_{2t+2r'-1}$, then we get the matrix F as above. To verify that F has the structure claimed, note that the first $r' = r - t + 2$ row indices of F'', that is, $R_{2t}, \ldots, R_{2t+r'-1}$, do not intersect with any of the column indices of F', since the largest element in a column index of F' is $2t-1$, and the smallest element in these row indices is $x = \min\{2t, k - 2r' + 1\} = 2t$, as $r \le 2t-3$, and so $k - 2r' + 1 = k - 2(r - t + 2) + 1 = 4t - r - 3 \ge 2t$.

As to the row indices $R_{2t+r'}, \ldots, R_{2t+2r'-1}$, they intersect the last t column indices of F', whereas, the column indices $C_{2t+r'}, \ldots, C_{2t+2r'-1}$ intersect with row indices $R_1, \ldots, R_{t-1}$ of F'. See also Figure 3 for an example. □

4.4 A construction of maximal isolation sets for $k \ge 4t-3$

Lemma 4.5 *Let $t \ge 2$ and $k \ge 4t-3$. There exists an isolation matrix in $A_{k,t}$ of size $k \times k$.*

Proof: Let $k' = 4t-3$ and let F be an isolation matrix of size $k' \times k'$, with row and column indices that are subsets of size t of $[k']$ as defined in the proof of Lemma 4.4. Now add $k - k'$ rows and $k - k'$ columns to F with the following indices:

$F =$

	4 3 2 1	5 4 3 2	6 5 4 3	7 6 5 4	1 7 6 5	2 1 7 6	3 2 1 7	3 2 1 9	3 2 1 10	3 2 1 11	3 2 1 12
$10,9,8,1$	**1**	0	0	0	1	1	1	1	1	1	1
$10,9,8,2$	1	**1**	0	0	0	1	1	1	1	1	1
$10,9,8,3$	1	1	**1**	0	0	0	1	1	1	1	1
$10,9,8,4$	1	1	1	**1**	0	0	0	1	1	0	0
$10,9,8,5$	0	1	1	1	**1**	0	0	1	1	0	0
$10,9,8,6$	0	0	1	1	1	**1**	0	1	1	0	0
$10,9,8,7$	0	0	0	1	1	1	**1**	1	1	0	0
$11,10,8,9$	0	0	0	0	0	0	0	**1**	1	1	0
$12,11,8,10$	0	0	0	0	0	0	0	0	**1**	1	1
$7,12,8,11$	0	0	0	1	1	1	1	0	0	**1**	1
$9,7,8,12$	0	0	0	1	1	1	1	1	0	0	**1**

Figure 3: An isolation matrix of size $(2r+3) \times (2r+3) = 11 \times 11$ in $A_{k,t}$, where $k = 12$, $t = 4$ and $r = k - 2t = 4$.

- For $i = 1, \ldots, k - k'$, add the row indices $\{k' + i, 2t - 1, 2t, 2t + 1, \ldots, 3t - 3\}$.
- For $i = 1, \ldots, k - k'$, add the column indices $\{k' + i, 1, 2, \ldots, t - 1\}$.

The resulting matrix is an isolation matrix of size $k \times k$. See Figure 4 for an example.

□

$F =$

	3 2 1	4 3 2	5 4 3	1 5 4	2 1 5	2 1 6	2 1 7	2 1 8	2 1 9	2 1 10	2 1 11
$7,6,1$	**1**	0	0	1	1	1	1	1	1	1	1
$7,6,2$	1	**1**	0	0	1	1	1	1	1	1	1
$7,6,3$	1	1	**1**	0	0	1	1	0	0	0	0
$7,6,4$	0	1	1	**1**	0	1	1	0	0	0	0
$7,6,5$	0	0	1	1	**1**	1	1	0	0	0	0
$8,7,6$	0	0	0	0	0	**1**	1	1	0	0	0
$9,8,7$	0	0	0	0	0	0	**1**	1	1	0	0
$5,9,8$	0	0	1	1	1	0	0	**1**	1	0	0
$6,5,9$	0	0	1	1	1	1	0	0	**1**	0	0
$6,5,10$	0	0	1	1	1	1	0	0	0	**1**	0
$6,5,11$	0	0	1	1	1	1	0	0	0	0	**1**

Figure 4: A maximal isolation matrix of size $k \times k = 11 \times 11$ in $A_{k,t}$, where $k = 11, t = 3$.

5 Bounds on the maximal size of isolation sets

As we saw, the constructions stated in Theorem 1.3 and proved in Section 4, are maximal for any $t \geq 2$ and $k \geq 4t-3$, since we get an isolation set of size k in this case. It is also easy to verify that our construction is maximal for $k = 2t$ (see also [1]).

In this section we prove Theorem 1.4, which provides an upper bound on the size of isolation sets in $A_{k,t}$ for $k = 2t+1$. This upper bound implies that our construction is also maximal for $k = 2t+1$. We first prove the following claims.

Claim 5.1 *Let $k = 2t+1$ and let F be an isolation matrix in $A_{k,t}$. Then F cannot contain a submatrix of size 2×2 that is the all-zero matrix.*

Proof: Assume, by contradiction, that F has a submatrix of size 2×2 that is the all-zero matrix, and assume that this submatrix is defined by row indices x, y and column indices z, w. Assume, without loss of generality, that $x = \{1, 2, \ldots, t\}$. Since $x \cap z = x \cap w = \emptyset$ and $z \neq w$, we must have that $z \cup w = \{t+1, \ldots, 2t+1\}$. But we have also that $y \cap z = y \cap w = \emptyset$, and therefore, $y = \{1, 2, \ldots, t\}$. Thus, $y = x$ and this is a contradiction. □

Claim 5.2 *Let $k = 2t+1$ and let F be an isolation matrix in $A_{k,t}$. Then every row and column of F has at most three zeros.*

Proof: Assume, by contradiction, that F has a row with four zeros, and assume, without loss of generality, that it is the first row and that the zeros are in positions $2, 3, 4, 5$ of this row. Consider the following submatrix W of F defined by the first five rows of F and columns $2, 3, 4, 5$ of F:

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{pmatrix}$$

Let W' be the submatrix containing the last four rows of the submatrix W. Note that W' is an isolation matrix of size 4×4. If W' contains two zeros in one of its rows, then with the zeros in the first row of W, we get that F contains a submatrix of size 2×2 that is the all-zero matrix, in contradiction to Claim 5.1. Thus, W' contains at most one zero in each one of its rows. But since W' is an isolation matrix of size 4×4, it must contain at least $\binom{4}{2} = 6$ zeros, and again we have a contradiction. □

Claim 5.3 *Let $G = (V_1, V_2, E)$ be a 3-regular bipartite graph, where $|V_1| = |V_2| = 6$. Then G contains a 4-cycle.*

Proof: Let $V_1 = \{x_1, \ldots, x_6\}$, $V_2 = \{y_1, \ldots, y_6\}$, and assume, without loss of generality, that $(x_1, y_1), (x_1, y_2), (x_1, y_3) \in E$ and $(y_1, x_2), (y_1, x_3) \in E$. If one of y_2 or y_3 is a neighbor of one of x_2 or x_3, then we are done since we have a 4-cycle (for example, if $(x_2, y_2) \in E$, then $(x_1, y_1, x_2, y_2, x_1)$ is a 4-cycle).

Thus, consider now the case that y_2 and y_3 are not neighbors of x_2 and x_3. Therefore, each of x_2 and x_3 has two neighbors from y_4, y_5, y_6, and so they have a common neighbor, say y_4. Since they are both also neighbors of y_1, we get a 4-cycle $(x_2, y_1, x_3, y_4, x_2)$. □

Using the above claims we can now prove Theorem 1.4.

Proof of Theorem 1.4: Let $k = 2t + 1$, $t \geq 2$, and assume, by contradiction, that there is an isolation submatrix F of size 6×6 in $A_{k,t}$. Denote the rows of F by $X_1, \ldots, X_6$ and the columns of F by $Y_1, \ldots, Y_6$. Since F is an isolation matrix, then $X_i \circ Y_i = e_i$ for every $1 \leq i \leq 6$, where e_i is the i^{th} standard basis vector, and $\circ$ is the Hadamard (entry-wise) product.

First notice that F cannot have a column Y_i with five ones, since then X_i must have four zeros (as $X_i \circ Y_i = e_i$), and this is impossible by Claim 5.2. Therefore, every column of F has at most four ones. A similar argument holds for the rows of F. Furthermore, if there exists a row/column with two ones then it has four zeros and again we get a contradiction. Thus, every row and column of F has at least three ones and at most four ones, and at least two zeros and at most three zeros. We have the following two cases:

Case 1: Every row and column in F has three ones. Let G be the bipartite 3-regular graph whose adjacency matrix is the complement of F (that is, each zero in F is an edge of the graph). Then by Claim 5.3, the graph G has a 4-cycle. Thus, F has a submatrix of size 2×2 that is all zeros, and we get a contradiction by Claim 5.1.

Case 2 : There exists at least one column in F with four ones. Assume, without loss of generality, that it is Y_1 and that $Y_1 = (1, 1, 1, 1, 0, 0)$. But, $X_1 \circ Y_1 = e_1$ and by Claim 5.2 every row of F contains at most three zeros. Thus, $X_1 = (1, 0, 0, 0, 1, 1)$.

Now consider the structure of the submatrix W of F defined by rows X_2, X_3, X_4 and columns Y_2, Y_3, Y_4 of F. Notice that W is an isolation matrix of size 3×3. First we claim that there cannot be two zeros in any of the rows of W (otherwise, we will get a submatrix of size 2×2 of zeros with the zeros in X_1). Hence, each row of W has at most one zero. Also there cannot be two zeros in any of the columns of W, since then we will get a 2×2 all ones submatrix on the diagonal of W, in contradiction to W being an isolation matrix. Thus, each one of the rows and columns of W must contain at most one zero. But since W is an isolation matrix of size 3×3 it should have at least $\binom{3}{2} = 3$ zeros, and so each one of the rows and columns of W must contain exactly one zero and two ones. Therefore, without loss of generality, F has the following structure:

$$F = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 1 & 1 \\ 1 & \mathbf{1} & 0 & 1 & & \\ 1 & 1 & \mathbf{1} & 0 & & \\ 1 & 0 & 1 & \mathbf{1} & & \\ 0 & & & & \mathbf{1} & \\ 0 & & & & & \mathbf{1} \end{pmatrix}$$

Similar considerations as those above, show that there cannot be two zeros in positions $2, 3, 4$ of X_5 or of X_6 (otherwise, there will be a submatrix of size 2×2 of zeros with the first row of F), and there cannot be three ones in positions $2, 3, 4$ of X_5 (otherwise, we get that $Y_5 = (1, 0, 0, 0, 1, 1)$ and therefore $Y_6 = (1, 1, 1, 1, 0, 1)$, or otherwise we get a submatrix of size 2×2 of zeros. But then Y_6 contains five ones and again we get a contradiction). A similar argument holds for X_6. Thus, X_5 and X_6 each must contain one zero and two ones in positions $2, 3, 4$. Hence, without loss of generality, F is of the following form (where columns Y_5 and Y_6 were determined according to X_5, X_6, so that $X_5 \circ Y_5 = e_5$, $X_6 \circ Y_6 = e_6$, and we do not get a submatrix of size 2×2 that is all-zeros):

$$F = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 1 & 1 \\ 1 & \mathbf{1} & 0 & 1 & 0 & 0 \\ 1 & 1 & \mathbf{1} & 0 & 0 & 1 \\ 1 & 0 & 1 & \mathbf{1} & 1 & 0 \\ 0 & 1 & 1 & 0 & \mathbf{1} & \\ 0 & 1 & 0 & 1 & & \mathbf{1} \end{pmatrix}$$

Now denote the row and column indices of F by $x_1, \ldots, x_6$ and $y_1, \ldots, y_6$, respectively, where each index is a subset of size t of $[k] = [2t + 1]$, and assume, without loss of generality, that $x_1 = \{1, \ldots, t\}$. From the structure of F we can deduce the following about its row and column indices:

- Since $x_1 \cap y_2 = x_1 \cap y_3 = x_1 \cap y_4 = \emptyset$, then $y_2, y_3, y_4 \subset \{t + 1, \ldots, 2t + 1\}$, $|y_i \cap y_j| = t - 1$ for $2 \leq i \neq j \leq 4$, and $|y_2 \cap y_3 \cap y_4| = t - 2$. Let $S = y_2 \cap y_3 \cap y_4$ and assume, without loss of generality, that $S = \{t + 1, t + 2, \ldots, 2t - 2\}$, and $y_2 = S \cup \{2t - 1, 2t\}, y_3 = S \cup \{2t - 1, 2t + 1\}, y_4 = S \cup \{2t, 2t + 1\}$. Note that if $t = 2$, then $S = \emptyset$.

- Since $x_2 \cap y_3 = \emptyset$, $x_2 \cap y_2 \neq \emptyset$, $y_2 \cap y_3 = S \cup \{2t - 1\}$ and $y_2 = S \cup \{2t - 1, 2t\}$ then $2t \in x_2$. In a similar way, $2t - 1 \in x_3$, $2t + 1 \in x_4$, $2t - 1 \in x_5$ and $2t \in x_6$.

- Furthermore, since $x_5 \cap y_1 = x_6 \cap y_1 = \emptyset$, then there exists a subset T of size $|T| = t - 1$ such that $T \subseteq x_5 \cap x_6$. Since $x_5 \cap y_4 = x_6 \cap y_3 = \emptyset$ then $T \cap y_4 = T \cap y_3 = \emptyset$. Thus, $T \subseteq \{1, 2, \ldots, t\}$.

Finally, since F is an isolation matrix, then either $x_5 \cap y_6 = \emptyset$ or $x_6 \cap y_5 = \emptyset$. Assume first that $y_6 \cap x_5 = \emptyset$. From the above discussion, in this case F has the following structure, where the row and column indices of F are denoted above and to the left of the matrix:

	y_1	$y_2 = S, 2t-1, 2t$	$y_3 = S, 2t-1, 2t+1$	$y_4 = S, 2t, 2t+1$	y_5	y_6
$x_1 = 1, \dots, t$	**1**	0	0	0	1	1
$2t \in x_2$	1	**1**	0	1	0	0
$2t-1 \in x_3$	1	1	**1**	0	0	1
$2t+1 \in x_4$	1	0	1	**1**	1	0
$x_5 = T, 2t-1$	0	1	1	0	**1**	0
$x_6 = T, 2t$	0	1	0	1		**1**

But then $Q \cap y_6 = \emptyset$, where $Q = \{2t, 2t+1, 2t-1\} \cup T$, and this is a contradiction, since then $y_6 \subseteq [k] \setminus Q$ and $|[k] \setminus Q| = t-1$.

In a similar way, if $x_6 \cap y_5 = \emptyset$, and since also $x_3 \cap y_5 = \emptyset$, then $(\{2t-1, 2t\} \cup T) \cap y_5 = \emptyset$. On the other hand, $x_5 \cap y_5 \neq \emptyset$ and $x_5 = T \cup \{2t-1\}$ and again we have a contradiction. ☐

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