ON STRONGLY EDGE-CRITICAL GRAPHS OF GIVEN DIAMETER

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Dedicated to the memory of Alan Rahilly, 1947 – 1992

ABSTRACT:

Let G be a simple undirected graph with edge set E(G) and diameter k. G is said to be **strongly t-edge-critical** or simply **(k,t)-critical** if for any $E' \subseteq E(G)$, $G-E'$ has diameter greater than k if and only if $|E'| \geq t$. (k,1)-Critical graphs have been studied by many authors. P. Kys conjectured that there is no (k,t)-critical graph for $k \geq 2$, $t \geq 2$. To date this conjecture has been established for : $k = 2$; $k = 3$; $k = 4$, $t \geq 3$; and for $k \geq 2$, $t \geq k$. In this paper, we prove the conjecture for $k \geq 2$, $t \geq 3$ and for $k = 4$ and 5.

1. INTRODUCTION

All graphs considered in this paper are finite loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [1]. Thus G is a graph with vertex set V(G), edge set E(G) and minimum degree $\delta(G)$. The **distance** $d_G(x,y)$ between two vertices x and y in G is defined as the length of the shortest (x,y)-path in G; if there is no path connecting x and y we define $d_G(x,y)$ to be infinite. The **diameter** of a graph G, denoted d(G), is defined to be the maximum distance in G; that is

$$d(G) = \max_{x,y \in V(G)} \left\{ d_G(x,y) \right\} .$$

Note that for any $E' \subseteq E(G)$, $d(G-E') \geq d(G)$.

Let G be a graph having diameter k. G is said to be **strongly t-edge-critical** or simply **(k,t)-critical** if for any $E' \subseteq E(G)$, $G-E'$ has diameter greater than k if and only if $|E'| \geq t$. Denote the class of (k,t)-critical graphs by $\mathcal{G}(k,t)$.

(k,1)-critical graphs do exist. For example : $\mathcal{G}(k,1)$ contains the cycle of length 2k and 2k + 1; $\mathcal{G}(2,1)$ contains the well known Petersen graph and the class of complete bipartite graphs. The class $\mathcal{G}(k,1)$ has been studied by many authors - see for example [2-6, 8]. There are many open problems concerning this class, the most well known being the conjecture of Plesnik [8] and Simon and Murty [2] that a graph $G \in \mathcal{G}(2,1)$ has at most $\lfloor \frac{1}{4} \nu^2 \rfloor$, $\nu = |V(G)|$, edges and this bound is attained if and only if

$$G \cong K_{\lfloor \frac{1}{2}\nu \rfloor, \lceil \frac{1}{2}\nu \rceil} .$$

This conjecture has recently been established by Füredi [3] for extremely large ν.

For $t \geq 2$ the class $\mathcal{G}(k,t)$ has been studied only by Kys [7]. He conjectured that $\mathcal{G}(k,t) = \phi$ for $k \geq 2$, $t \geq 2$. Further, he established the conjecture for about half the cases : for k = 2; k = 3; k = 4 and $t \geq 3$; and for $t \geq k \geq 2$. In this paper, we prove that the conjecture holds for : $k \geq 2$, $t \geq 3$; and for k = 4 and 5. This leaves the only unresolved cases as : $k \geq 6$, $t = 2$.

We present our main results in Section 3. In the next section we study the properties of (k,t)-critical graphs which are crucial in establishing our main results.

2. PROPERTIES OF (k,t)-CRITICAL GRAPHS

Let G be a graph of diameter k and u any vertex of G. The **eccentricity** of u, denoted $ec_G(u)$, is defined as :

$$ec_G(u) = \max_{v \in V(G)} \left\{ d_G(u,v) \right\} .$$

Let $L_i(u)$ denote the vertices of G that are at a distance i from u, $i = 0,1,2,\ldots,ec_G(u)$. We call $\{L_i(u)\colon i = 0,1,\ldots,ec_G(u)\}$ the **distance decomposition** of V(G) from the vertex u.

We denote the length of a path P in G by $|P|$. Further, for $E' \subseteq E(G)$, $P \cap E'$ denotes the set of edges of G which belong to P and E'. We now state a number of results of Kys [7] which we make use of in our work.

Lemma 2.1: If $G \in \mathcal{G}(k,t)$, then $\delta(G) \geq t$. □

Lemma 2.2: If $\mathcal{G}(k,t) = \phi$, then $\mathcal{G}(k,t+1) = \phi$. □

Lemma 2.3 : Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and $E' = \{e_1, e_2, \ldots, e_t\}$ be any set of t edges of G. Then for any two vertices m and n of G with $d_{G-E'}(m,n) > k$ there are t (m,n)-paths $P_1, P_2, \ldots, P_t$ in G such that $|P_i| \leq k$ and $P_i \cap E' = \{e_i\}$, $i = 1,2,\ldots,t$. □

Lemma 2.4 : Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and u a vertex of G having $ec_G(u) = k$. Then no two vertices of $L_k(u)$ are joined in G. □

Lemma 2.5 : Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and u, x be vertices of G with $d_G(u,x) = k$. Let E′ be a set of t edges of G containing the edges uv and xy with $v \in L_1(u)$ and $y \in L_{k-1}(u)$. If for $m \in L_r(u)$ and $n \in L_s(u)$, $d_{G-E'}(m,n) > k$, then $r + s = k$. Furthermore, if every edge of $E' \setminus \{uv, xy\}$ is incident to u or x, then $d_G(m,n) = k$. □

Note that the m and n in the above lemma exist for some r and s since G is (k,t)-critical.

To establish our main results we need, in addition to the above mentioned lemmas, a number of further properties concerning the class $\mathcal{G}(k,t)$. Before presenting these new results we need to introduce some further terminology.

Let P be an (a,b)-path in a graph G. We say that the vertex x **preceeds** y on P if the (a,y)-section of P, denoted by P(a,y), contains the vertex x.

Our first lemma is essentially an extension of Lemma 2.5.

Lemma 2.6 : Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and u, x be vertices of G with $d_G(u,x) = k$. Let E′ be a set of t edges of G containing the edges uv and xy with $v \in L_1(u)$ and $y \in L_{k-1}(u)$. If for $m \in L_r(u)$ and $n \in L_s(u)$, $d_{G-E'}(m,n) > k$, then there exists an (m,n)-path P_1 in G of length at most k containing the edge uv such that either

(i) $|P_1(m,v)| = r - 1$ and $|P_1(u,n)| = s$

or

$$\text{(ii)} \qquad |P_1(m,u)| = r \text{ and } |P_1(v,n)| = s - 1.$$

Proof : Lemma 2.3 implies the existence of an (m,n)-path P_1 of length at most k containing the edge uv. So we need only establish that P_1 satisfies condition (i) or (ii). Suppose that v preceeds u on P_1. Then clearly $|P_1(m,v)| \geq r - 1$ and $|P_1(u,n)| \geq s$. Further

$$|P_1(m,v)| = |P_1| - |P_1(u,n)| - 1 \leq k - s - 1,$$

and hence, since by Lemma 2.5, $r + s = k$,

$$|P_1(m,u)| \leq r - 1.$$

This proves (i). When u preceeds v on P_1 the same argument yields (ii). This completes the proof of the lemma. □

Corollary : Assume the hypothesis of Lemma 2.6 and let uw be an edge of $E'\setminus\{uv,xy\}$. If P_2 is an (m,n)-path of length at most k in G containing the edge uw, then w preceeds u on P_2 if condition (i) of Lemma 2.6 holds.

Proof : Suppose that condition (i) of Lemma 2.6 holds and u preceeds w on P_2. Then condition (ii) of Lemma 2.6 holds for P_2. But then, by Lemma 2.3

$$P_2(m,u) \cup P_1(u,n)$$

contains an (m,n)-path in G-E′ of length at most $r + s = k$, a contradiction. This completes the proof. □

Remark 1 : If the length of P_i, i = 1, 2 is exactly k, then at most two edges of P_i join vertices of $L_j(u)$ to vertices of $L_{j+1}(u)$, $0 \leq j \leq$

$k - 1$. Furthermore, there is exactly one edge of P_i between $L_j(u)$ and $L_{j+1}(u)$ for $r \le j \le s - 1$.

In the proofs that follow we make frequent use of the following simple fact which follows from Lemma 2.4.

Lemma 2.7 : Let $G \in \mathcal{G}(k,t)$, $k \ge 2$, $t \ge 2$. If $d_G(u,x) = k$, then $d_G(v,x) = k - 1$ for every $v \in N_G(u)$. □

Our next two lemmas are important in establishing a lower bound on the degree of vertices of $G \in \mathcal{G}(k,t)$ having eccentricity k.

Lemma 2.8 : Let $G \in \mathcal{G}(k,t)$, $k \ge 2$, $t \ge 2$, and u, x be vertices of G with $d_G(u,x) = k$. Let P_1 be a (v,x)-path, $v \in L_1(u)$, in G of length $k - 1$ and E′ a set of t edges of $G \backslash \{uv \cup E(P_1)\}$ containing the edges uw and xy. If for $m \in L_r(u)$ and $n \in L_s(u)$, $d_{G-E'}(m,n) > k$ and $r + s = k$, then $r \ge 1$ and $s \ge 1$. Moreover, if $t \ge 3$ and there are at least two edges of E′ incident to u, then $r \ge 2$ and $s \ge 2$.

Proof : Without any loss of generality suppose that $r \le s$. We need to prove that $r \ne 0$. By Lemma 2.3 there exists (m,n)-paths Q_1 and Q_2 in G of length at most k such that

$$Q_1 \cap E' = \{uw\}$$

and

$$Q_2 \cap E' = \{xy\}.$$

If $r = 0$, then $s = k$ and thus $m = u$ and $n \in L_k(u)$. Since P_1 is a (v,x)-path in G-E′ of length $k - 1$ and $uv \notin E'$, $n \ne x$. But then the path Q_2 which contains the edge xy cannot be of length at most k, a contradiction. Hence $r \neq 0$, proving the first part of the lemma.

Now suppose that $t \geq 3$ and $uz \in E'$, $z \neq w$. Let Q_3 be the (m,n)-path in G of length at most k such that

$$Q_3 \cap E' = \{uz\}.$$

Suppose that $r = 1$. Then $s = k - 1$ and so $m \in L_1(u)$ and $n \in L_{k-1}(u)$. If $m = w$, then Q_3 has length greater than k, since $uw \notin Q_3$. Hence $m \neq w$ and, similarly, $m \neq z$. Furthermore, every (m,n)-path in G containing uw or uz of length at most k must contain the edge mu. But then

$$d_{G-E''}(m,n) \geq d_{G-E'}(m,n) > k$$

where

$$E'' = \{um\} \cup E' \setminus \{uw, uz\} ,$$

contradicting the fact that G is (k,t)-critical. This proves that $r \neq 1$ thus completing the proof of the lemma. □

Lemma 2.9 : Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and $E' = \{e_1, e_2, \ldots, e_t\}$ be any set of t edges of G. If for any two vertices m and n of G with $d_G(m,n) = k$ and $d_{G-E'}(m,n) > k$ there are t (m,n)-paths $P_1, P_2, \ldots, P_t$ such that $P_i \cap E' = \{e_i\}$, $i = 1,2,\ldots,t$, then the paths $P_1, P_2, \ldots, P_t$ are pairwise edge-disjoint.

Proof : Clearly if $e' \in P_i \cap P_j$, $i \neq j$, then $d_{G-E''}(m,n) \geq d_{G-E'}(m,n) > k$, where $E'' = \{e'\} \cup E' \setminus \{e_i, e_j\}$, contradicting the fact that $G \in \mathcal{G}(k,t)$. This proves the lemma. □

We are now ready to prove the main result of this section.

Theorem 2.1 : Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$. If $ec_G(u) = k$, then $d_G(u) \geq 2t - 2$.

Proof : Let $L_1(u) = \{u_1, u_2, \ldots, u_\ell\}$ and $x \in L_k(u)$. Then by Lemma 2.1 $\ell \geq t$ and hence we only need to consider the case $t \geq 3$. Since $G \in \mathcal{G}(k,t)$, there are in G at least t edge-disjoint (u,x)-paths of length k. Let $P_1, P_2, \ldots, P_t$ be any t such paths and without any loss of generality suppose that $uu_i \in P_i$, $i = 1,2,\ldots,t$.

Now consider the t edges

$$E' = \{uu_1, uu_2, \ldots, uu_{t-1}, xy\}$$

where $y \notin P_t$. Then, by lemmas 2.5 and 2.8, there exist vertices $m \in L_r(u)$ and $n \in L_s(u)$ with $d_{G-E'}(m,n) > k$, $r + s = k$ and $s \geq r \geq 2$. Further, $d_G(m,n) = k$. Lemma 2.3 implies the existence of (m,n)-paths $Q_1, Q_2, \ldots, Q_t$, in G, of length k with $Q_i \cap E' = \{uu_i\}$ for $i = 1,2,\ldots,t-1$ and $Q_t \cap E' = \{xy\}$. These t paths are, by Lemma 2.9, pairwise edge-disjoint. Now since each Q_i, $i = 1,2,\ldots,t-1$, contains 2 edges incident to u, $d_G(u) \geq 2(t-1)$, as required. □

For the case when $G \in \mathcal{G}(k,2)$, $k = 4$ or 5 we have the following lower bound on the degree of a vertex of G having eccentricity k.

Lemma 2.10 : Let $G \in \mathcal{G}(k,2)$, $k = 4$ or 5. If $ec_G(u) = k$, then $d_G(u) \geq 3$.

Proof : Suppose to the contrary that $d_G(u) \leq 2$. Then, by Lemma 2.1, $d_G(u) = 2$. Let $L_1(u) = \{v,w\}$, $x \in L_k(u)$ and P_1 and P_2 be the two edge-disjoint (u,x)-paths in G. Without any loss of generality let

$uv \in P_1$ and $uw \in P_2$. Now consider the edges $E' = \{uv, xy\}$, where $y \notin P_2$. Then, by lemmas 2.5 and 2.8, there exist vertices $m \in L_r(u)$ and $n \in L_s(u)$ with $d_{G-E'}(m,n) > k$, $r + s = k$ and $s \geq r \geq 1$.

As in the proof of Theorem 2.1 there exist (m,n)-paths Q_1 and Q_2 in G of length k with $Q_1 \cap E' = \{uv\}$ and $Q_2 \cap E' = \{xy\}$.

If v preceeds u on Q_1 then, since $d_{G-uw}(u,n) \leq k$, we have $d_{G-uw}(v,n) \leq k - 1$. Let R denote a (v,n)-path of length at most $k - 1$ in G-uw. Now since $k = 4$ or 5 and $s \geq r \geq 1$, we have $r = 1$ or 2. If $r = 1$, then $m = v$ and hence $d_{G-E'}(m,n) = d_{G-uw}(m,n) \leq k - 1$, a contradiction. If $r = 2$, then $mv \in E(G)$, and hence,

$$R \cup \{mv\}$$

is an (m,n)-path of length at most k in G-E', a contradiction. Hence v does not preceed u on Q_1. A similar argument will establish that u cannot preceed v on Q_1. Hence the lemma. □

3. MAIN RESULTS

In this section we prove that $\mathcal{G}(k,t) = \phi$ for : $k \geq 2$ and $t \geq 3$; and $(k,t) = (4,2)$ and $(5,2)$. Thus the only unresolved case of Kys' conjecture is $k \geq 6$, $t = 2$.

Theorem 3.1 : $\mathcal{G}(k,t) = \phi$ for $k \geq 2$ and $t \geq 3$.

Proof : In view of Lemma 2.2 we need only prove that $\mathcal{G}(k,3) = \phi$ for $k \geq 2$. Assume to the contrary that $\mathcal{G}(k,3) \neq \phi$, $k \geq 2$, and let $G \in \mathcal{G}(k,3)$.

Let u be a vertex of G with $ec_G(u) = k$. Let $L_1(u) = \{u_1, u_2, \ldots, u_\ell\}$ and $x \in L_k(u)$. Theorem 2.1 implies that $\ell \geq 4$. Since

$G \in \mathcal{G}(k,3)$, there are at least three edge-disjoint (u,x)-paths of length k. Let P_1, P_2 and P_3 be three such paths and assume without any loss of generality that $uu_i \in P_i$, $i = 1,2,3$. Now consider the edges

$$E' = \{uu_1, uu_2, xy\},$$

where $y \notin P_3$. As in the proof of Theorem 2.1, there exist vertices $m \in L_r(u)$ and $n \in L_s(u)$ with $d_{G-E'}(m,n) > k$, $r + s = k$, $s \geq r \geq 2$, $d_G(m,n) = k$ and pairwise edge-disjoint (m,n)-paths Q_1, Q_2 and Q_3, in G, of length k with $Q_i \cap E' = \{uu_i\}$, for $i = 1,2$, and $Q_3 \cap E' = \{xy\}$.

Since $s \geq r \geq 2$, $k = r + s \geq 4$, thus we have nothing to prove for $k \leq 3$. For $k \geq 4$ we establish our contradiction by considering the distance decomposition of vertex m. Clearly $u \in L_r(m)$ and $x \in L_s(m)$. Lemma 2.6 and its Corollary implies that either $u_1, u_2 \in L_{r-1}(m)$ (when u_1 preceeds u on Q_1) or u_1, $u_2 \in L_{r+1}(m)$ (when u preceeds u_1 on Q_1). Further, y is in $L_{s-1}(m)$ or $L_{s+1}(m)$.

Choose vertices m_1, $m_2 \in L_1(m)$ and $n_1 \in L_{k-1}(m)$ such that $m_1 \notin Q_1 \cup Q_2 \cup Q_3$, $m_2 \in Q_2$ and $n_1 \notin Q_1 \cup Q_2 \cup Q_3$. Such vertices exist since, by Theorem 2.1, both m and n have degree at least four. Let

$$E'' = \{mm_1, mm_2, nn_1\}.$$

We will establish that $d(G-E'') = k$, contradicting the criticality of G. Suppose to the contrary that $d(G-E'') > k$.

Then there exist vertices $a \in L_{r^*}(m)$ and $b \in L_{s^*}(m)$ with $d_{G-E''}(a,b) > k$ and $r^* + s^* = k$. Further, by lemmas 2.3, 2.5, 2.8 and 2.9 we have : $r^* \geq 2$, $s^* \geq 2$; and pairwise edge-disjoint (a,b)-paths R_1, R_2 and R_3, in G, of length k with $R_i \cap E'' = \{mm_i\}$ for $i = 1,2$ and $R_3 \cap E'' = \{nn_1\}$.

Let H be the subgraph of G formed by taking the union of the three paths R_1, R_2 and R_3. Observe that H is a connected graph of diameter k containing m and n. We will establish the required contradiction by showing that H contains an (m,n)-path $\hat{Q}$ of length at most k such that $\hat{Q} \cap E' = \phi$. Note that such a $\hat{Q}$ would also be an (m,n)-path of length at most k in G-E', a contradiction.

We assume without any loss of generality that $s^* \geq r^*$. Now we distinguish three cases according to the value of r^*.

Case 1 : $2 \leq r^* \leq r - 1$

In this case $s^* \geq s + 1$ since $k = r^* + s^* = r + s$. Since $r \leq s$ we have $r^* \leq r - 1 < r \leq s < s^* \leq k - 2$. The situation is depicted in Figure 3.1 below. Note that in all our figures we write L_i for $L_i(m)$.

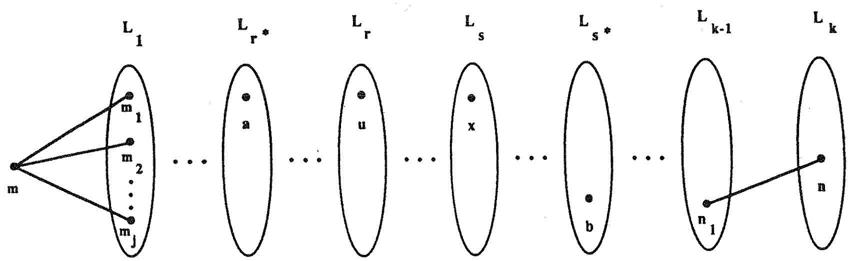

Figure 3.1

Consequently, since y is in $L_{s-1}(m)$ or $L_{s+1}(m)$, $x \in L_s(m)$ and $|R_3| = k$, $xy \notin R_3(b,n)$. Further, by Remark 1, the section $R_3(b,n)$ contains neither uu_1 nor uu_2. Now if $R_2 \cap E' = \phi$, then

$$R_2(m,b) \cup R_3(b,n)$$

is an (m,n)-path of length

$$|R_2(m,b)| + |R_3(b,n)| = s^* + k - s^* = k$$

in G-E', a contradiction. Hence $R_2 \cap E' \neq \phi$.

Suppose $uu_1 \in R_2$. If m_2 preceeds m on R_2, then the subgraph

$$R_2(a,m_2) \cup Q_2(m_2,u) \cup R_2(u,b)$$

contains (see Figure 3.2) an (a,b)-path of length at most

$$(r^* - 1) + (r - 1) + (s^* - r) = r^* + s^* - 2 < k$$

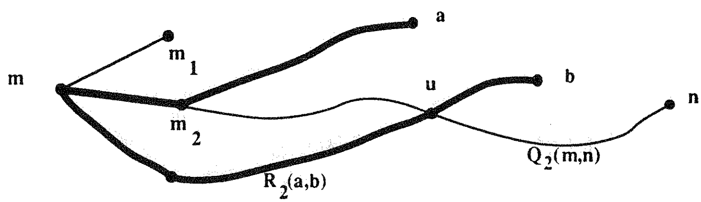

Figure 3.2

having no edges of E″, a contradiction. Hence m preceeds m_2 on R_2. But then the subgraph

$$R_2(a,m) \cup Q_1(m,u) \cup R_2(u,b)$$

contains (see Figure 3.3) an (a,b)-path of length at most

$$r^* + r + s^* - r = k,$$

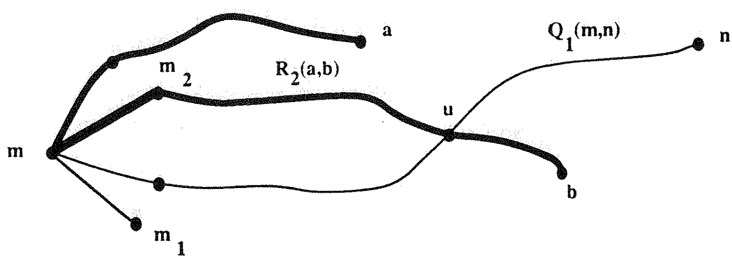

Figure 3.3

containing no edges of E″, again a contradiction. So $uu_1 \notin R_2$. Similarly $uu_2 \notin R_2$.

The only possibility then is for $xy \in R_2$. In this case $xy \notin R_1 \cup R_3$. Consequently, if $R_1 \cap \{uu_1, uu_2\} = \phi$, then the subgraph

$$R_1(m,b) \cup R_3(b,n)$$

contains an (m,n)-path of length at most $s^* + k - s^* = k$ in G-E′, a contradiction. Hence $R_1 \cap \{uu_1, uu_2\} \neq \phi$.

Suppose $uu_1 \in R_1$. If m preceeds m_1 on R_1, then the subgraph

$$R_1(a,m) \cup Q_1(m,u) \cup R_1(u,b)$$

contains (see Figure 3.4) an (a,b)-path of length at most

$$r^* + r + s^* - r = k$$

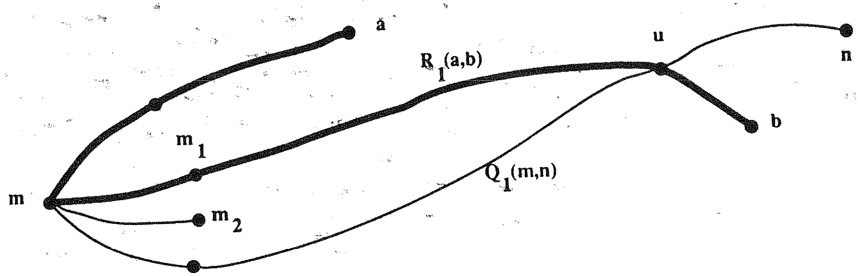

Figure 3.4

having no edges of E″, a contradiction. Therefore m_1 preceeds m on R_1. But then, by Lemma 2.6 and its Corollary, m_2 preceeds m on R_2. Consequently the subgraph

$$R_2(a,m_2) \cup Q_2(m_2,u) \cup R_1(u,b)$$

contains (see Figure 3.5) an (a,b)-path of length at most

$$r^* - 1 + r - 1 + s^* - r = k - 2$$

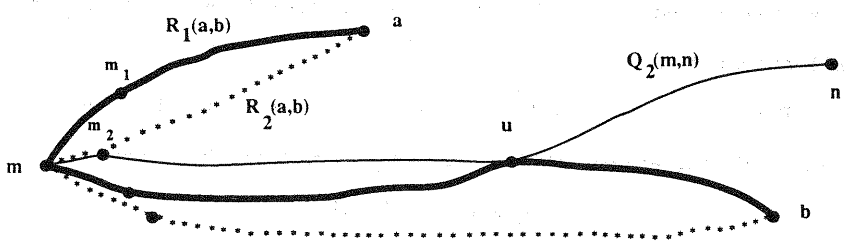

Figure 3.5

having no edge of E″, again a contradiction. hence $uu_1 \notin R_1$. Similarly $uu_2 \notin R_1$. Therefore $R_1 \cap \{uu_1, uu_2\} = \phi$ and hence $xy \notin R_2$. This completes the proof for Case 1.

Case 2 : r* = r

In this case s* = s and so $a, u \in L_r(m)$ and $b, x \in L_s(m)$. Note that a could be u and b could be x. Suppose first that a = u. Then $b \neq x$, since otherwise $Q_1(u,m) \cup Q_3(m,x)$ would be a (u,x)-path in G−E″ of length k. Consequently, since $r \le s$, $|R_3| = k$, $b \in L_s(m)$ and in view of Remark 1, $R_3(b,n) \cap E' = \phi$. Therefore if $R_2 \cap E' = \phi$, then as in Case 1

$$R_2(m,b) \cup R_3(b,n)$$

is a (m,n)-path of length k in G−E′. Hence $R_2 \cap E' \neq \phi$.

Suppose that $uu_1 \in R_2$. If m_2 preceeds m on R_2, then the subgraph

$$Q_1(a,m) \cup R_2(m,b)$$

contains an (a,b)-path of length k having no edges of E″, a contradiction. Hence m preceeds m_2 on R_2. But then the subgraph

$$Q_2(u,m_2) \cup R_2(m_2,b)$$

contains an (a,b)-path of length $(r^* - 1) + (s^* - 1) < k$ having no edges of E'', again a contradiction. Hence $uu_1 \notin R_2$. Similarly $uu_2 \notin R_2$. So the only possibility is for $xy \in R_2$ and hence $y \in L_{s-1}(m)$. But then, noting Remark 1, we must have $b = x$, a contradiction. This proves that $a \neq u$.

Again we will prove that $R_2 \cap E' = \phi$. Suppose this is not the case. Since $a, u \in L_r(m)$ and $a \neq u$, if $uu_1 \in R_2$ or $uu_2 \in R_2$, then m preceeds u on R_2. Similar to the proof of Case 1, we have $R_2 \cap \{uu_1, uu_2\} = \phi$ and $R_1 \cap \{uu_1, uu_2\} = \phi$. The only possibility is for $xy \in R_2$ and hence $xy \notin R_1 \cup R_3$. Recall that $b \in L_s(m)$. Consequently if $r < s$, then clearly $b \neq u$ and when $r = s$, then by a similar argument that used in case $a = u$, we can establish that $b \neq u$. Consequently, $R_3(b,n) \cap \{uu_1, uu_2\} = \phi$ and thus $R_3(b,n) \cap E' = \phi$. But then

$$R_1(m,b) \cup R_3(b,n)$$

is an (m,n)-path of length k in G-E′, a contradiction. Thus $xy \notin R_2$ and hence $R_2 \cap E' = \phi$. Now if $xy \notin R_3(b,n)$, then

$$R_2(m,b) \cup R_3(b,n)$$

is an (m,n)-path of length k in G-E′, a contradiction. Hence $xy \in R_3(b,n)$. Consequently, since $a \neq u$, $R_3(a,n) \cap E' = \phi$. But then

$$R_2(m,a) \cup R_3(a,n)$$

is an (m,n)-path of length k in G-E′, a contradiction. This completes the proof of the Case 2.

Case 3 : $r^* \geq r + 1$

In this case, since $r^* + s^* = r + s$ we have $r + 1 \leq r^* \leq$

$s^* \le s - 1$. Hence $r \le s - 2$. Now since, $x \in L_s(m)$ and $|R_1| = |R_2| = k$, we have $xy \notin R_1 \cup R_2$. Further, $R_3 \cap \{uu_1, uu_2\} = \phi$ since $|R_3| = k$, $u \in L_r(m)$, $r \le s - 2$ and $s^* \ge r + 1$. As in the previous cases we show that $R_2 \cap E' = \phi$.

Suppose that $uu_1 \in R_2$. If m_2 preceeds m on R_2, then one of the following subgraphs occurs :

$$R_2(a,u) \cup Q_1(u,m) \cup R_2(m,b)$$

or

$$R_2(a,m_2) \cup Q_2(m_2,u) \cup R_2(u,b) .$$

As each of these contains an (a,b)-path of length at most k having no edges of E″, we have a contradiction. Hence m preceeds m_2 on R_2. But then one of the following subgraphs occurs :

$$R_2(a,u) \cup Q_2(u,m_2) \cup R_2(m_2,b)$$

or

$$R_2(a,m) \cup Q_1(m,u) \cup R_2(u,b) .$$

As each of these contains an (a,b)-path of length at most k having no edge of E″, we again have a contradiction. Hence $uu_1 \notin R_2$. Similarly $uu_2 \notin R_2$ and so $R_2 \cap E' = \phi$.

Now if $xy \in R_3(a,n)$, then $R_2(m,b) \cup R_3(b,n)$ is an (m,n)-path of length k in G-E′, a contradiction. Hence $xy \notin R_3(a,n)$. If $xy \in R_3(b,n)$, then

$$R_2(m,a) \cup R_3(a,n)$$

is an (m,n)-path of length k in G-E′, again a contradiction. Consequently $R_3 \cap E' = \phi$. Hence $R_2 \cup R_3$ contains an (m,n)-path of length k containing no edges of E′, a contradiction. This completes the proof of the theorem. □

We now consider the case $(k,t) = (4,2)$. We begin with the following lemma.

Lemma 3.1 : Let $G \in \mathcal{G}(4,2)$ and $E' = \{uv, xy\}$ be edges of G such that $d_G(u,x) = d_{G-E'}(u,x) = 4$. If $m \in L_r(u)$ and $n \in L_s(u)$, $d_{G-E'}(m,n) > 4$, then either $r = 1$ or $s = 1$.

Proof : The situation here is very similar to that in the proof of Theorem 3.1. Thus there exists (m,n)-paths Q_1 and Q_2, in G, of length 4 with $Q_1 \cap E' = \{uv\}$ and $Q_2 \cap E' = \{xy\}$. Further, there are edges $E'' = \{mm_1, nn_1\}$ with $m_1, n_1 \notin Q_1 \cup Q_2$. There exist vertices $a \in L_{r^*}(m)$ and $b \in L_{s^*}(m)$ with $d_{G-E''}(a,b) > 4$, $r^* + s^* = 4$ and (a,b)-paths R_1 and R_2, in G, of length 4 with $R_1 \cap E'' = \{mm_1\}$ and $R_2 \cap E'' = \{nn_1\}$.

The subgraph $R_1 \cup R_2$ is a cycle of length 8 containing the vertices m and n. Consequently $E' \subseteq R_1 \cup R_2$, since otherwise there would exist an (m,n)-path of length 4 not containing edges of E'.

Now assume that $r \neq 1$ and $s \neq 1$. Then, by lemmas 2.5 and 2.8, $r \geq 2$, $s \geq 2$, and $r + s = 4$. Thus $r = s = 2$. We can without any loss of generality assume that v preceeds u on Q_1 and $r^* \leq s^*$. We now distinguish two cases according to the location of x and y on Q_2.

Case 1 : x preceeds y on Q_2

The situation is depicted in Figure 3.6

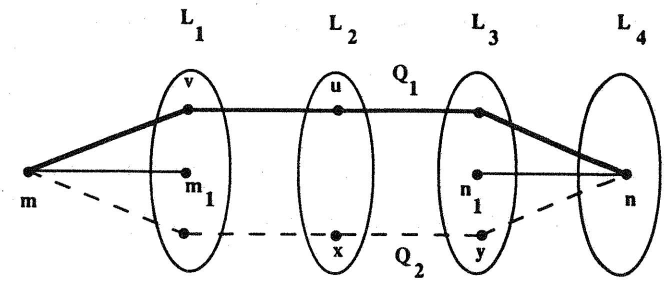

Figure 3.6

Suppose first that $r^* = 1$. Then $a \in L_1(m)$ and $b \in L_3(m)$. Since $nn_1 \in R_2$ and $|R_2| = 4$, $bn \in E(G)$. If $a = v$, then $R_1 \cap E' = \{xy\}$ since $E' \subseteq R_1 \cup R_2$ and R_1 has length 4 and passes through m_1. But then $y = b$ and hence $Q_1(v,n) \cup \{ny\}$ is an (a,b)-path of length 4 in $G-E''$, a contradiction. Hence $a \neq v$. By Lemma 2.10, $d_G(a) \geq 3$. Thus there exists a vertex $a_1 \in N_G(a)\backslash\{R_1 \cup R_2\}$. Hence, by Lemma 2.7 $d_G(a_1,b) = 3$. Since $d_{G-E''}(a,b) > 4$, the (a_1,b)-path $\hat{R}$ of length 3 must contain one of the edges E''. The only possibility is for $a_1 \in L_2(m)$, $nn_1 \in \hat{R}$ and thus $xy \notin \hat{R}$. But then $\{ma, aa_1\} \cup \hat{R}(a_1,n\}$ is an (m,n)-path of length 4 in $G-E'$, a contradiction. This proves that $r^* \neq 1$.

Next we suppose that $r^* = 2$. Then $a,b \in L_2(m)$. Suppose $a = u$. Since $Q_1(u,m) \cup Q_2(m,x)$ is a (u,x)-path of length 4 in $G-E''$, $b \neq x$. But then $xy \notin R_1 \cup R_2$ otherwise $|R_1|$ or $|R_2|$ is greater then 4, a contradiction. Hence $a \neq u$. By the same argument we establish that a, b, u and x are distinct vertices. Since $|R_1| = |R_2| = 4$, neither R_1 nor R_2 contains xy. But then $E' \nsubseteq R_1 \cup R_2$, a contradiction. This completes the proof of Case 1.

Case 2 : y preceeds x on Q_2

The situation is depicted in Figure 3.7

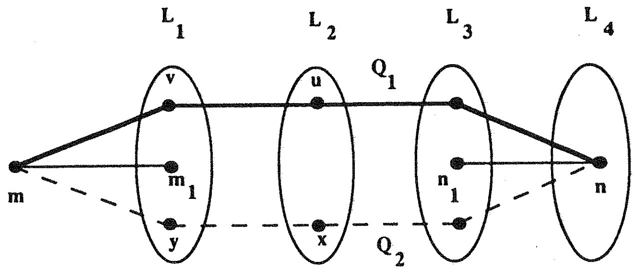

Figure 3.7

Clearly, if a = u (a = x), then b ≠ x (b ≠ u). Since $mm_1 \in R_1$, $|R_1| = 4$, $nn_1 \in R_2$ and $|R_2| = 4$ we must have $|R_1 \cap E'| \le 1$ and $|R_2 \cap E'| \le 1$. Further, if $|R_1 \cap E'| = 1$, then $R_2 \cap E' = \phi$. Hence $E' \not\subseteq R_1 \cup R_2$, a contradiction. This completes the proof of the lemma.

Theorem 3.2 : $\mathcal{G}(4,t) = \phi$ for $t \ge 2$.

Proof: In view of Lemma 2.2 we need only prove that $\mathcal{G}(4,2) = \phi$. Assume to the contrary that $\mathcal{G}(4,2) \ne \phi$ and let $G \in \mathcal{G}(4,2)$.

Letting u and x be vertices of G with $d_G(u,x) = 4$ and following the same line of argument as in the proof of Theorem 3.1 we define edge-disjoint (u,x)-paths P_1 and P_2 of length 4 with $uv \in P_1$ and $uw \in P_2$. Further, we define

$$E' = \{uv, xy\}$$

where $y \notin P_2$. Observe that $d_{G-E'}(u,x) = 4$. Hence, by Lemma 3.1 there exists vertices $m \in L_1(u)$ and $n \in L_3(u)$ with $d_{G-E'}(m,n) > 4$. We take E″, a, b, Q_1, Q_2, R_1 and R_2 as in the proof of Lemma 3.1. Further, we assume without any loss of generality that $r^* \le s^*$. We distinguish three cases according to the location of v and u on Q_1 and x and y on Q_2.

Case 1 : v preceeds u on Q_1 and x preceeds y on Q_2

Then y = n and m = v. Figure 3.8 depicts the situation.

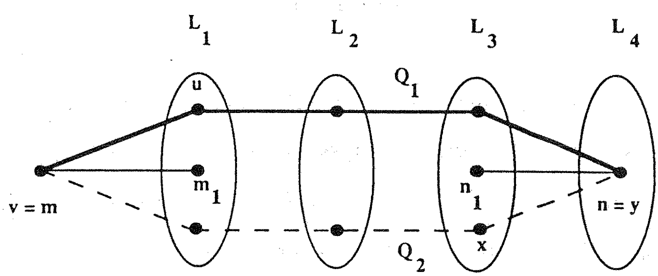

Figure 3.8

Observe that $xy \notin R_1$, since $mm_1 \in R_1$ and $|R_1| = 4$. Similarly $uv \notin R_2$. As in the proof of Lemma 3.1, $E' \subseteq R_1 \cup R_2$. Consequently $uv \in R_1$ and $xy \in R_2$.

First suppose that $r^* = 1$. Then $a = u$ or m_1 since $uv \in R_1$ and $|R_1| = 4$. Further $bn \in E(G)$ since $nn_1 \in R_2$ and $|R_2| = 4$. If $a = u$, then $b \ne x$ and a must preceed m on R_1. But then $R_1(m,b) \cup \{bn\}$ is an (m,n)-path of length 4 in G-E′, a contradiction. Therefore $a \ne u$. Hence $a = m_1$. Similarly $b = n_1$. Now every (a,b)-path T of length 4, in G, must contain exactly one edge of E″. Further, if $m_1m \in T$ ($n_1n \in T$), then $mu \in T$ ($xy \in T$), for otherwise $T(m,b) \cup \{bn\}$ ($\{mm_1\} \cup T(m_1,n)$) is an (m,n)-path of length 4 in G-E′, a contradiction. Now $d_{G-E'}(a,b) > 4$, $a \in L_2(u)$ and $b \in L_2(u)$, contradicting Lemma 3.1. Hence $r^* \neq 1$.

The only possibility is $r^* = s^* = 2$. Recall that $uv \in R_1$ and $xy \in R_2$. Without any loss of generality we may take $R_1 = (a, u, m, m_1, b)$. Because $d_G(u,x) = 4$, $R_2 = (b, x, y, n_1, a)$. Since $d_G(a) \geq 3$, there is a vertex $\alpha \notin R_1 \cup R_2$ that is adjacent to a. By Lemma 2.7, $d_G(\alpha,b) = 3$. Hence because of the property of (a,b)-paths mentioned above $\alpha \in L_1(m)$ or $L_3(m)$. Now $\alpha \notin L_1(m)$, since otherwise $\{m\alpha, \alpha a\} \cup R_2$ (a,n) is an (m,n)-path of length 4 in G-E′. Hence $\alpha \in L_3(m)$. But then (α, n, n_1, b) is an (α,b)-path of length 3 in G, implying that n_1 is joined to both a and b, a contradiction. This completes the proof for Case 1.

Case 2 : v preceeds u on Q_1 and y preceeds x on Q_2

Then v = m and $y \in L_2(m)$. Figure 3.9 depicts the situation.

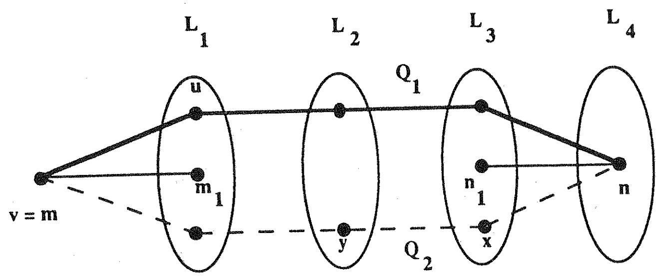

Figure 3.9

Observe that $uv \notin R_2$, since $nn_1 \in R_2$ and $|R_2| = 4$. Hence, since $E' \subseteq R_1 \cup R_2$, $uv \in R_1$.

Now suppose that $r^* = 1$. Then $a = u$ or m_1, since $uv \in R_1$ and $|R_1| = 4$. As in Case 1 above $a \neq u$. Consequently $a = m_1$. Since $d_G(a) \geq 3$, there is a vertex $\beta \notin R_1 \cup R_2$ that is adjacent to a. By Lemma 2.7, $d_G(\beta,b) = 3$. Let S be a (β,b)-path of length 3. Since $d_{G-E''}(a,b) > 4$, S must contain mm_1 or nn_1. Therefore, since $b \in L_3(m)$, $\beta \in L_2(m)$. Now, since $S = (\beta, n_1, n, b)$, (m, m_1, β, n_1, n) is an (m,n)-path of length 4 in $G-E'$, a contradiction. Hence $r^* \neq 1$.

The only possibility is for $r^* = s^* = 2$. Since $xy \in R_2$ one of a or b must be y. Suppose without any loss of generality, $b = y$. Then $R_2 =(y, x, n, n_1, a)$. If $am_1 \in E(G)$, then $\{mm_1, m_1a\} \cup R_2(a,n)$ is an (m,n)-path of length 4 in $G-E'$, a contradiction. Hence $am_1 \notin E(G)$ and thus $R_1 =(a, u, m, m_1, b)$. Now applying the same argument as in the corresponding case in Case 1 will yield the desired contradiction.

Case 3 : u preceeds v on Q_1 and y preceeds x on Q_2

The situation is depicted in Figure 3.10.

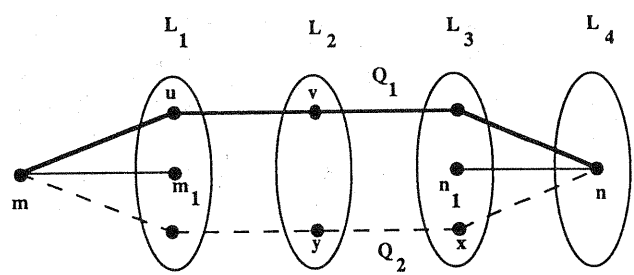

Figure 3.10

Suppose that $r^* = 1$. Then $a \in L_1(m)$ and $b \in L_3(m)$. Since $nn_1 \in R_2$ and $|R_2| = 4$, $bn \in E(G)$. If $a \neq u$ and m_1, then $uv \notin R_1 \cup R_2$, since $|R_1| = |R_2| = 4$. Consequently $E' \nsubseteq R_1 \cup R_2$, a contradiction. Hence $a = u$ or m_1. Now using a similar argument as in Case 1 above establishes $r^* \neq 1$.

The only possibility is $r^* = s^* = 2$. Then $a, b \in L_2(m)$. Suppose $a = v$. Since $Q_1(v,m) \cup Q_2(m,y)$ is a (v,y)-path of length 4 in $G-E''$, $b \neq y$. But then $xy \notin R_1 \cup R_2$, otherwise $|R_1|$ or $|R_2|$ is greater than 4, a contradiction. Hence $a \neq v$. By the same argument we establish that a, b, v and y are distinct vertices. Since $|R_1| = |R_2| = 4$, neither R_1 nor R_2 contains xy. Consequently $E' \nsubseteq R_1 \cup R_2$, a contradiction. This completes the proof of the theorem. □

The method of proof used in Lemma 3.1 and Theorem 3.2 can be applied to the case $k = 5$ with very little change. In fact, conclusion of the Lemma 3.1 is valid for $k = 5$. We do not detail the case analysis here but simply state the result. However, the methods do not extend beyond $k = 5$ and so the cases $k \geq 6$, $t = 2$ remain unresolved.

Theorem 3.3 : $\mathcal{G}(5,t) = \phi$ for $t \geq 2$. □

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